Advanced Algorithmic Techniques (COMP523)

Greedy Algorithms 3

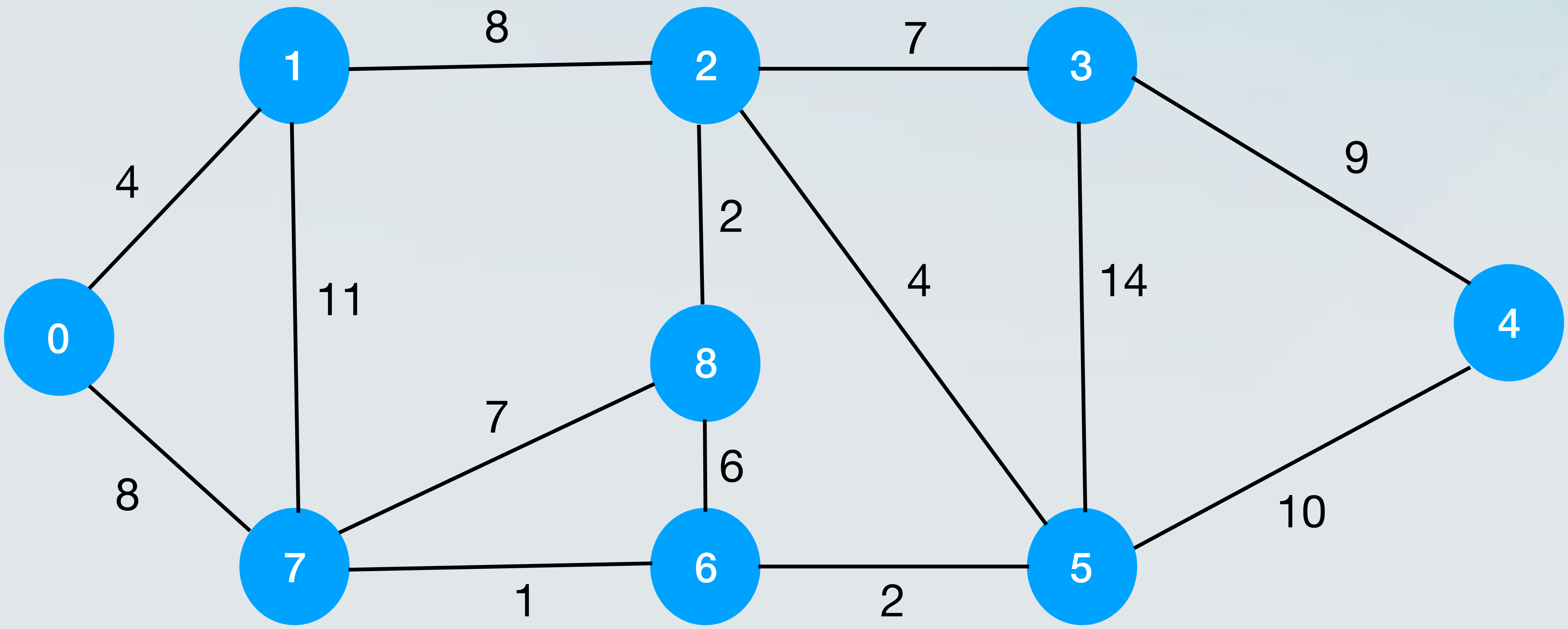
Recap and plan

- **Last lecture:**
  - Minimum Spanning Tree
  - Kruskal’s Algorithm
  - Prim’s Algorithm
- **This lecture:**
  - Prim’s Algorithm (cont.)
  - Clustering

Prim's Algorithm

- Start with an empty set of edges T.
- Start with a node s.
  - Add an edge e=(s,w) to T.
  - Which one?
  - The one with the minimum cost c_e.
- We continue like this.
  - We only consider edges to neighbours that are not in the spanning tree.

Example



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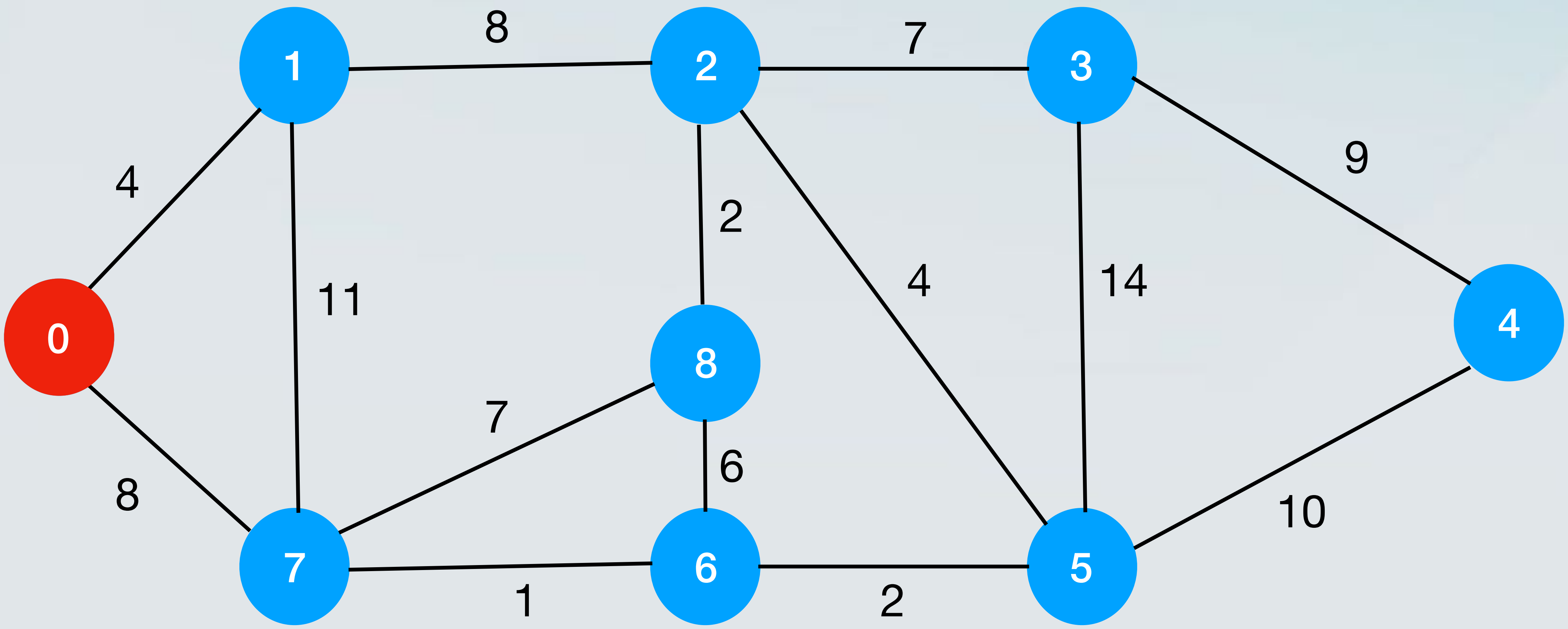

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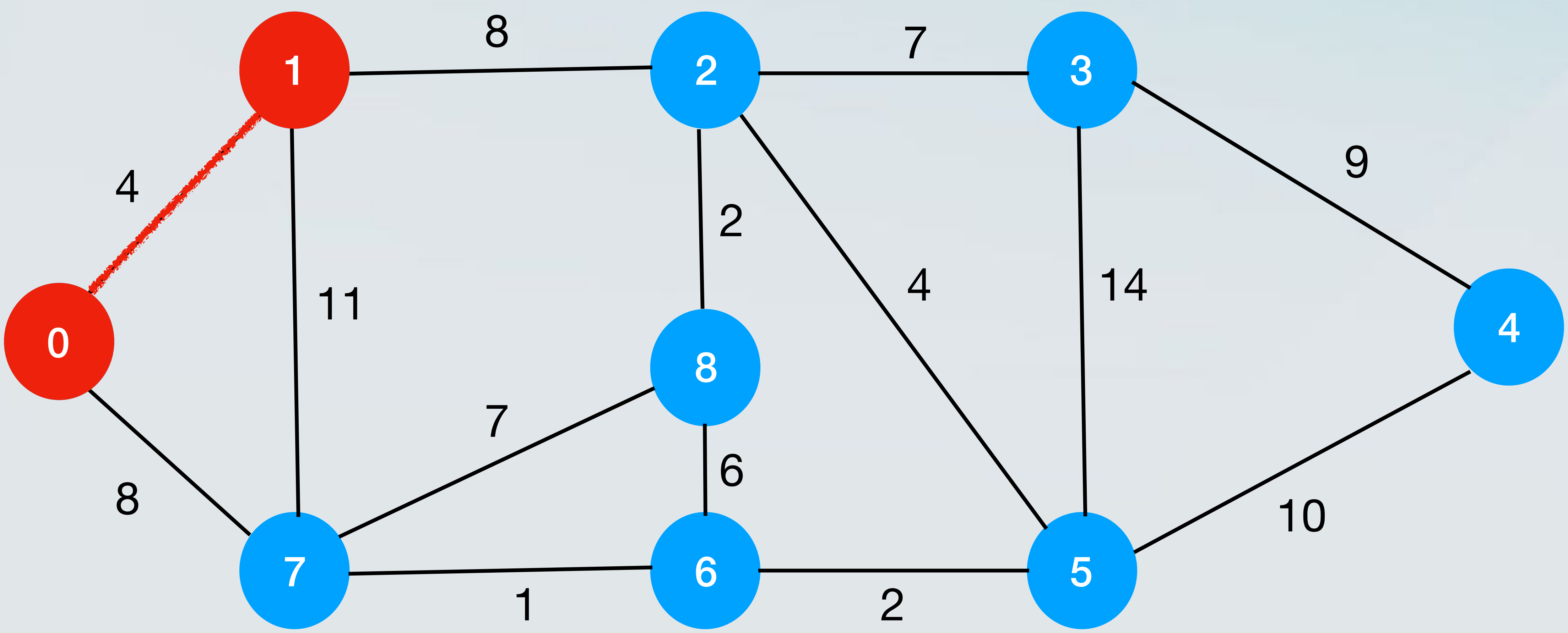

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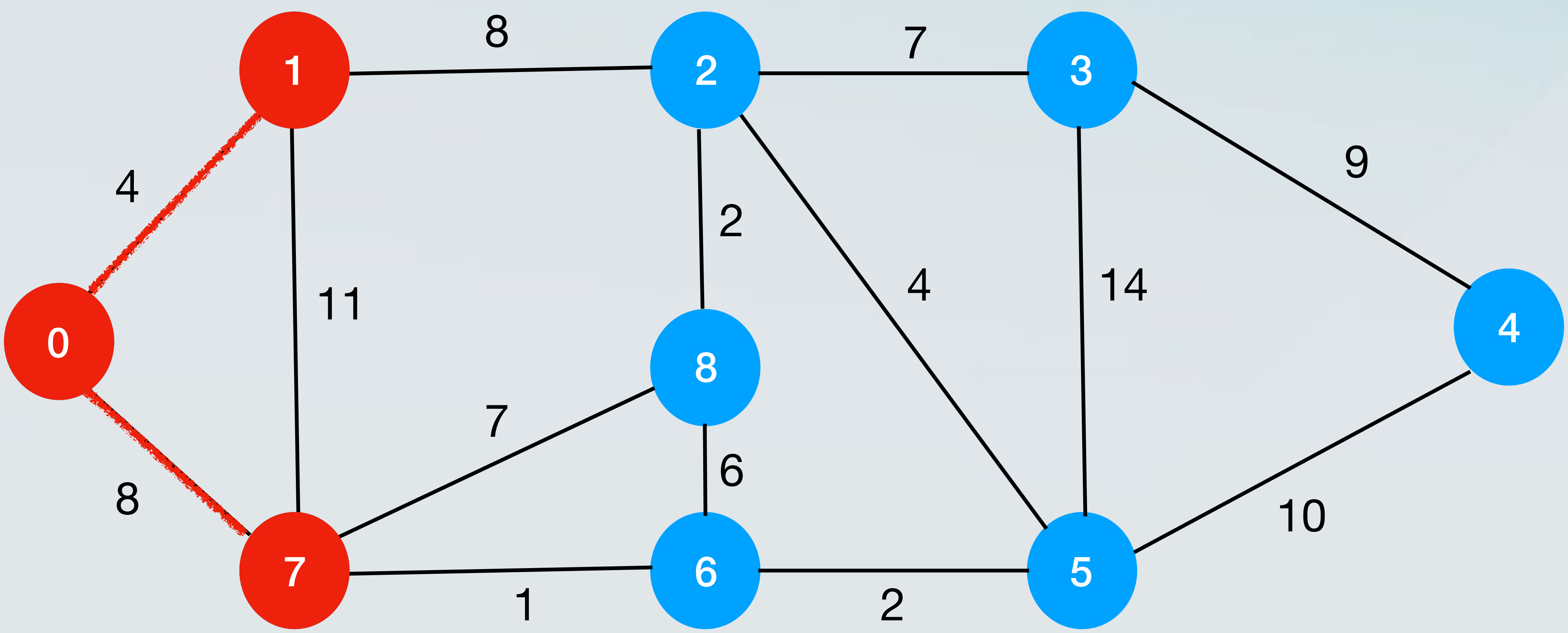

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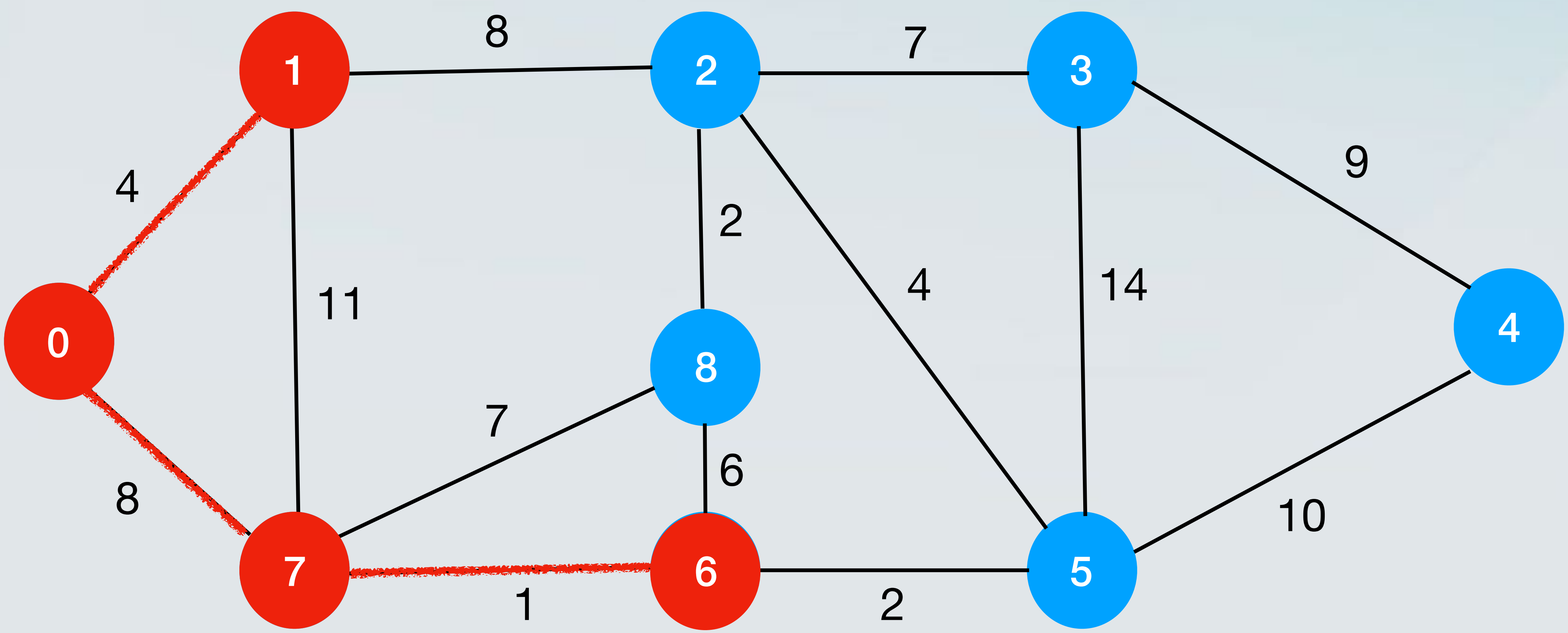

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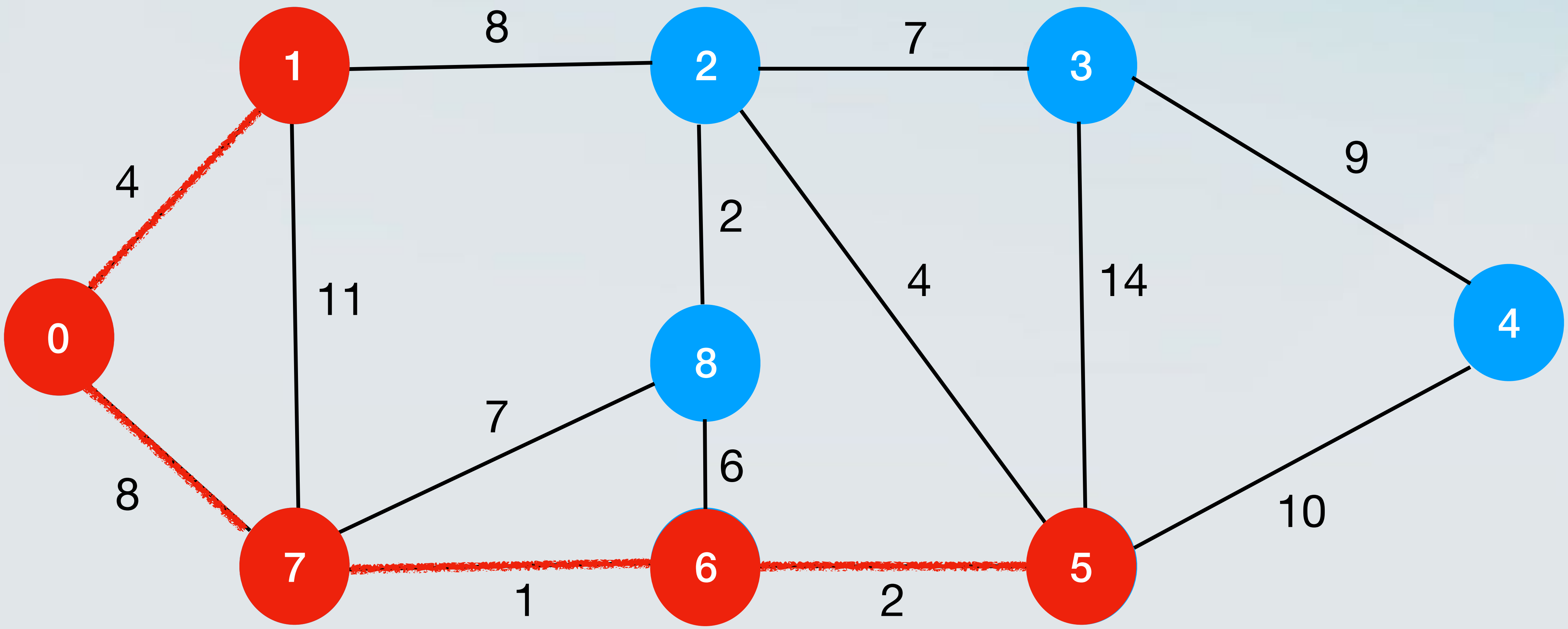

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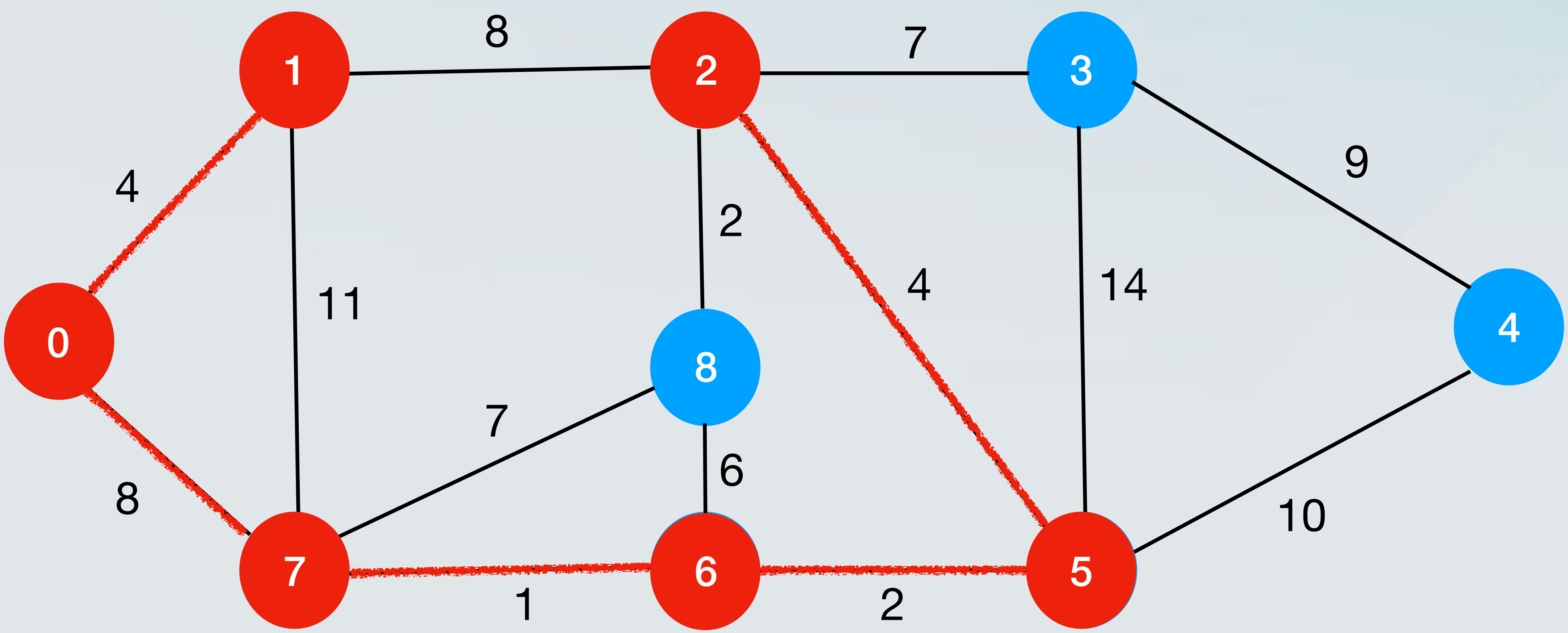

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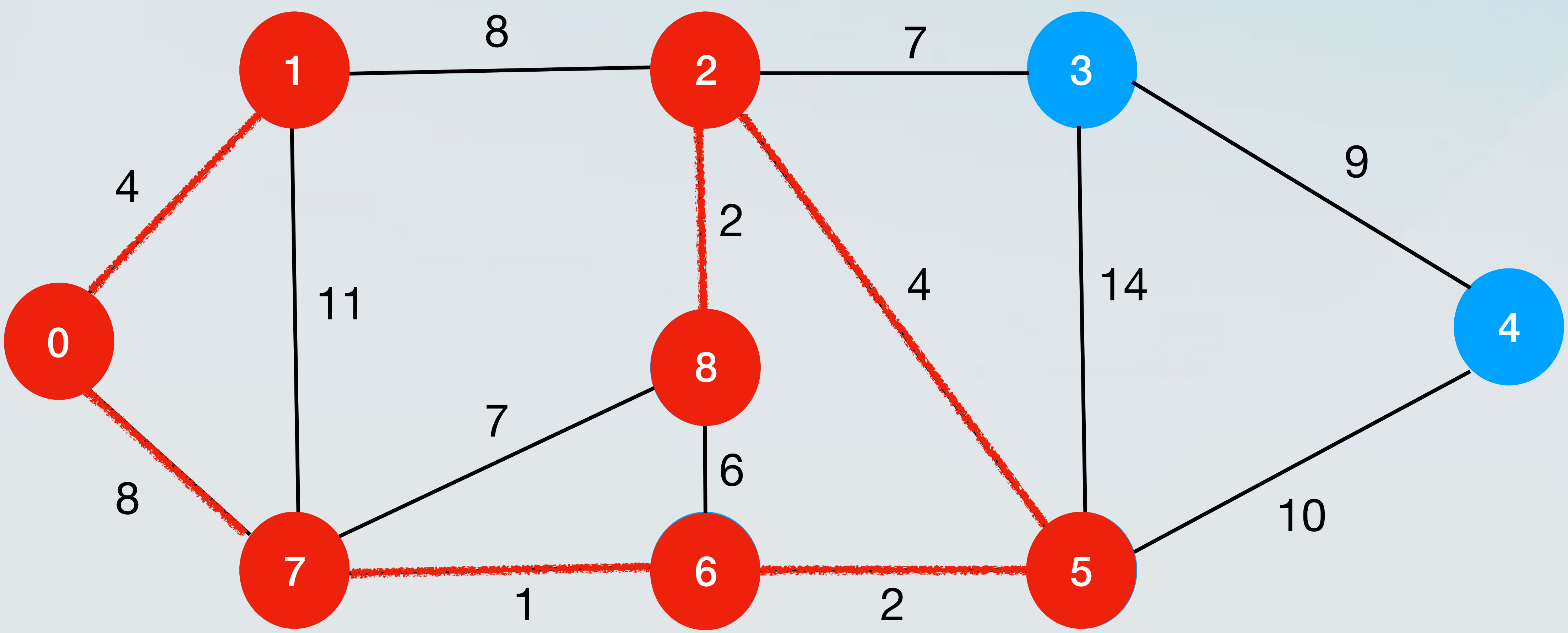

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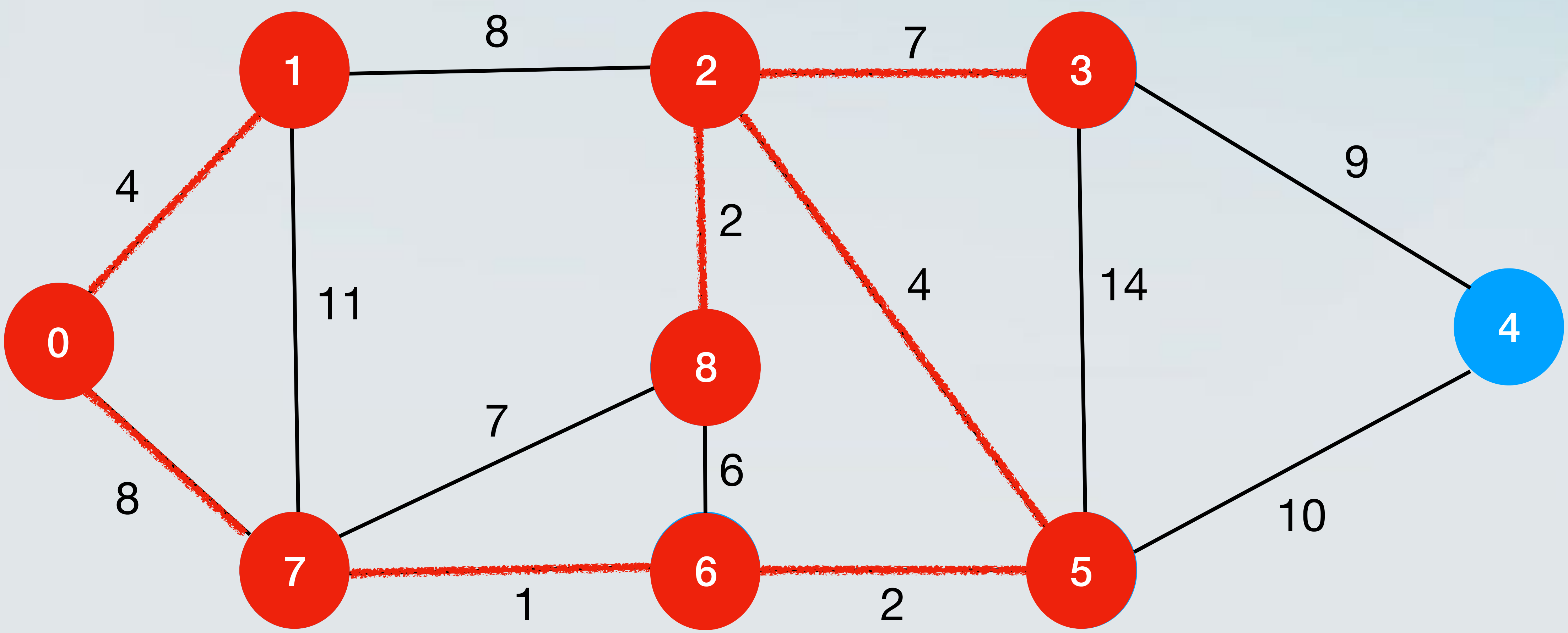

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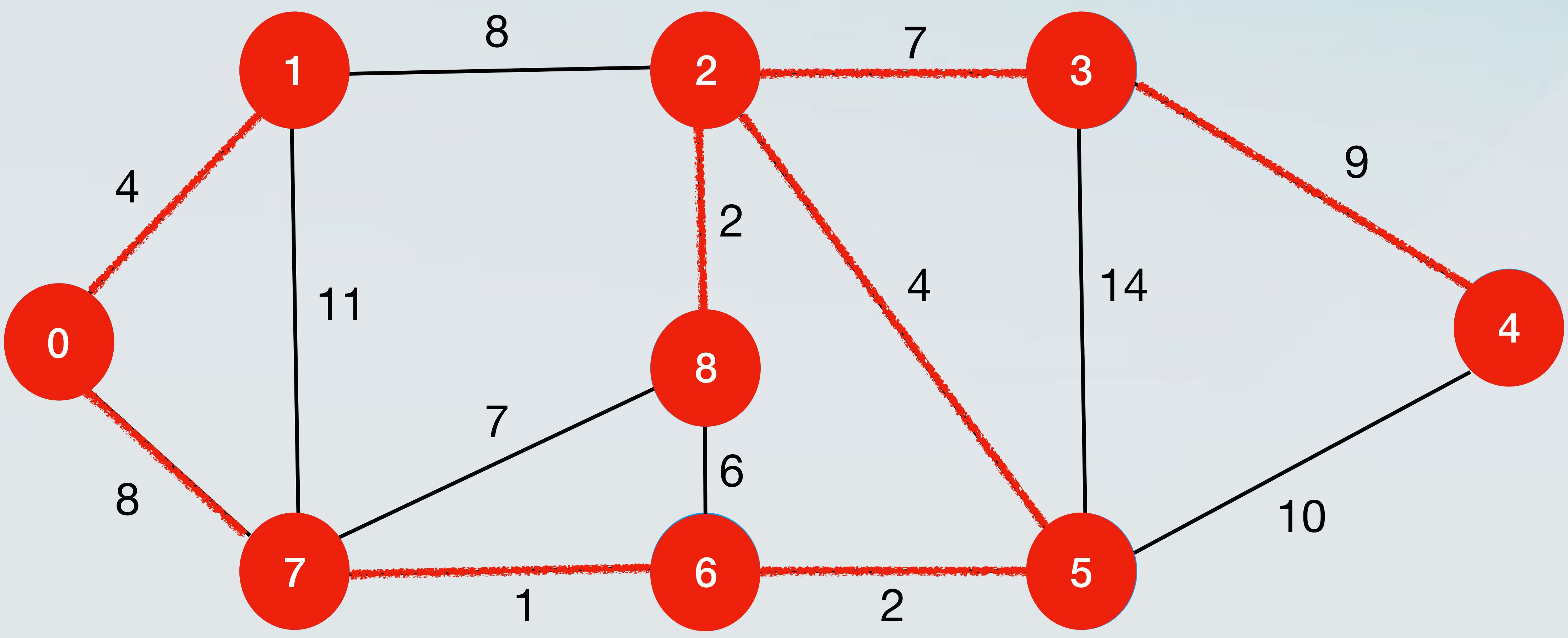

Optimality and running time

- Optimality argued in the last lecture.
- Running time?

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  - $O(n^2)$.

Data Structures

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Priority Queue

- Maintains
  - A set of elements S.
  - A *key* **key**(v) for each element v in S.
    - The key denotes the priority of v.
- Operations:
  - **Add**(v) - with priority key.
  - **Delete**(v)
  - **Extract_Min**(v)
  - **Change_key**(v)

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  - PQ operations can be implemented in $O(\log n)$ time.

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  - Running time O(m log n).

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  - Objects in a group exhibit high similarity.
- Applications: Many, e.g., machine learning.

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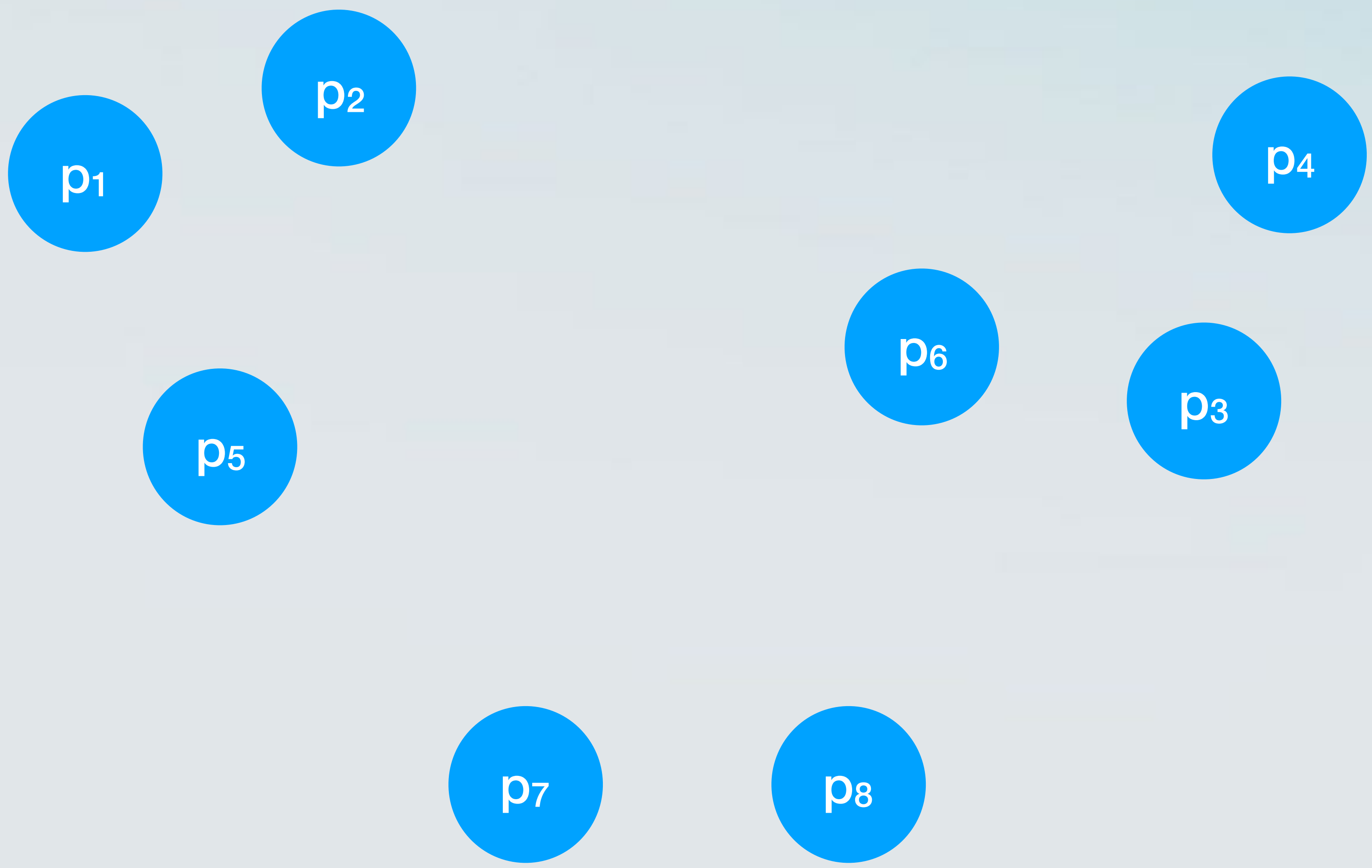
  - E.g., running time, algorithmic principle.

Clustering (concretely)



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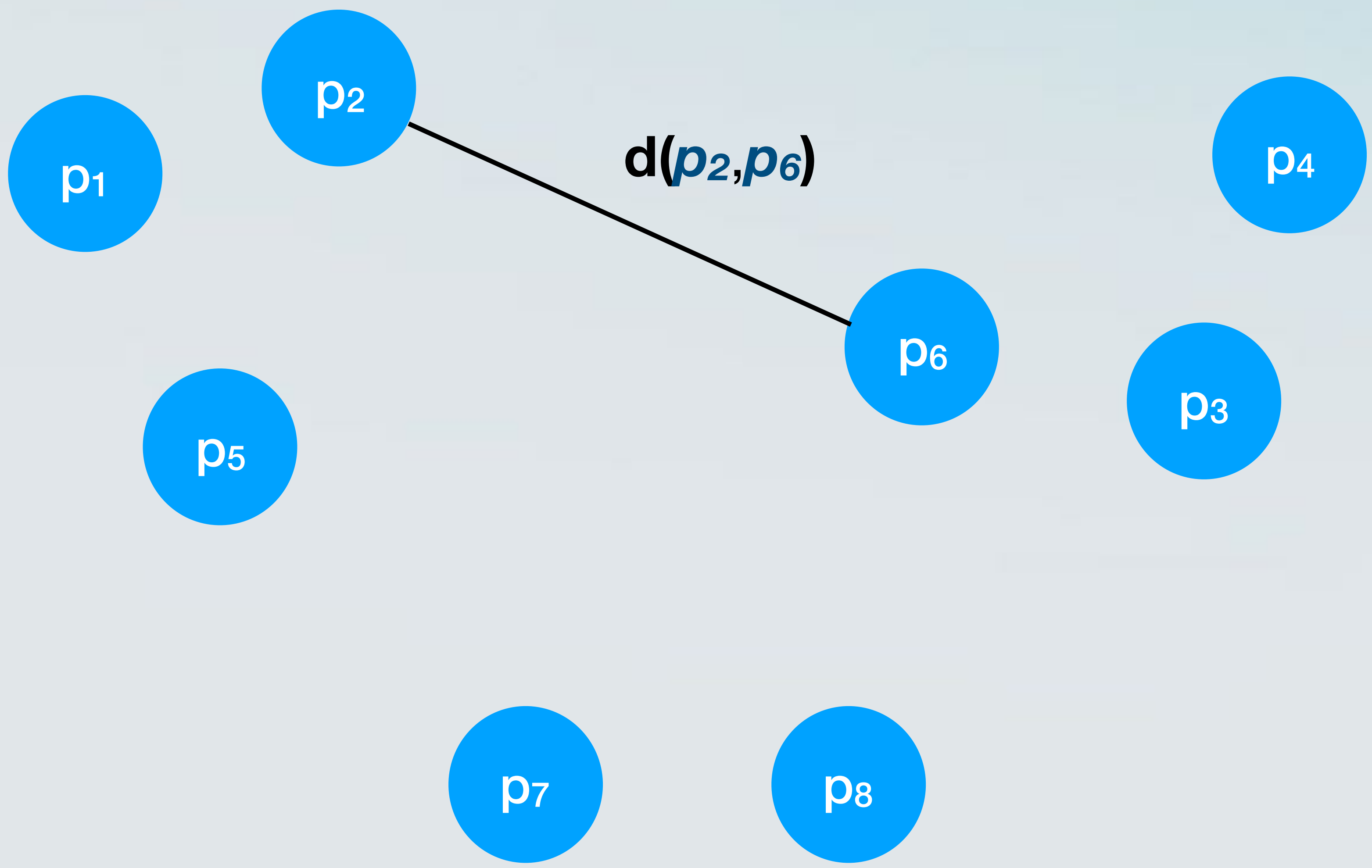

Properties of distance

- $d(p_i,p_i)=0$ for any $i = 1, \ldots , n$
- $d(p_i,p_j)>0$ for any $i \neq j$
- $d(p_i,p_j)=d(p_j,p_i)$ for any i, j

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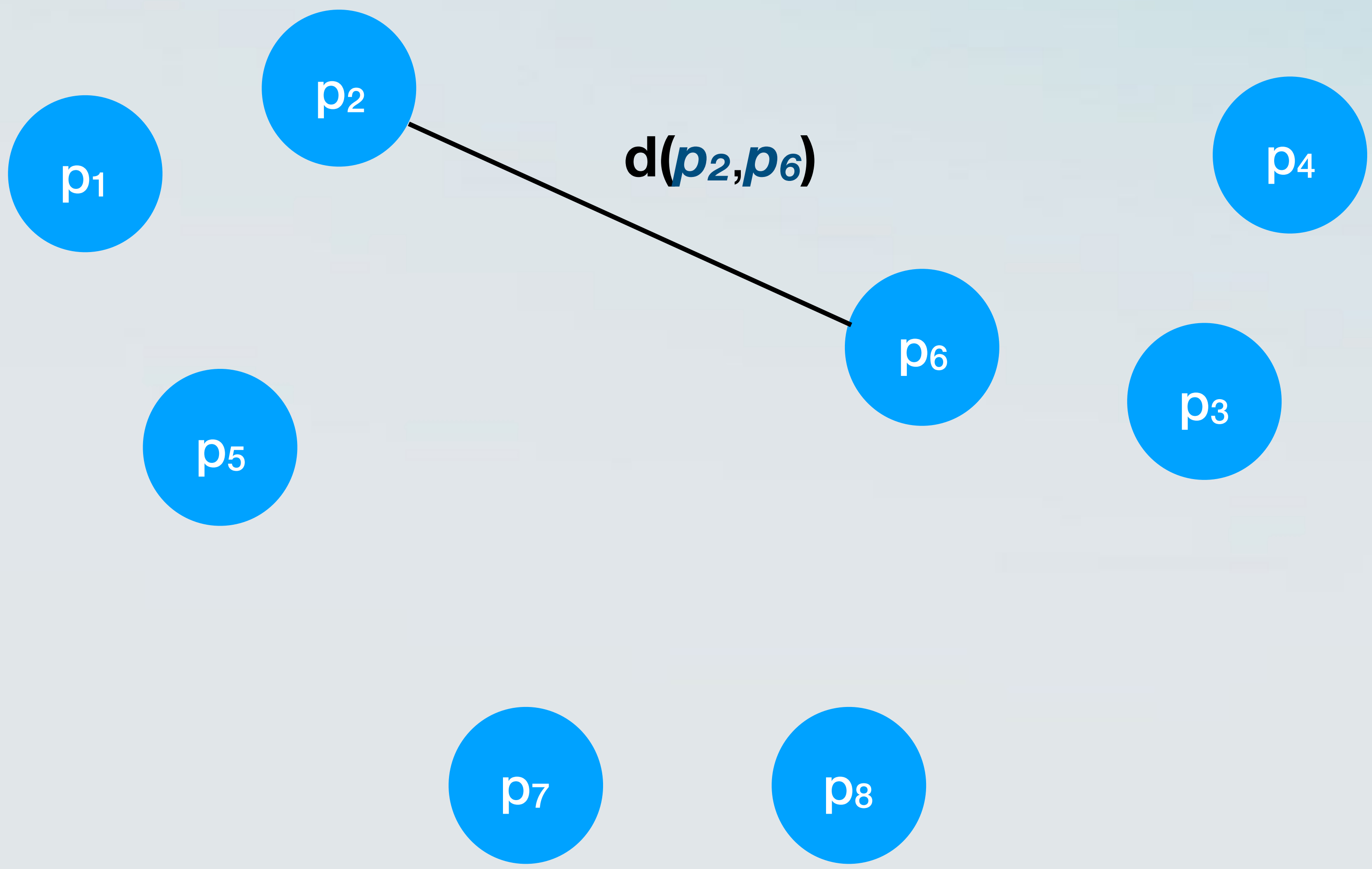

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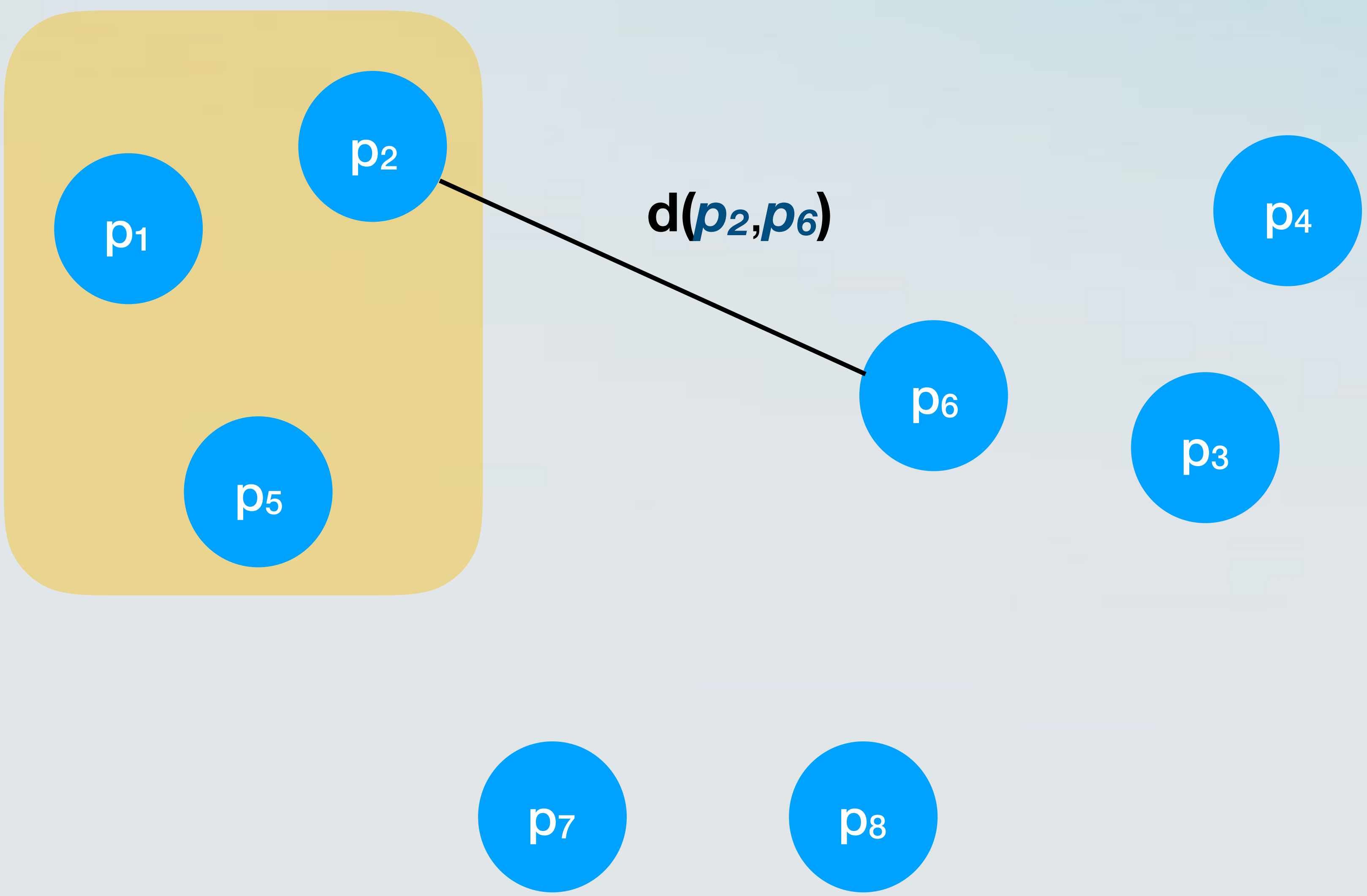

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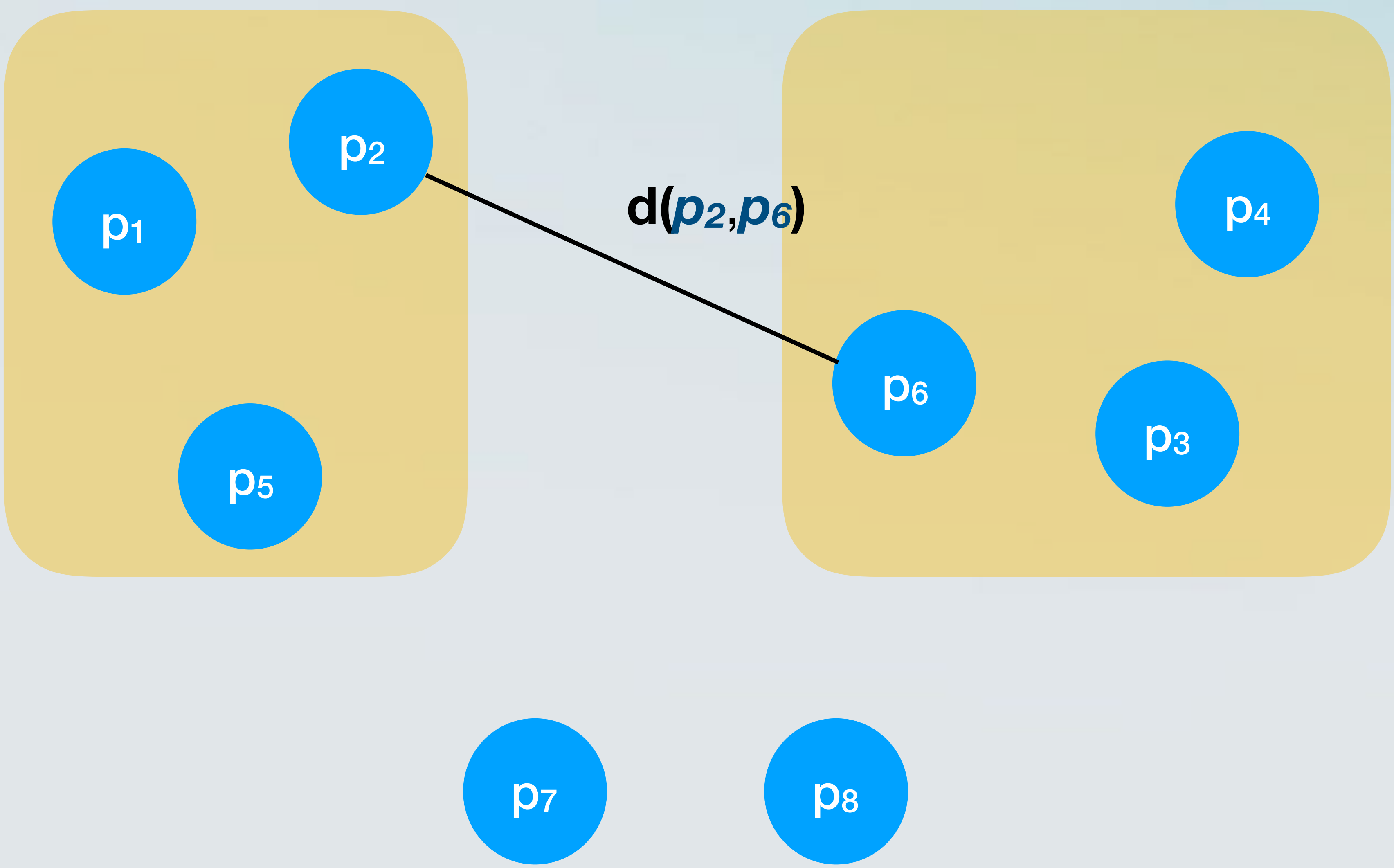

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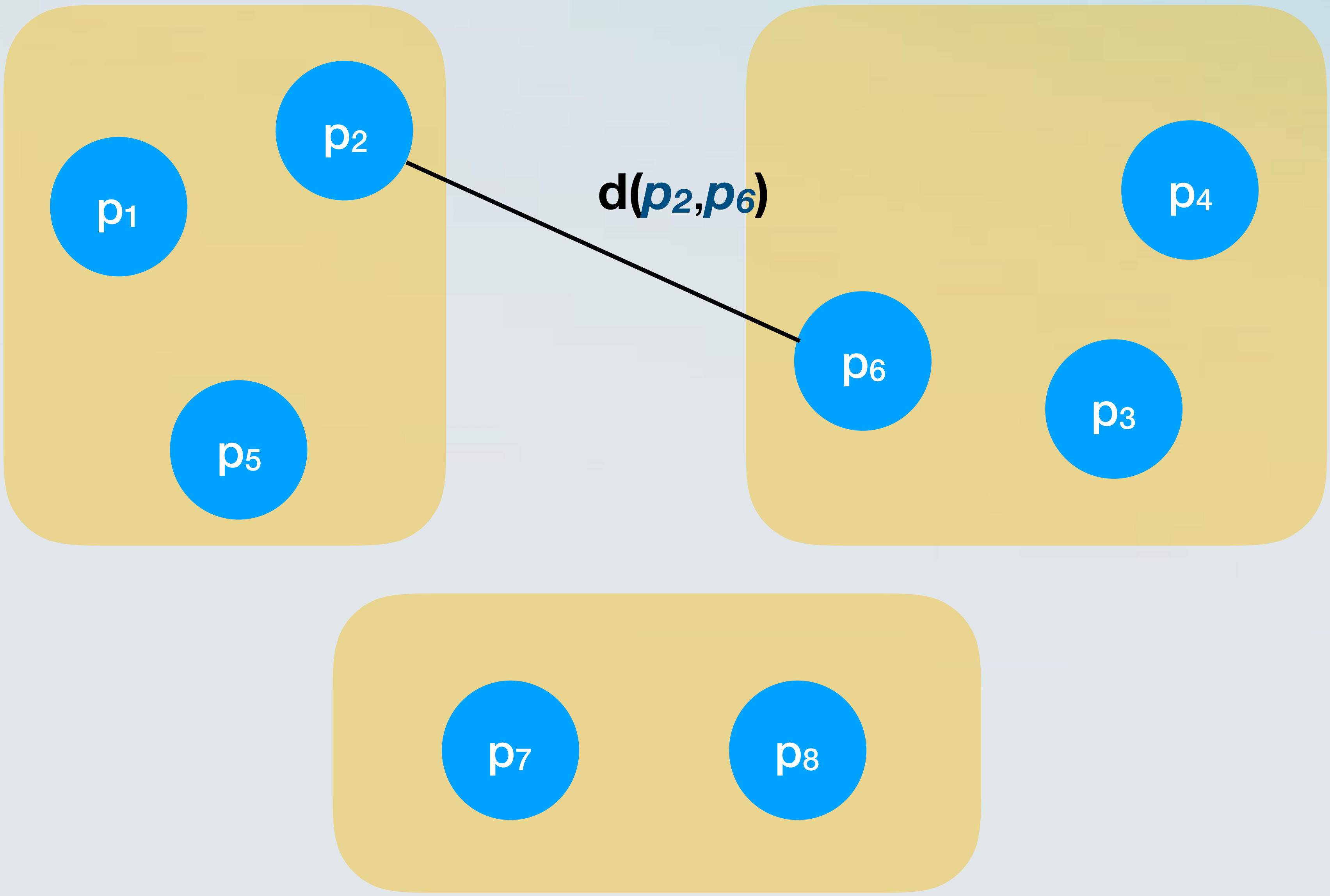

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- Definition: The *spacing* of a k-clustering is the minimum distance between any pair of points in different clusters.
- Goal: Among all possible k-clusterings, find one with the *maximum possible spacing*.

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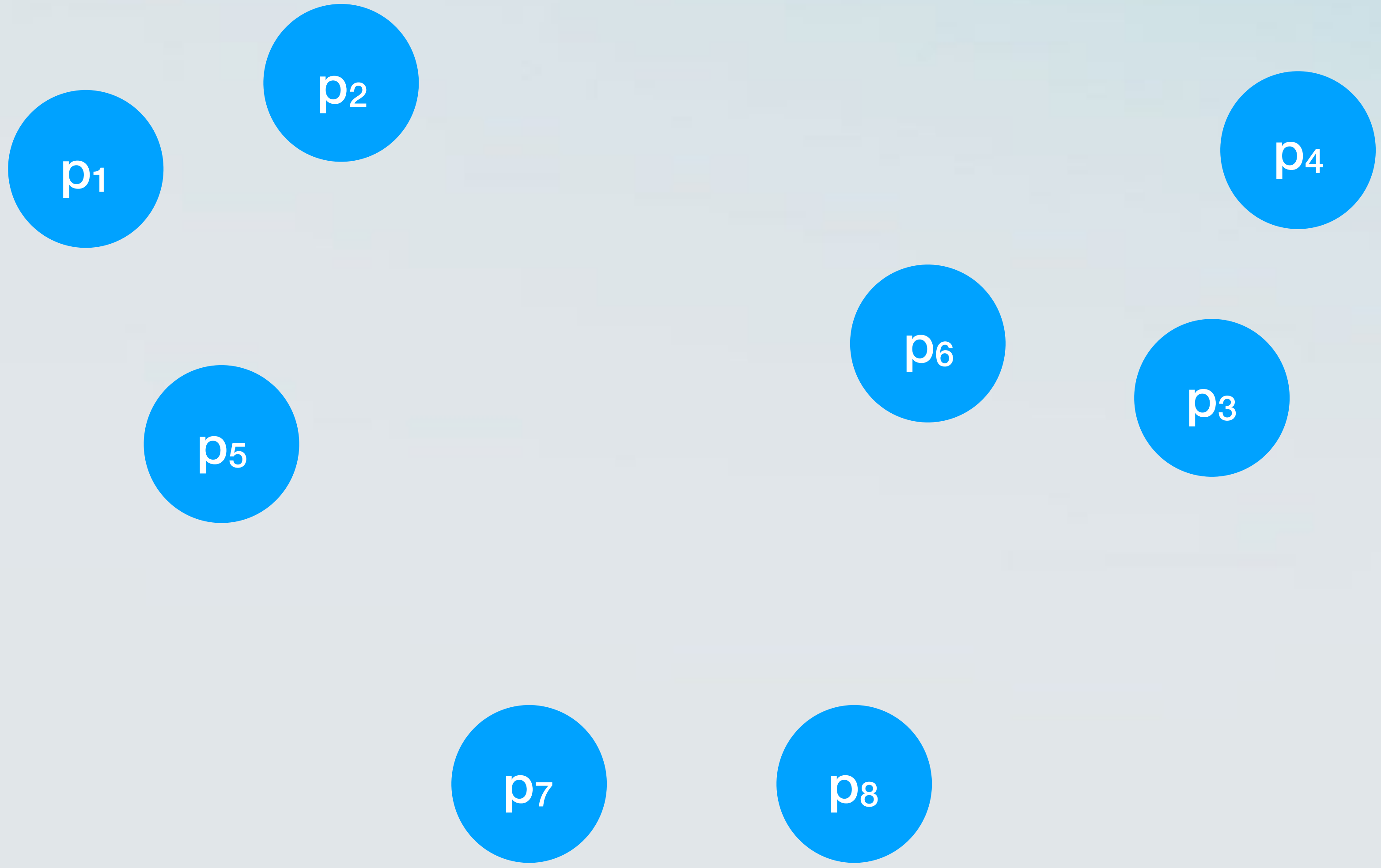

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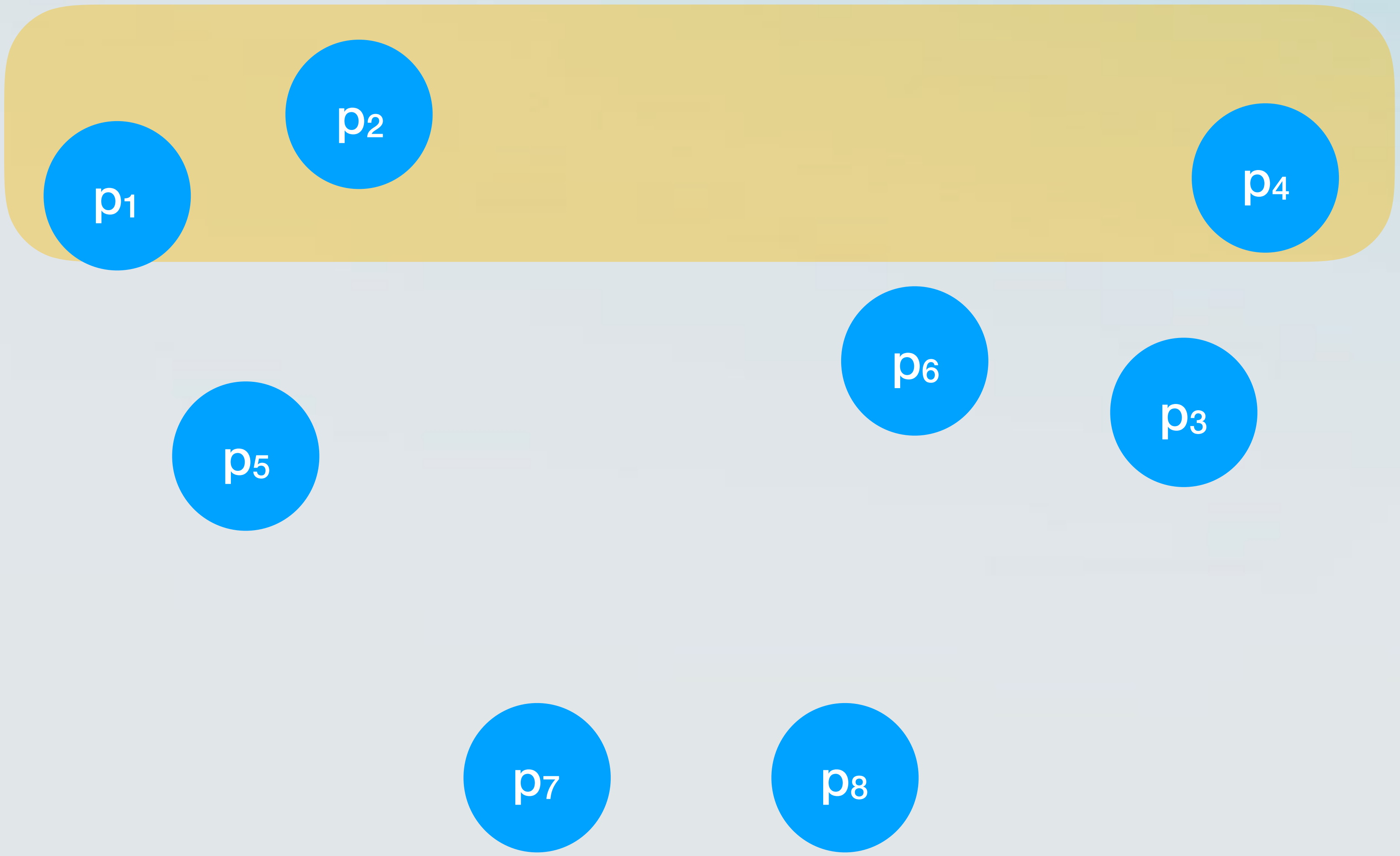

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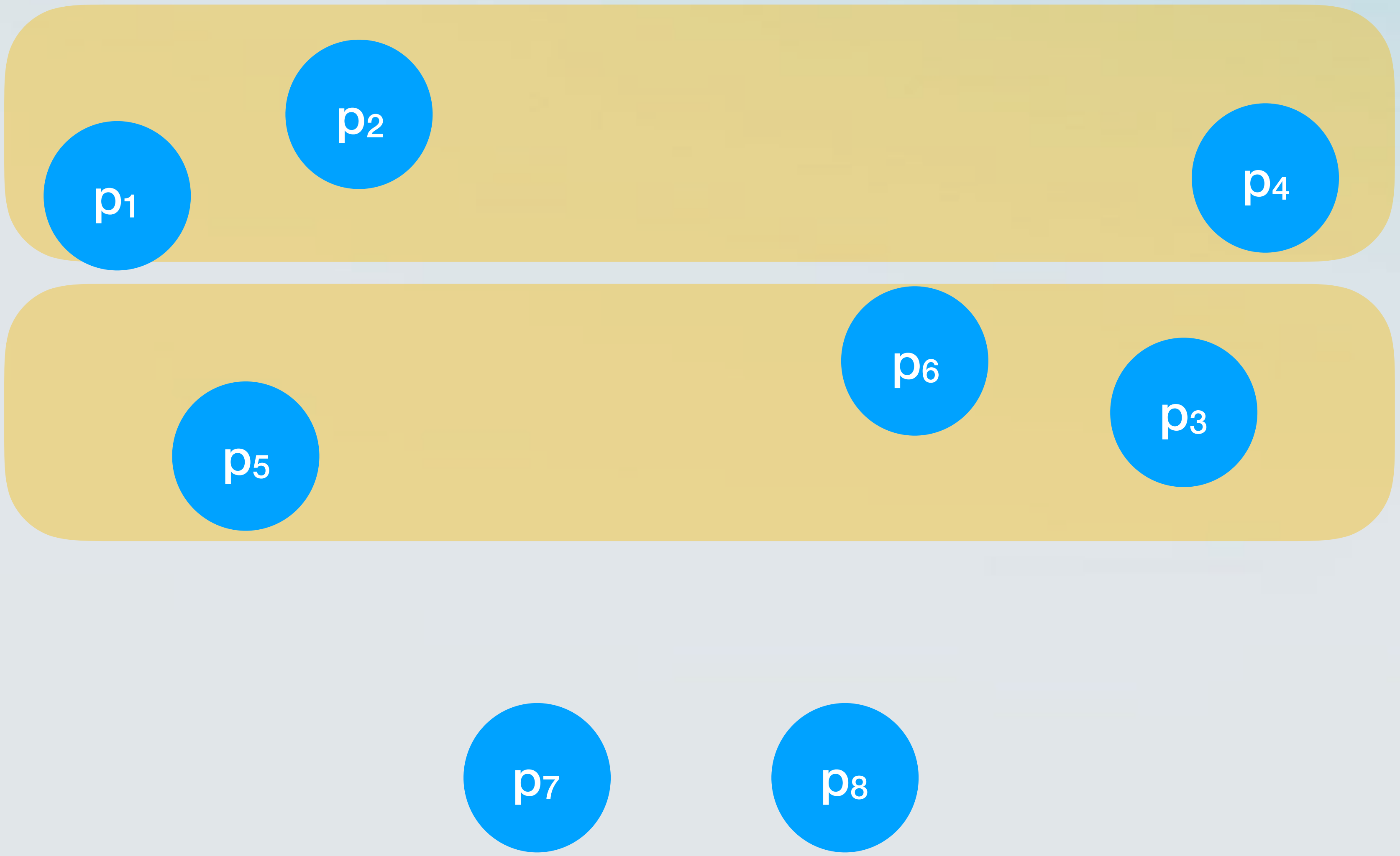

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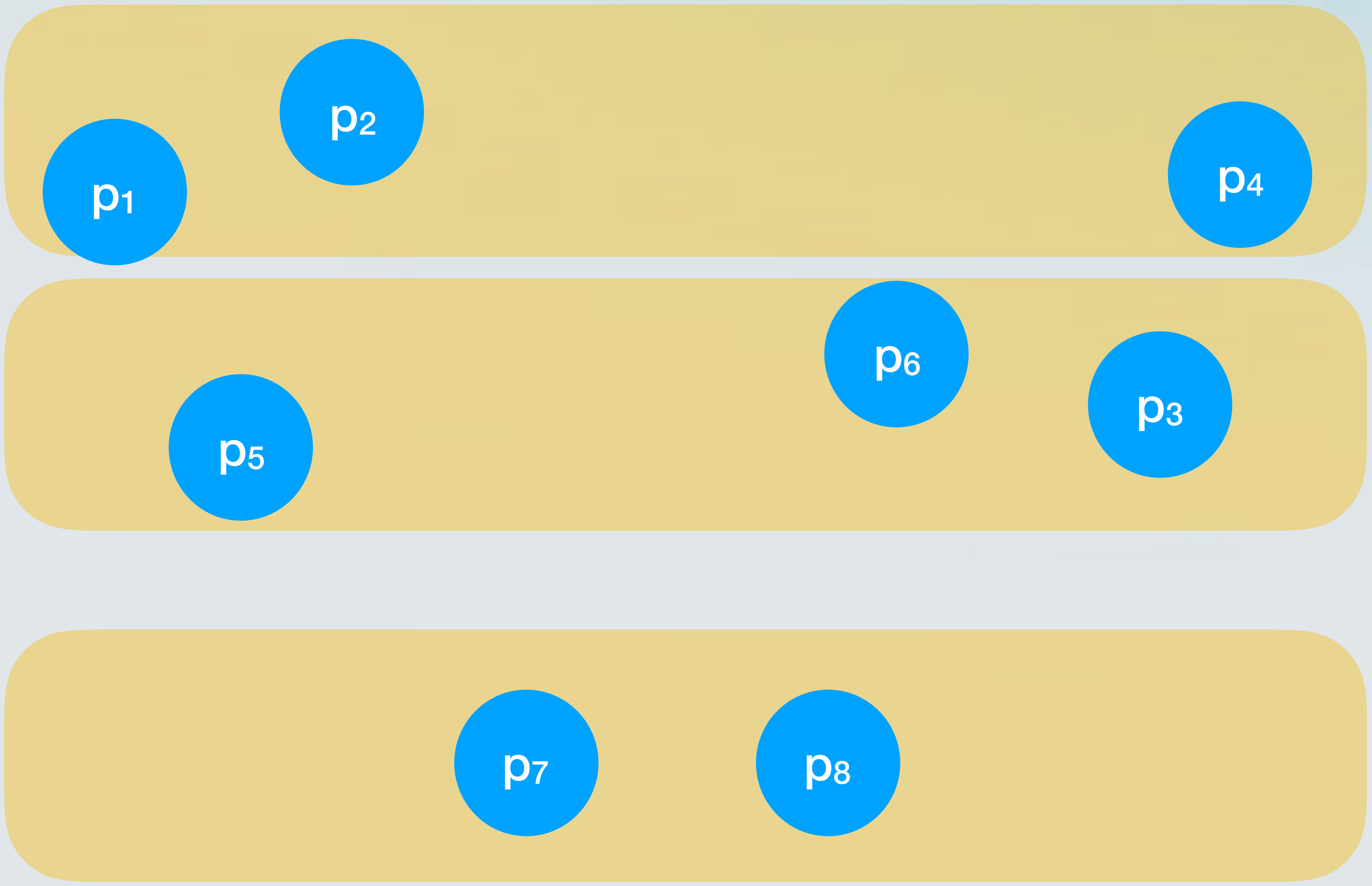

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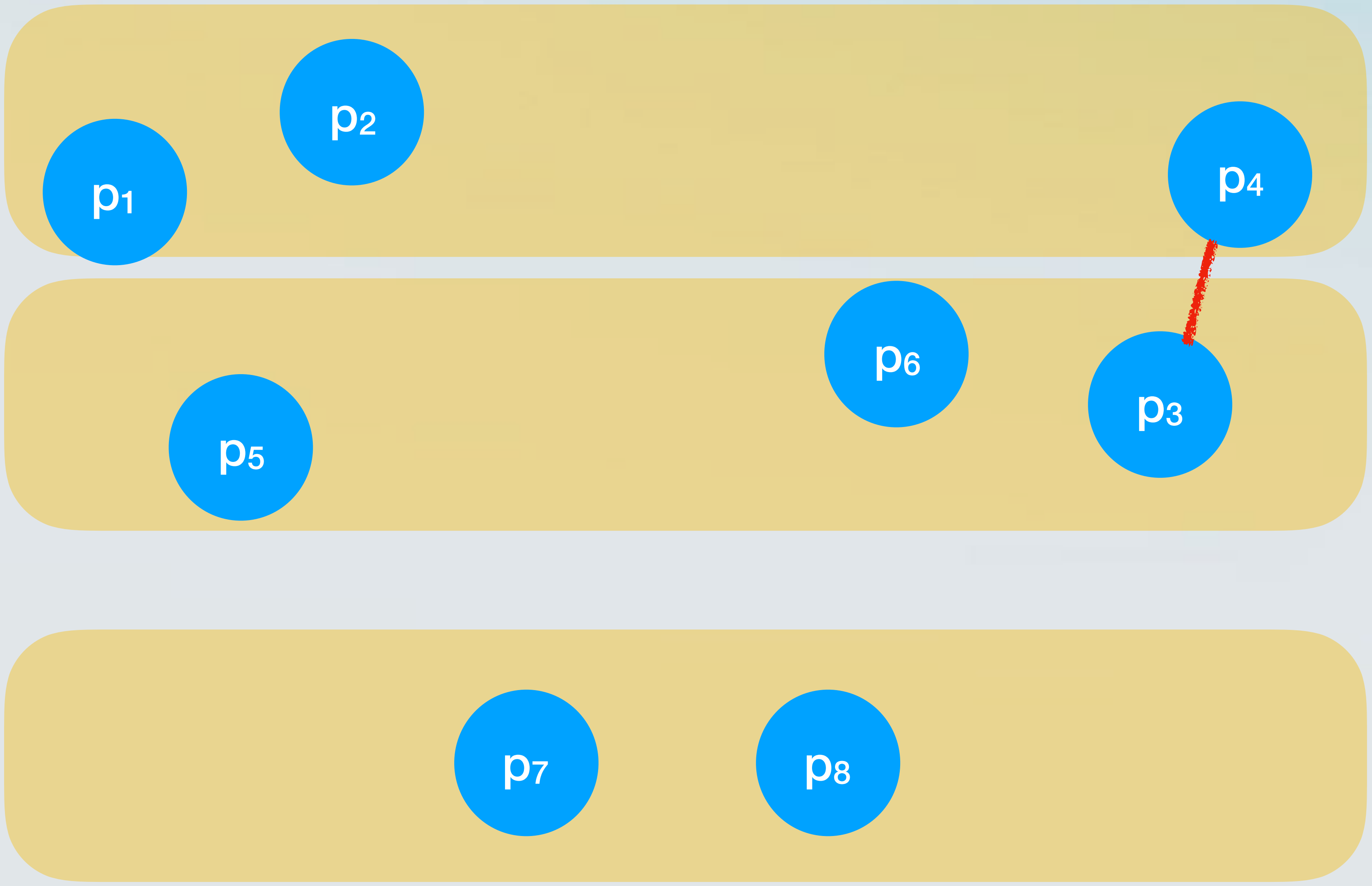

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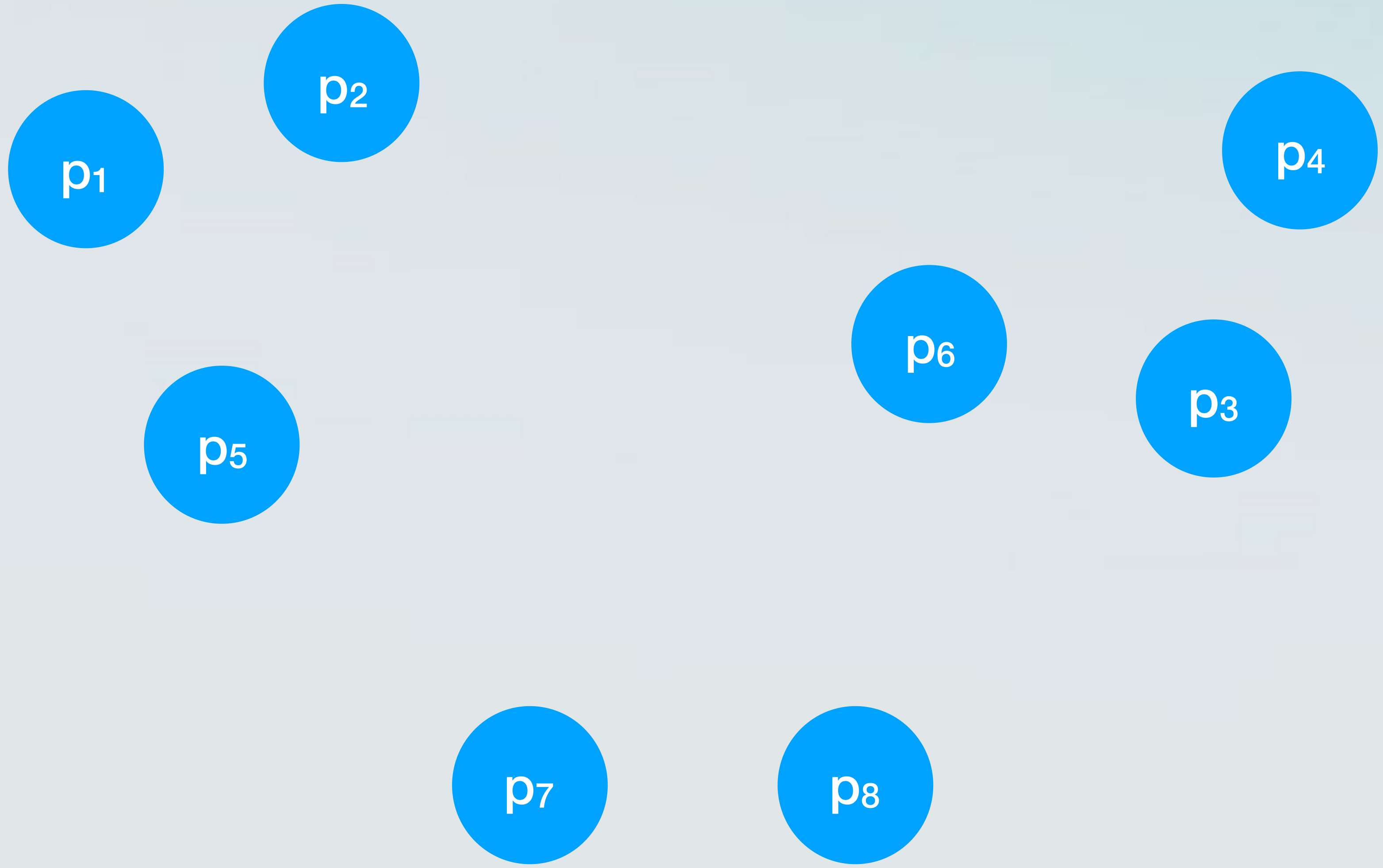

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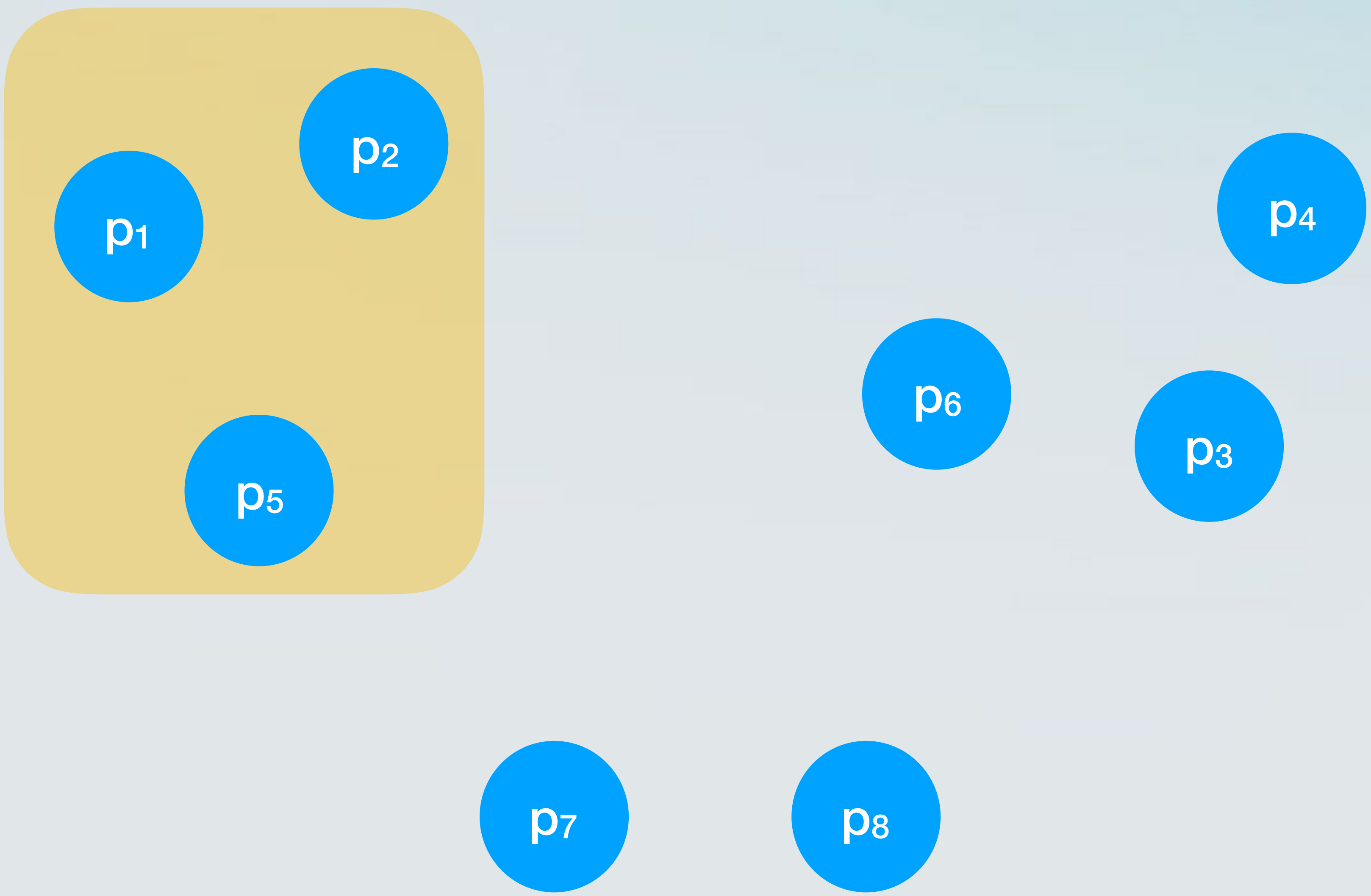

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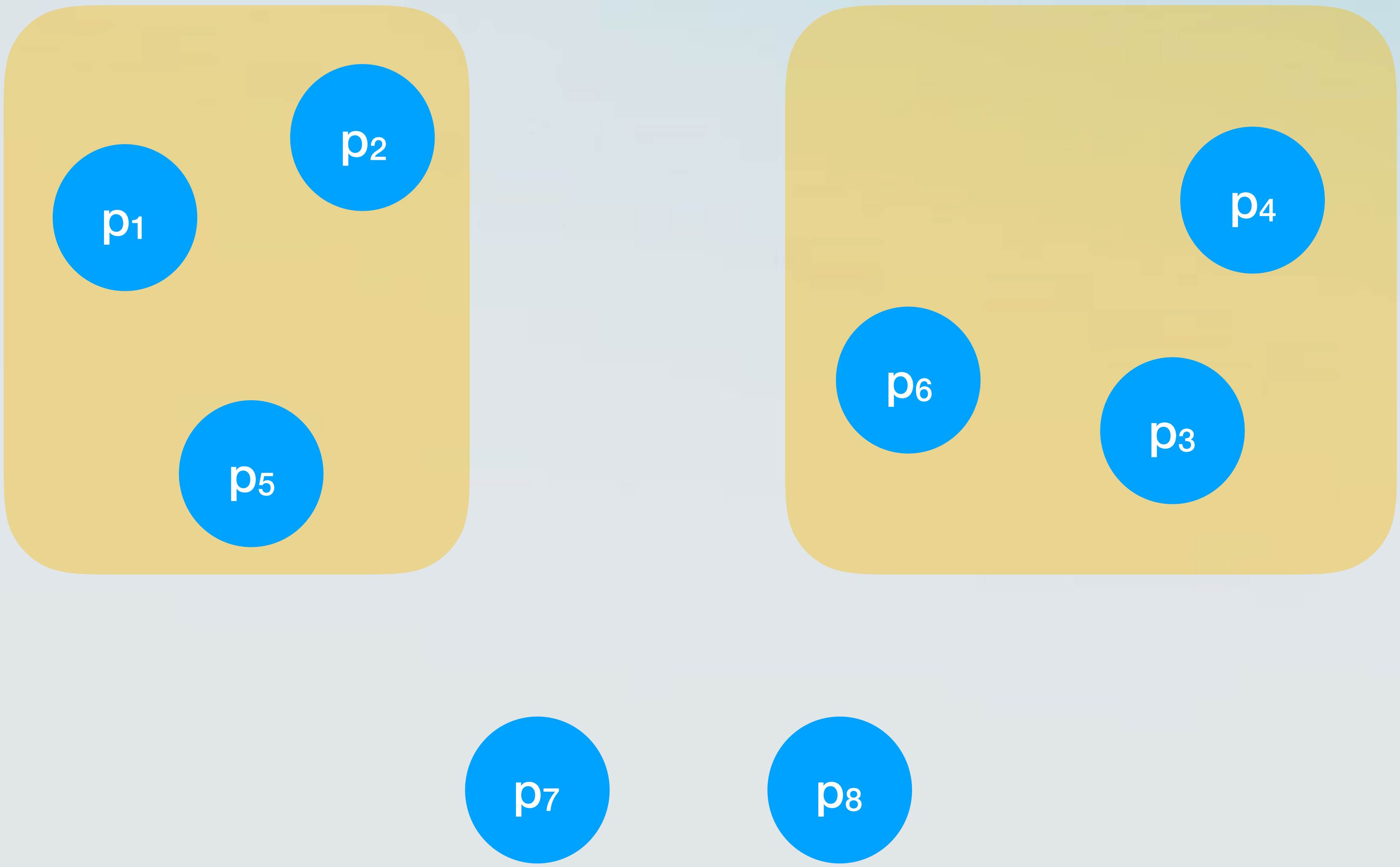

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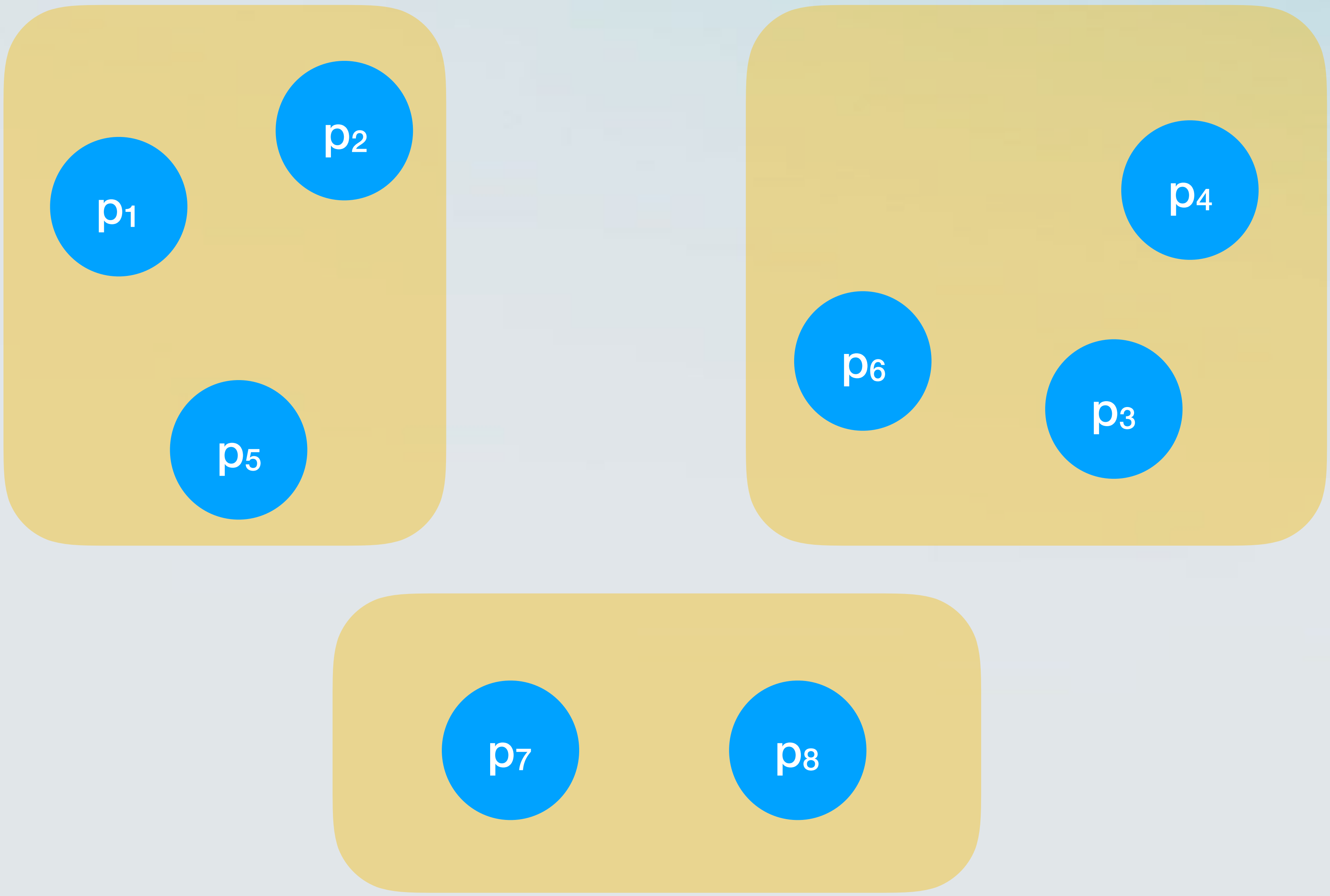

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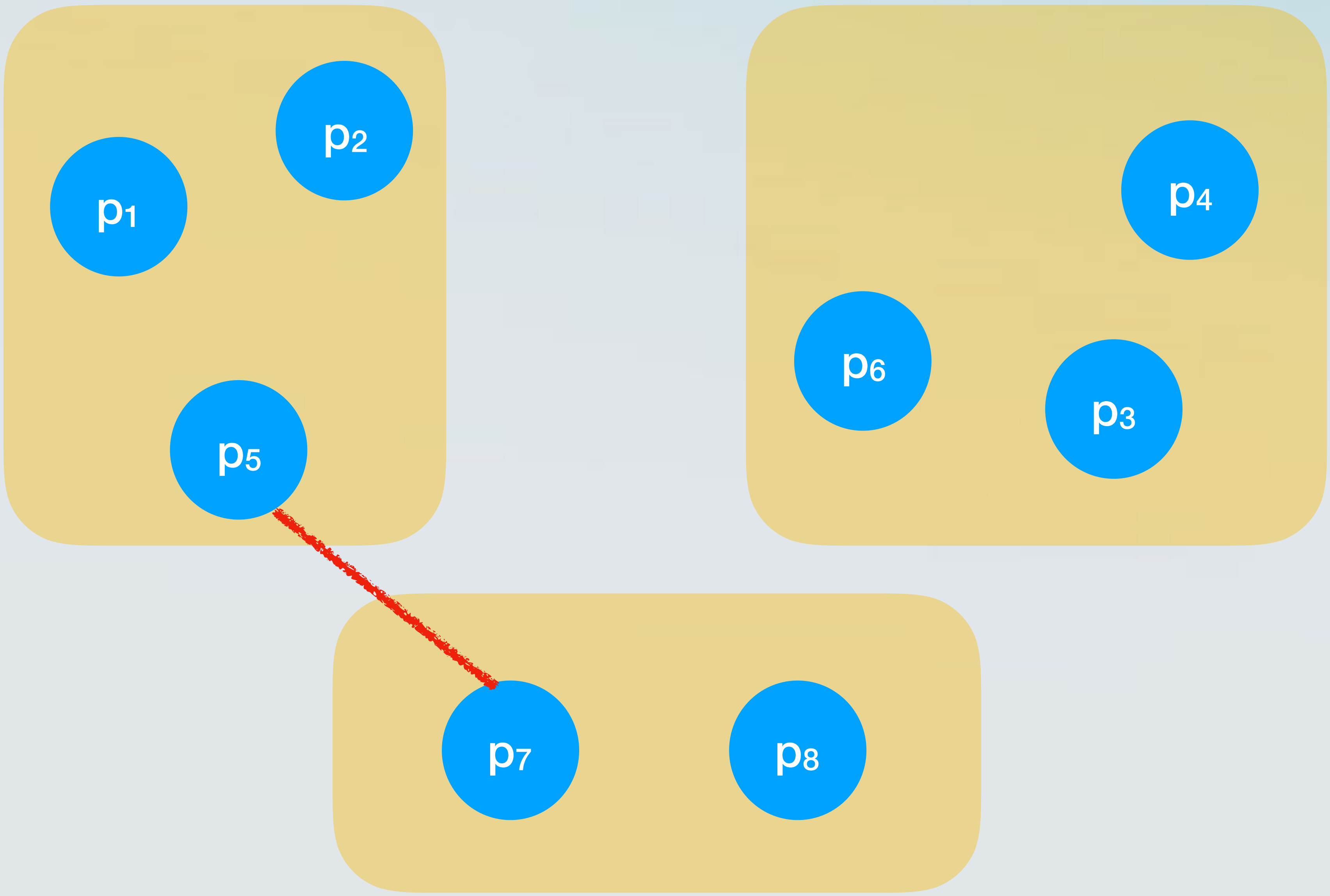

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- If the edge e under consideration connects two objects p_i and p_j already in the same component, skip it.

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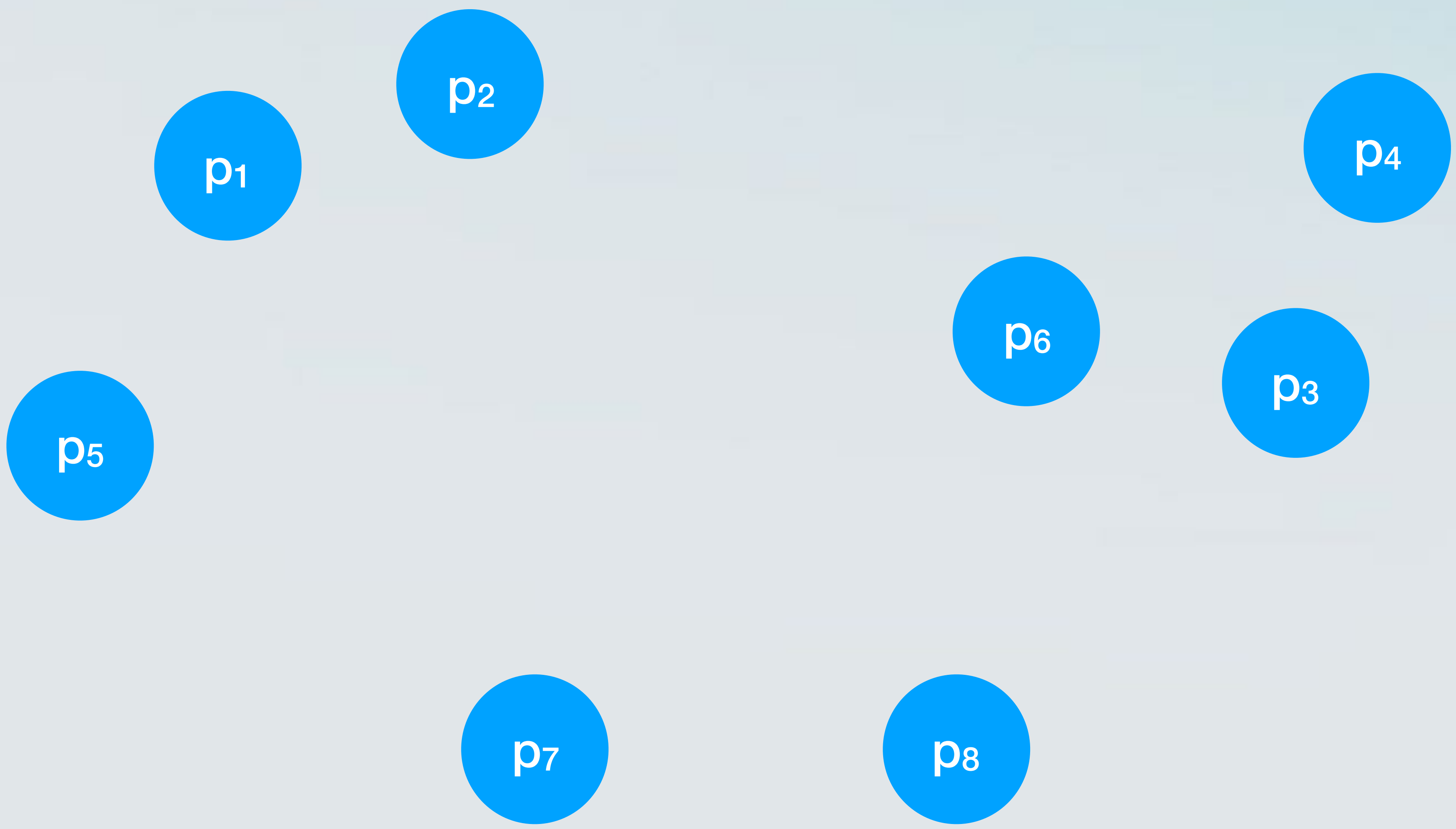
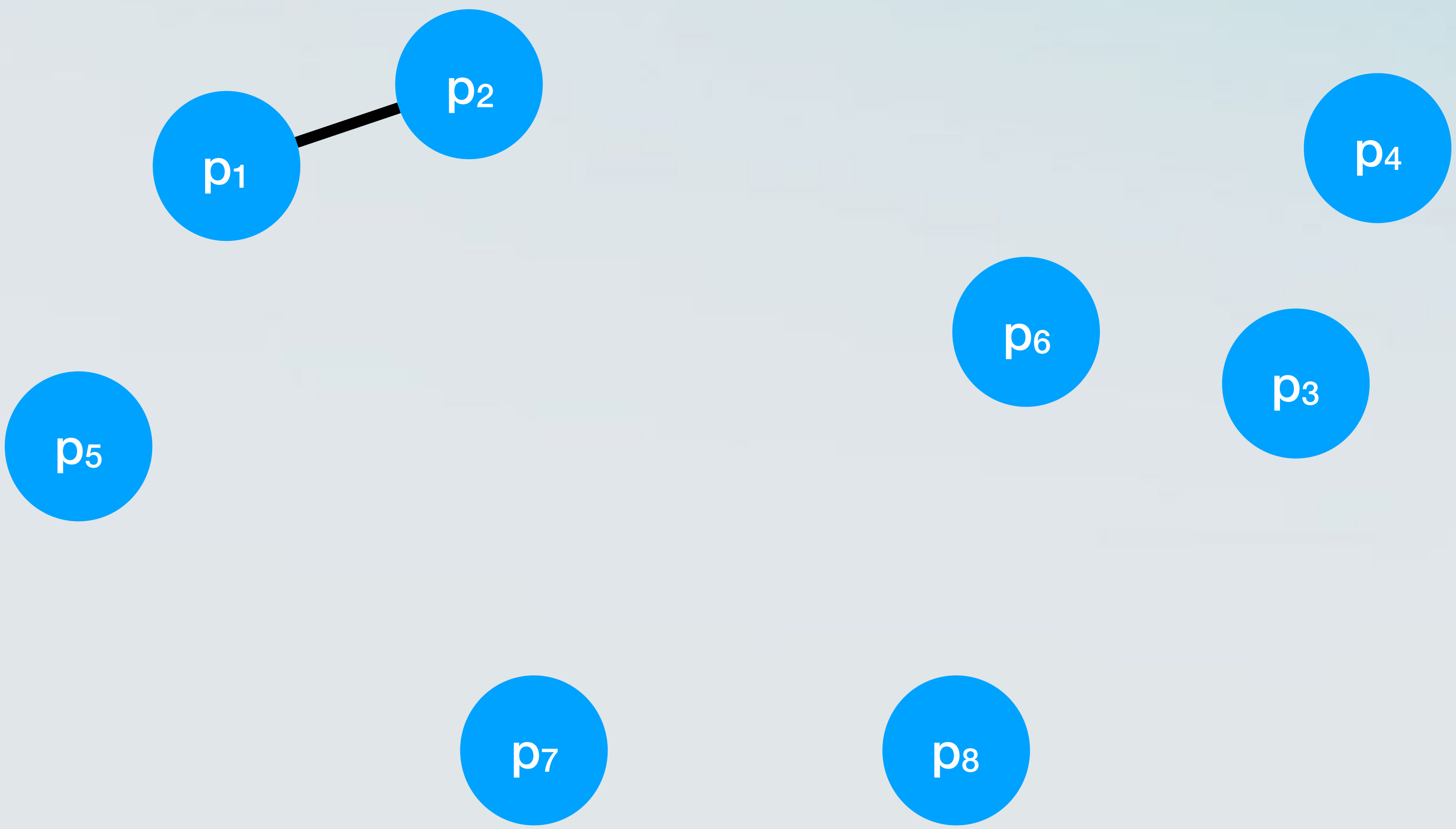


8 components

Clustering (concretely)



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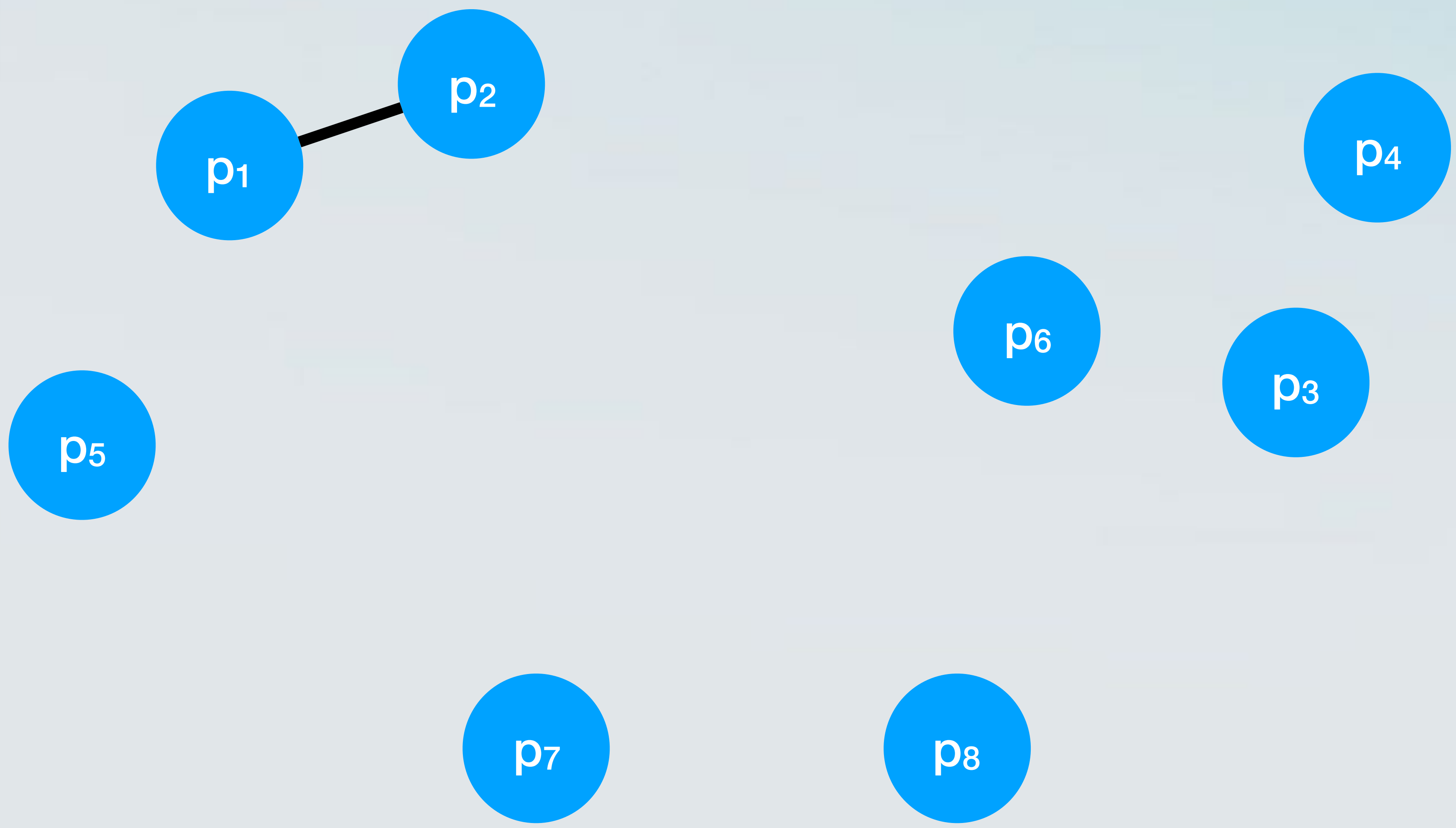

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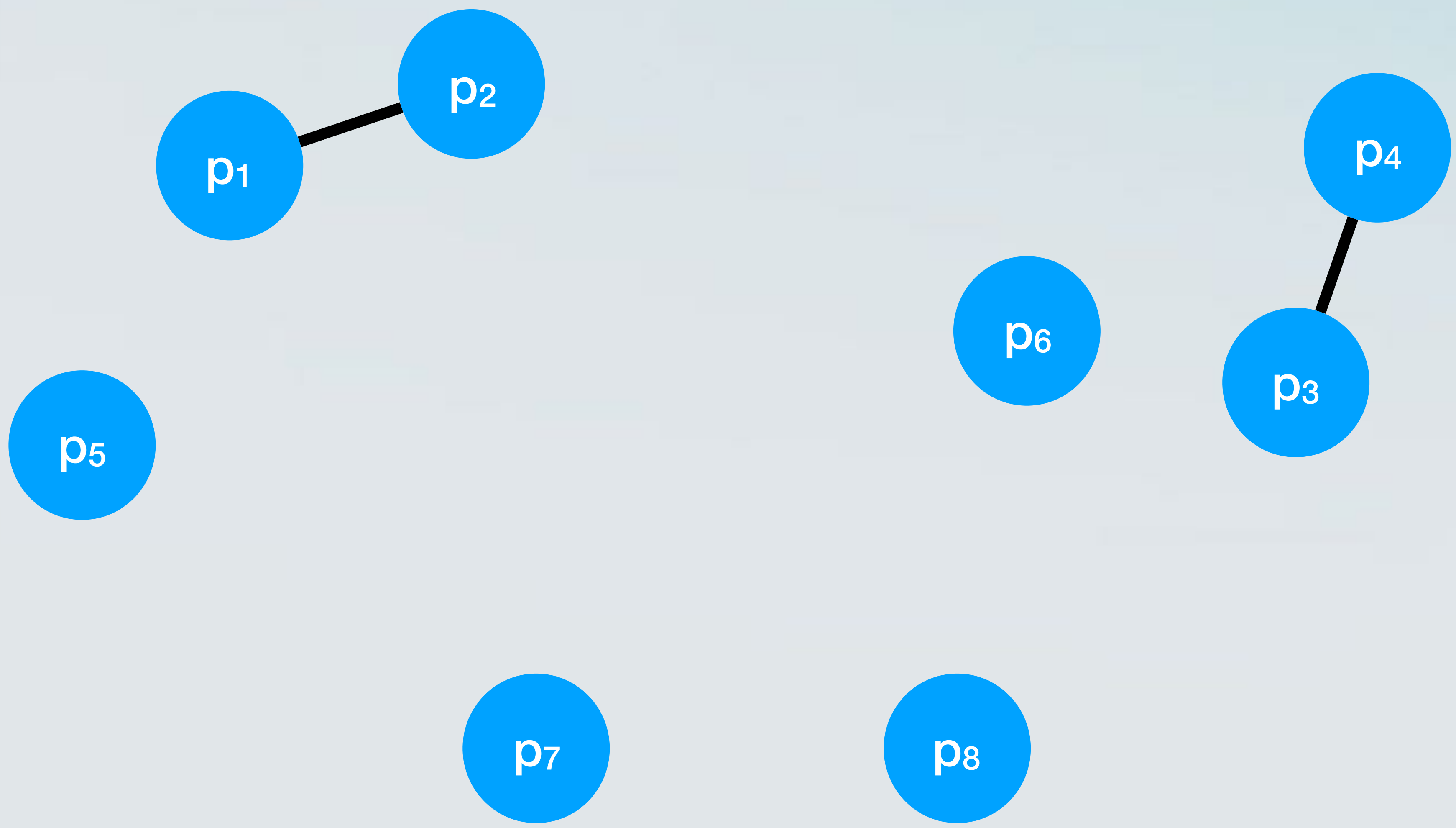

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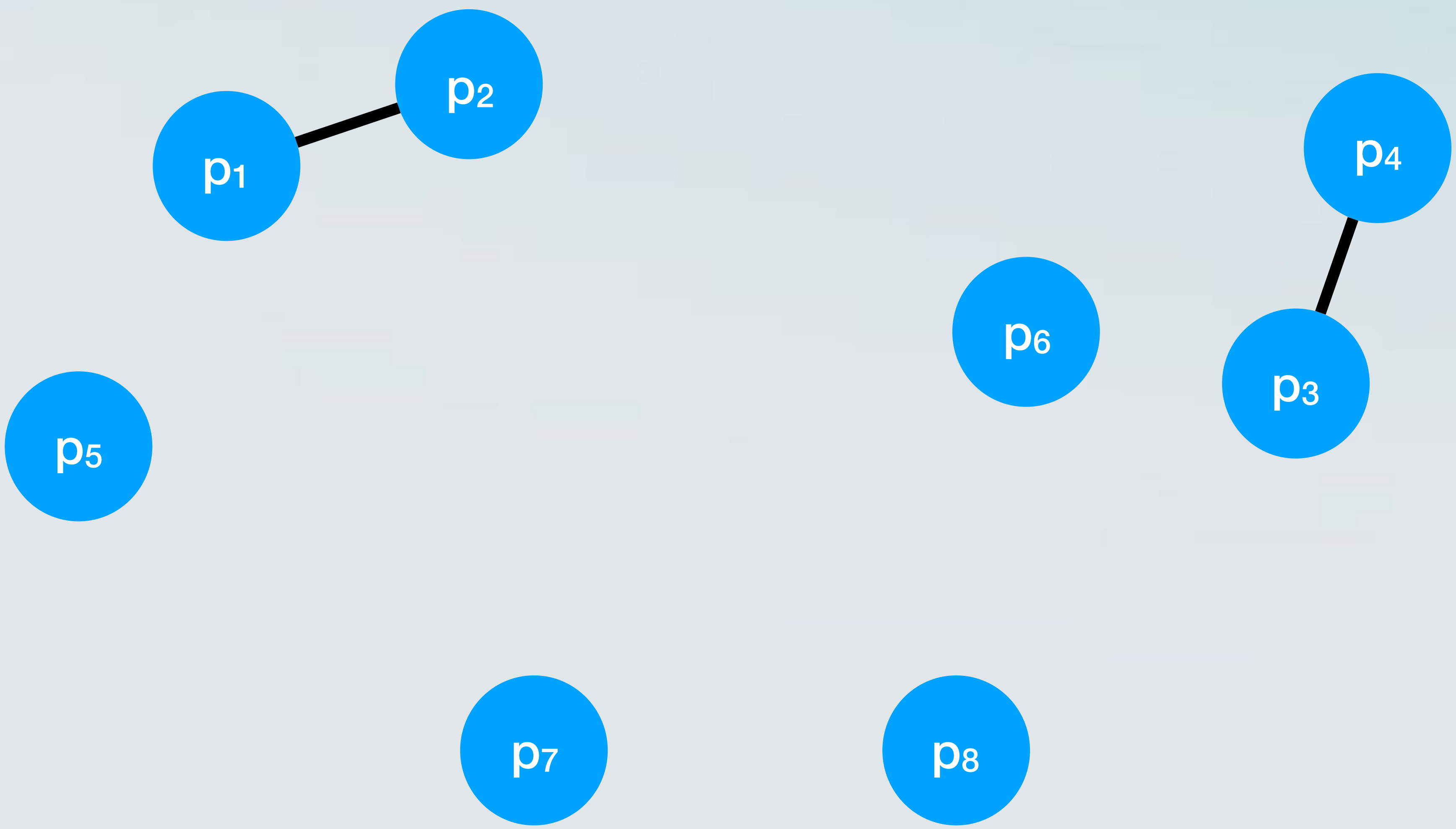

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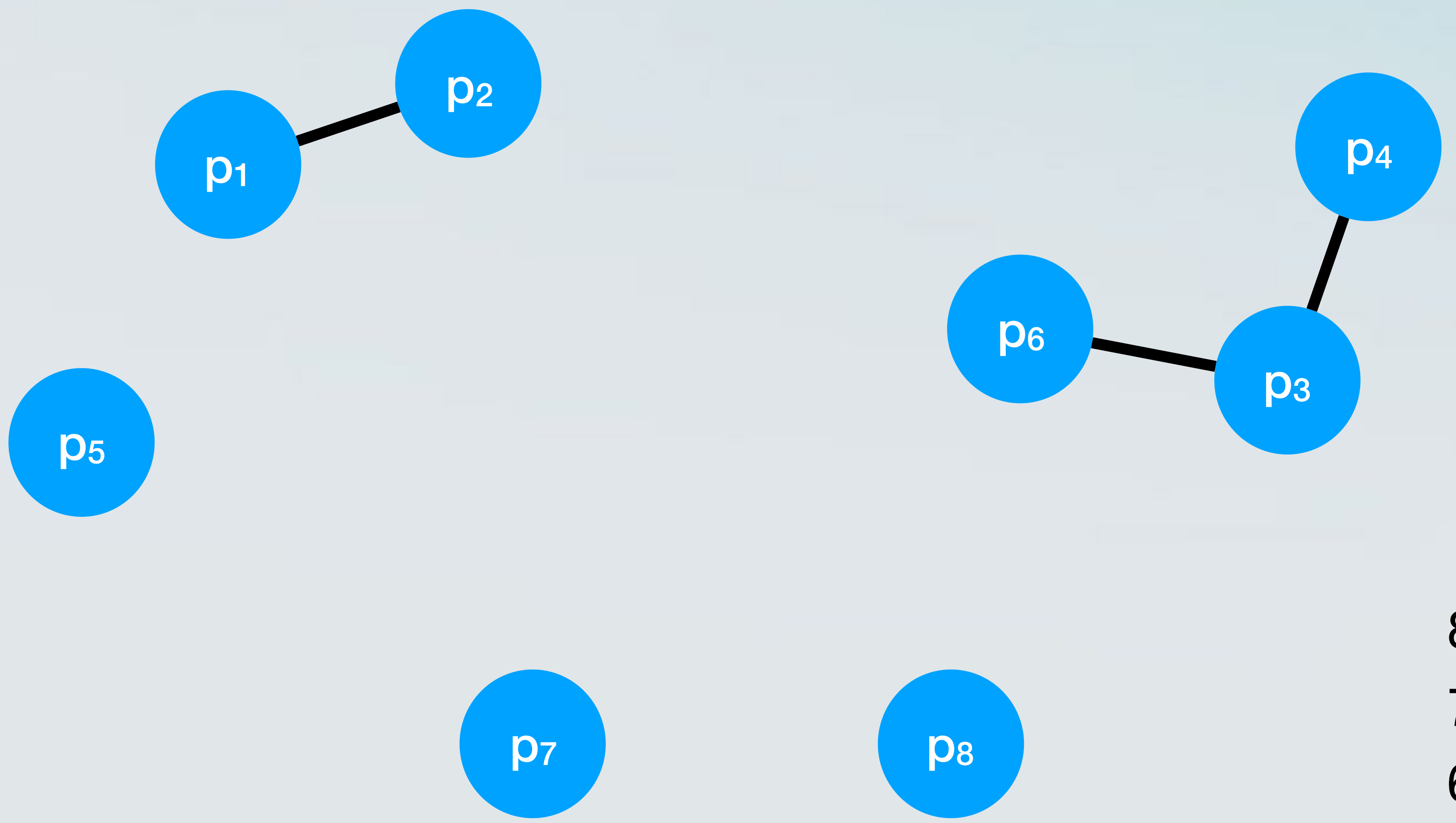

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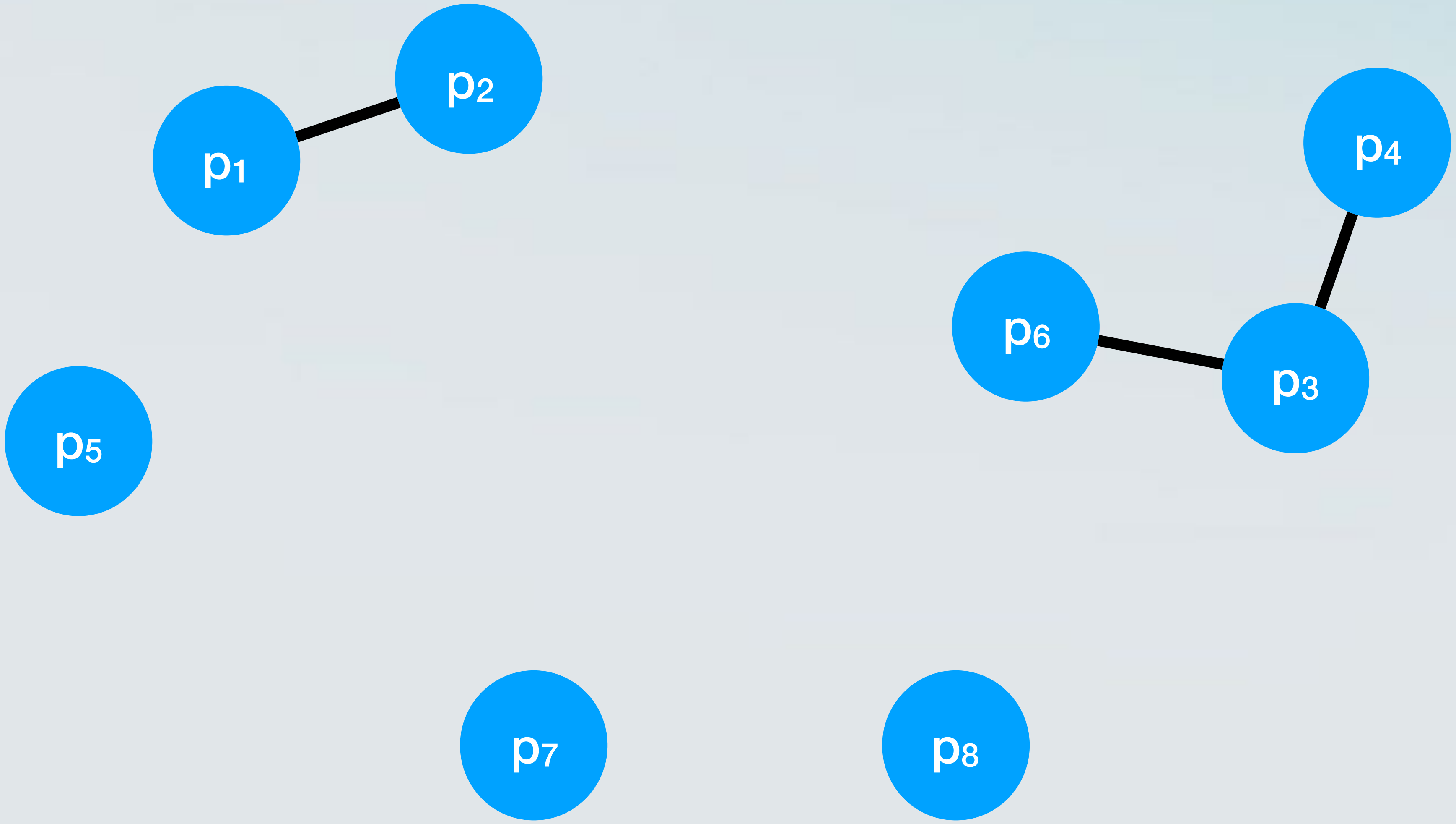

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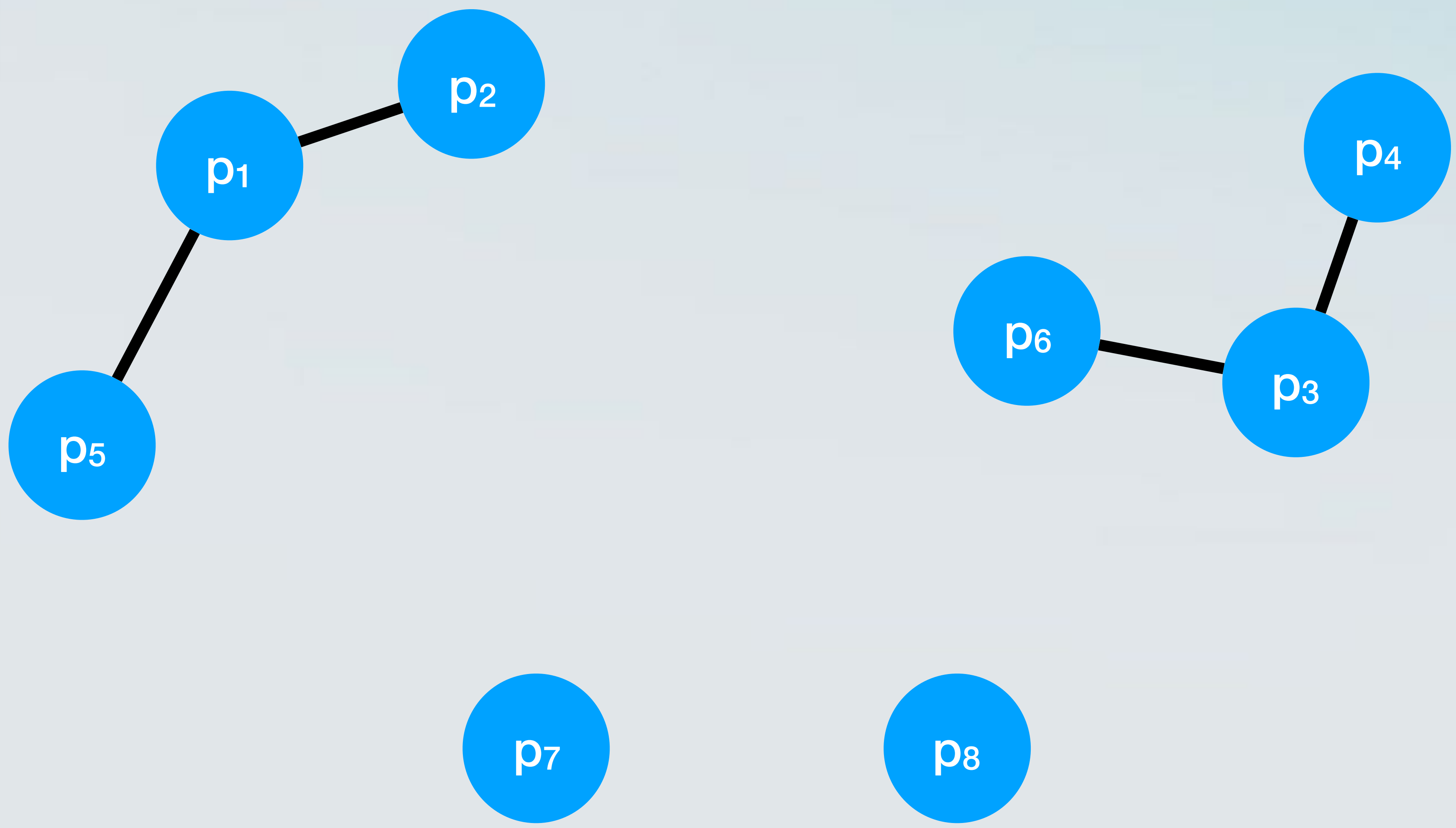

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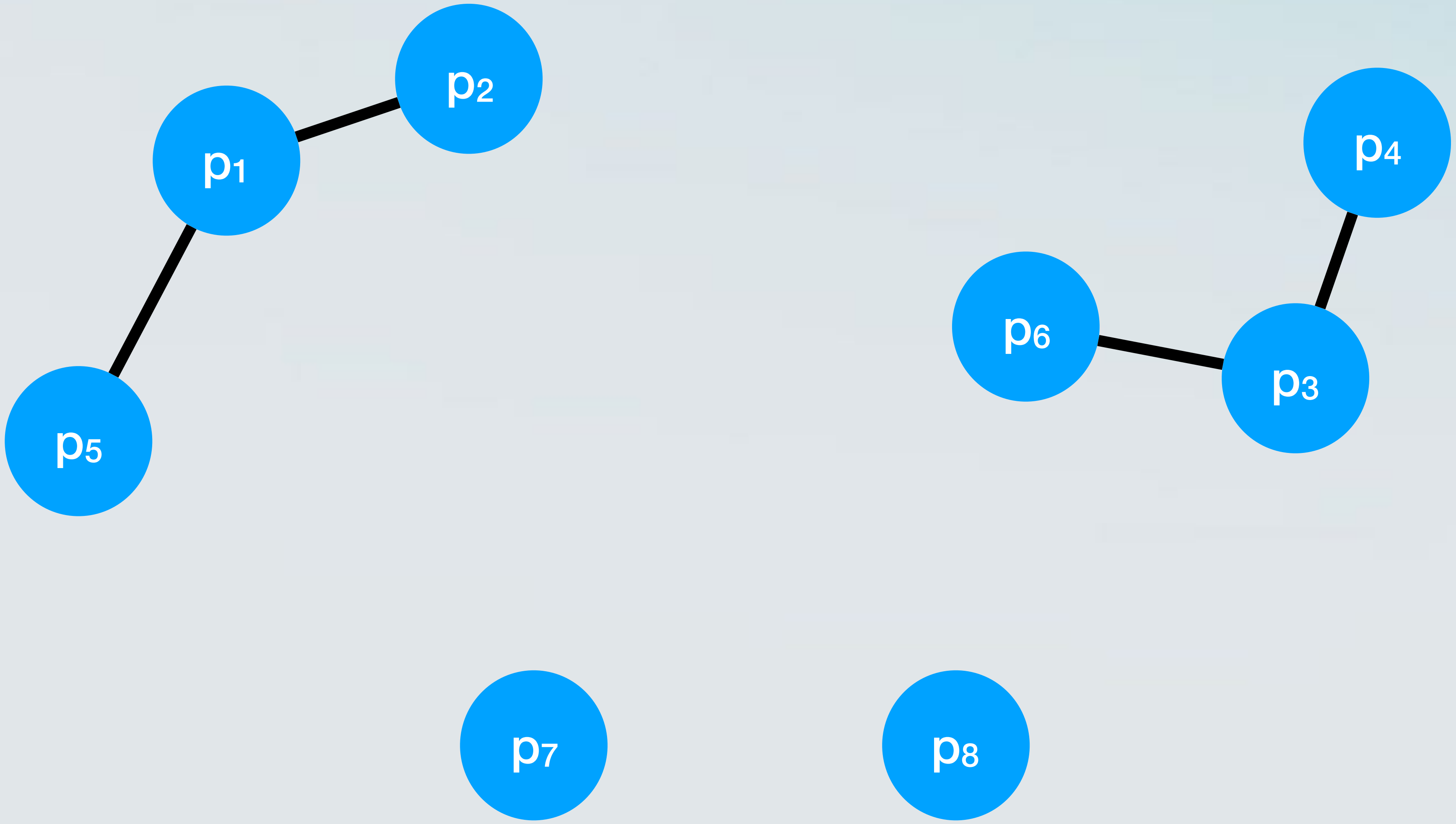

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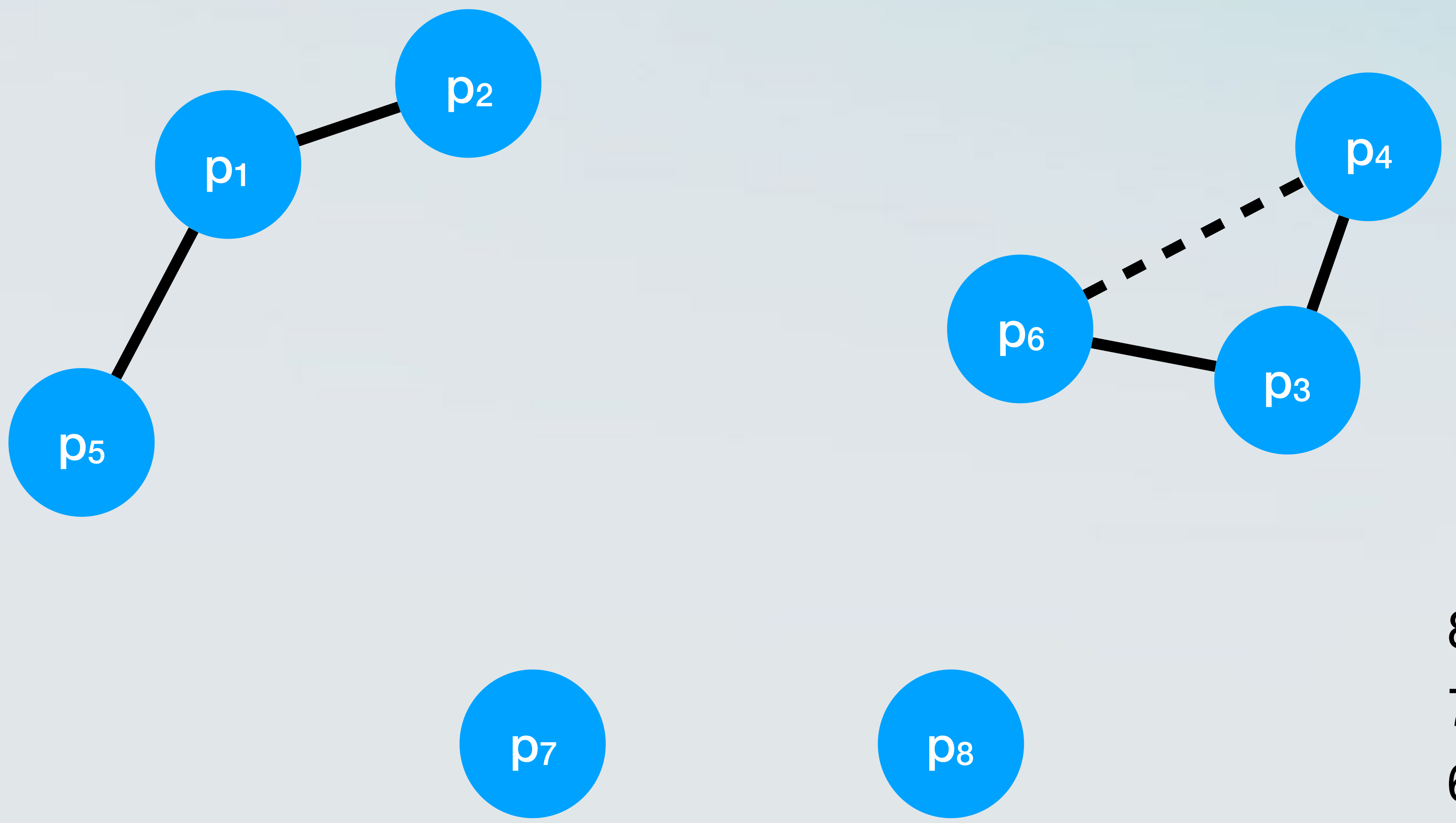

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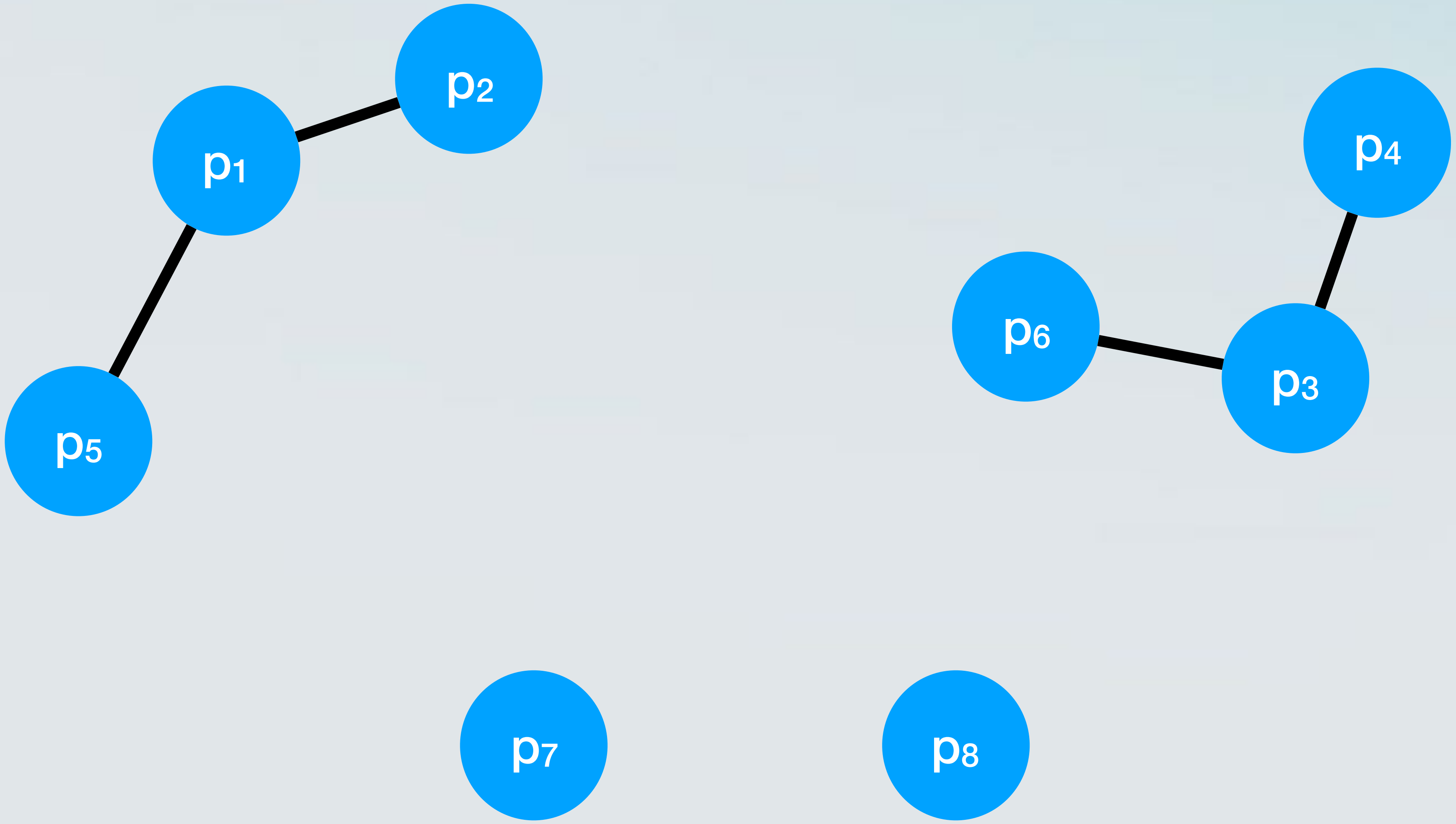

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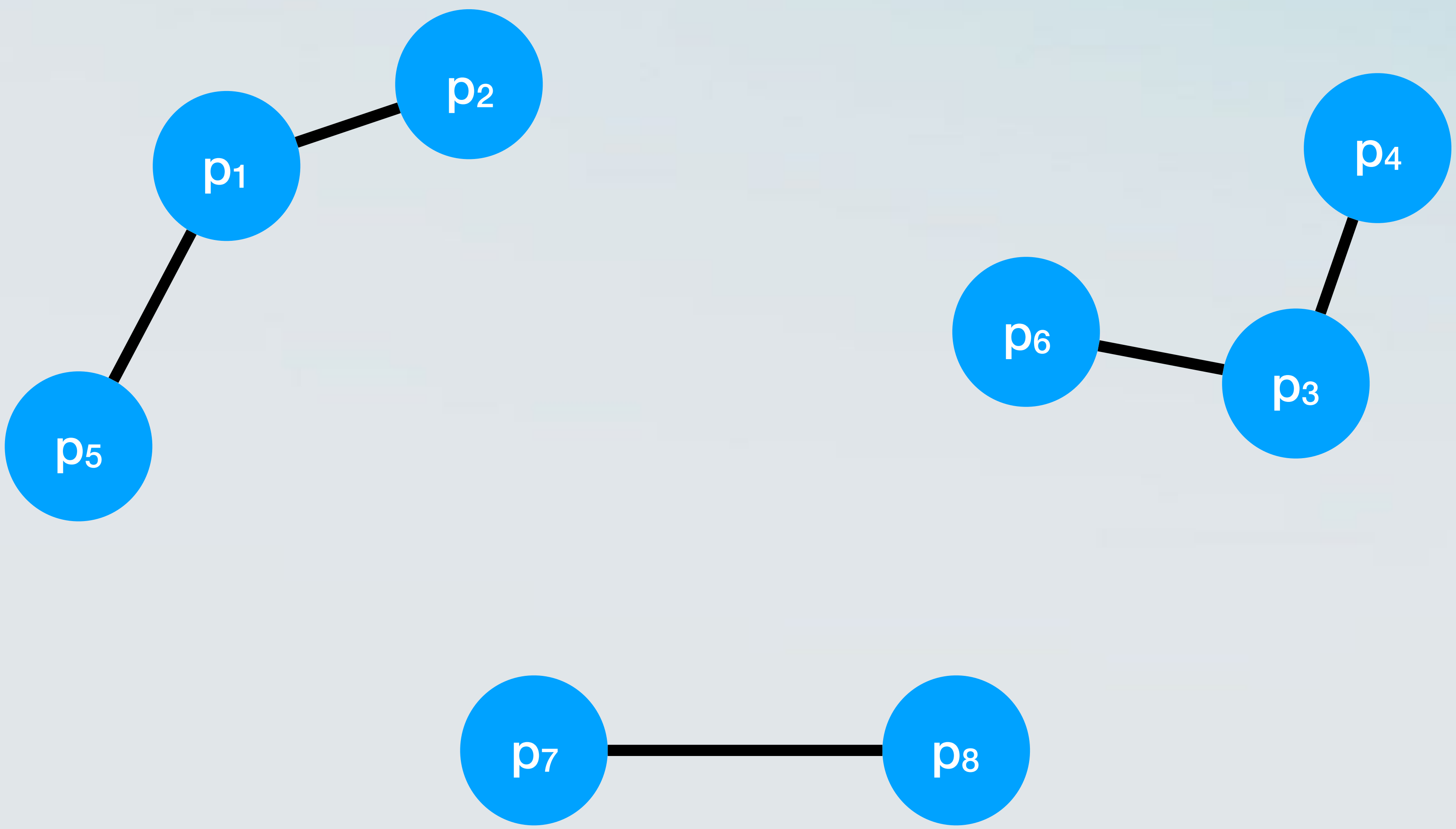

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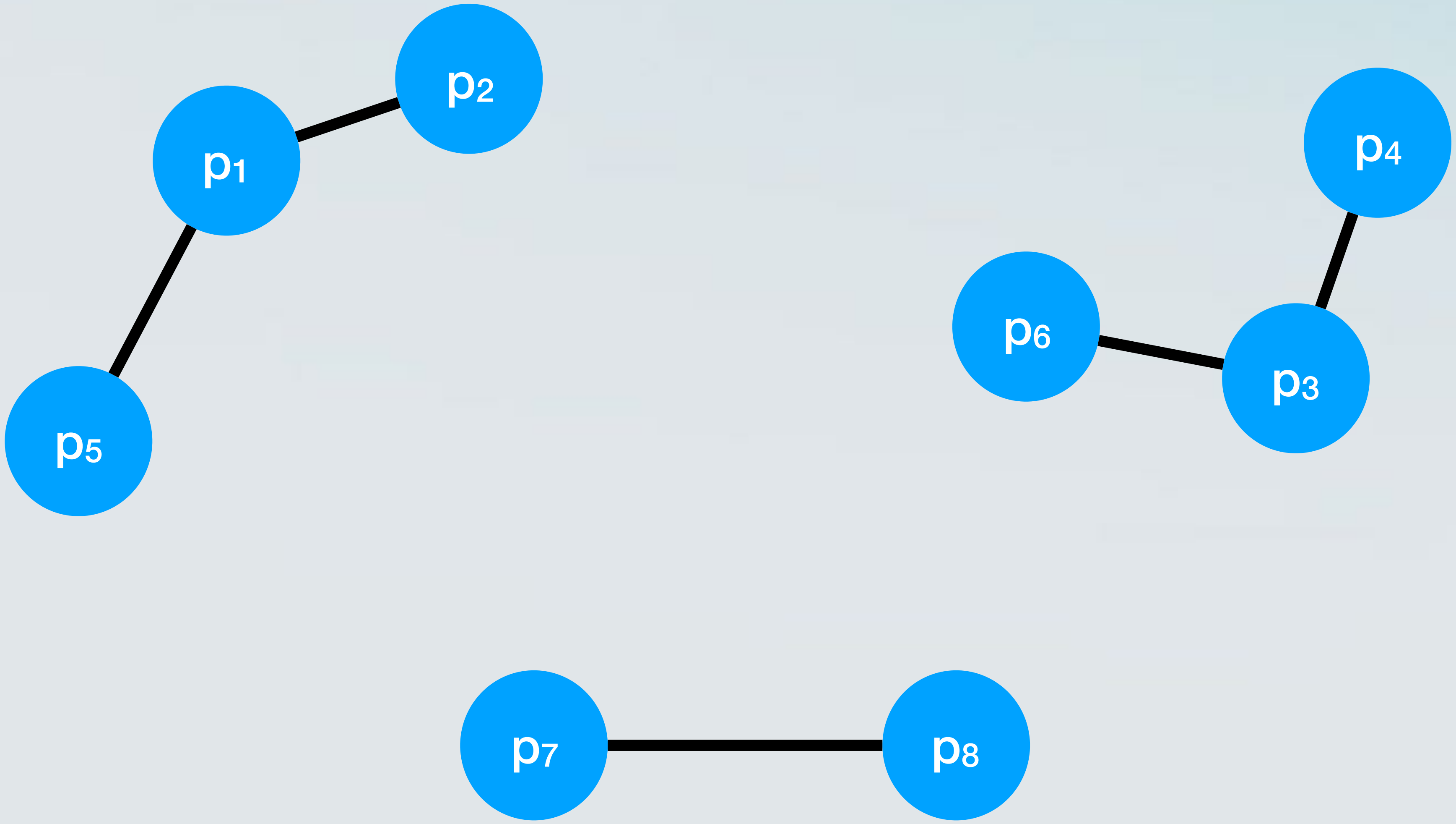

8 components
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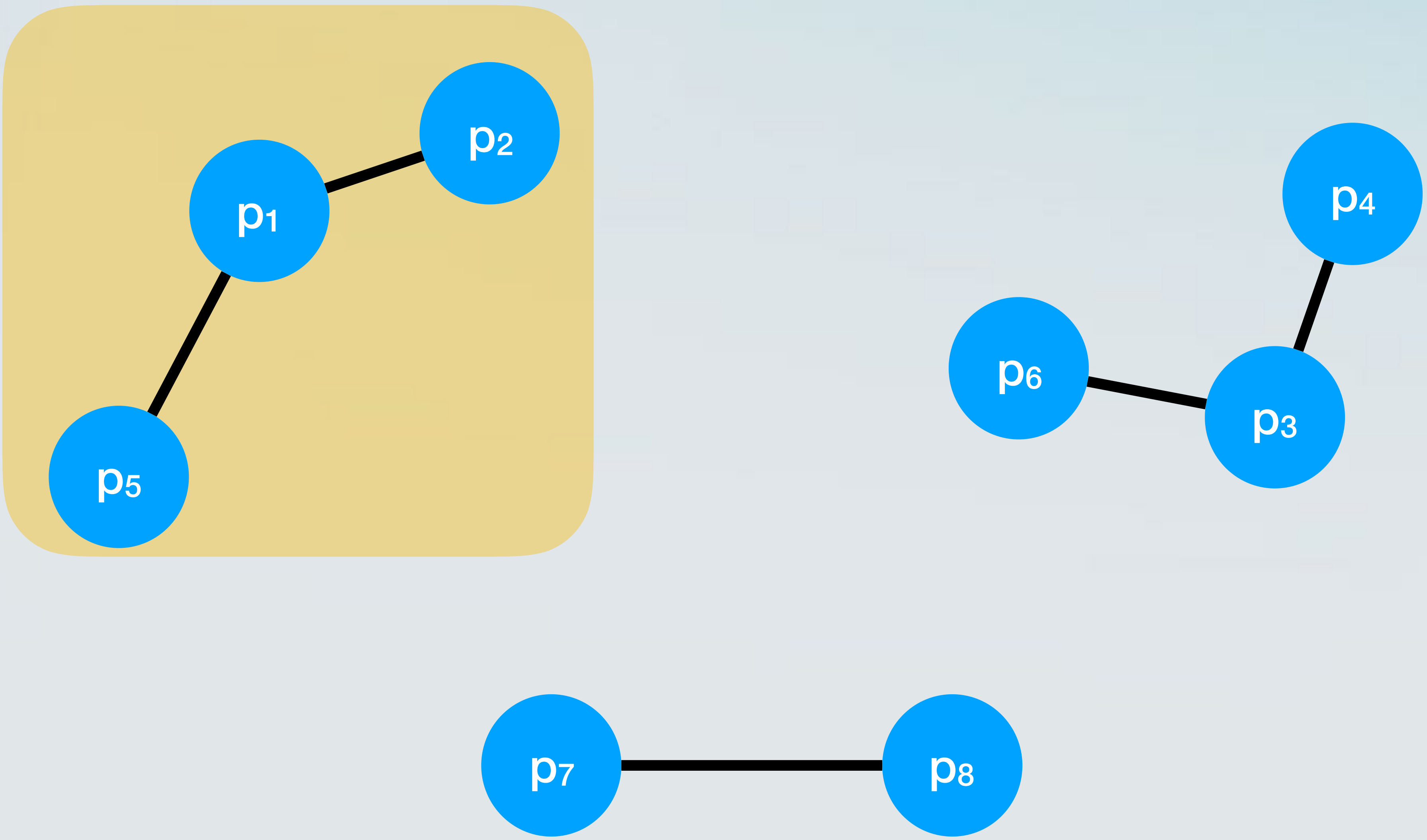

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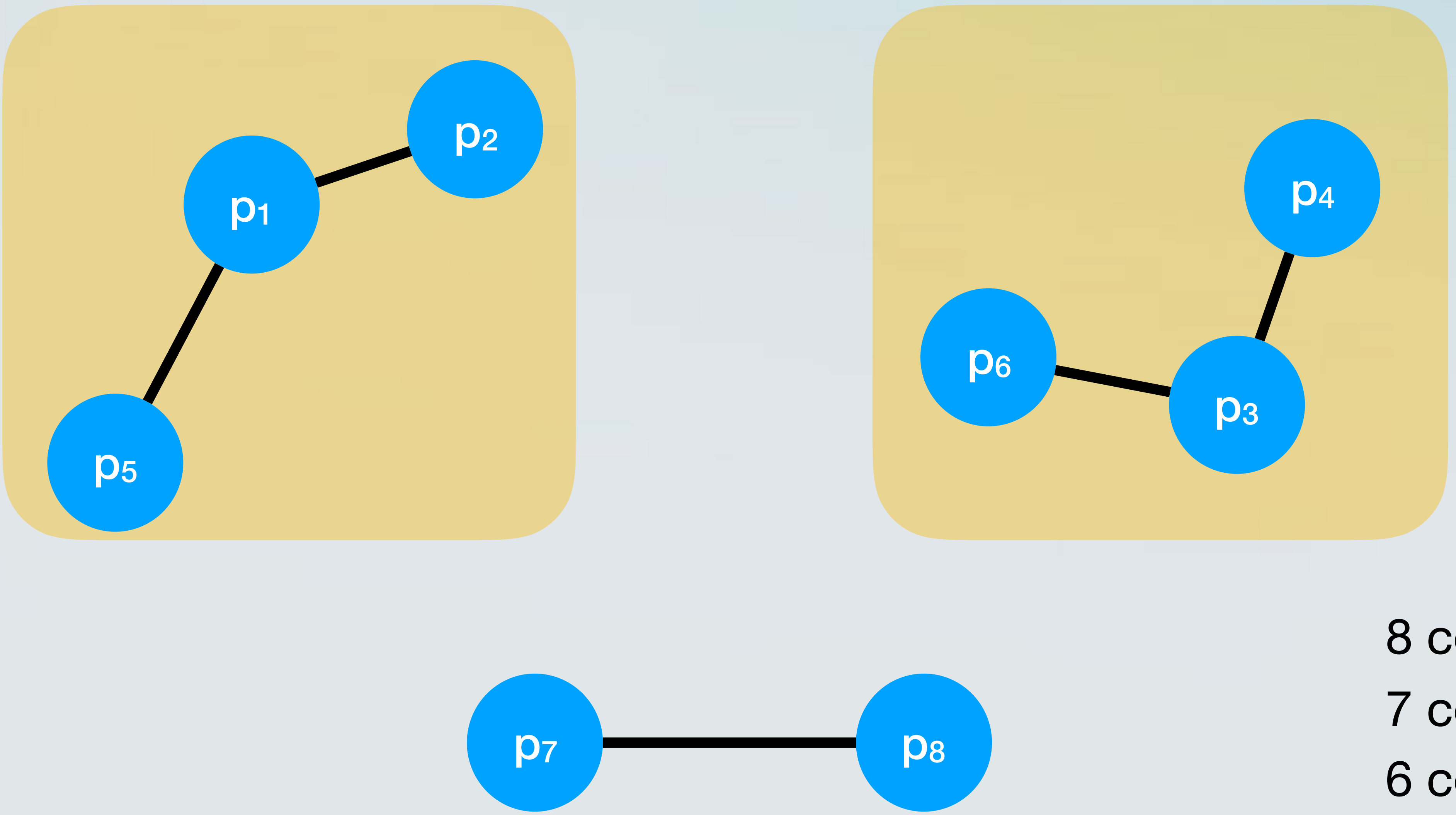

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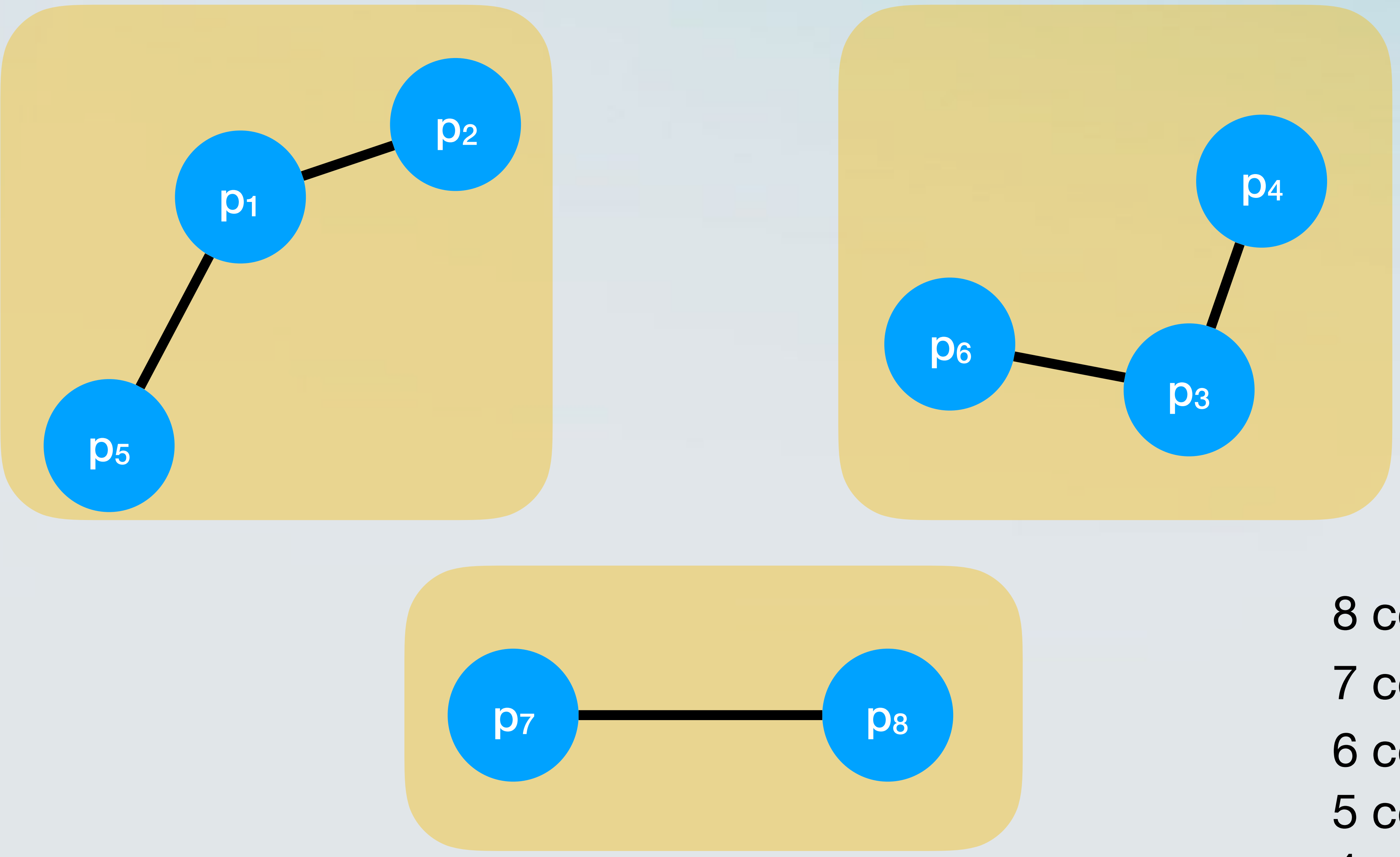

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- Continue like this until obtain k connected components.

Sound familiar?

- Pick two objects p_i and p_j with the smallest distance d(p_i,p_j).
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Correctness

- Lemma: Let $C_1, C_2, \ldots, C_k$ be the k connected components formed by deleting the k-1 most expensive edges from a minimum spanning tree T.

  These are a k-clustering of maximum spacing.

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- What is the *spacing* of C?
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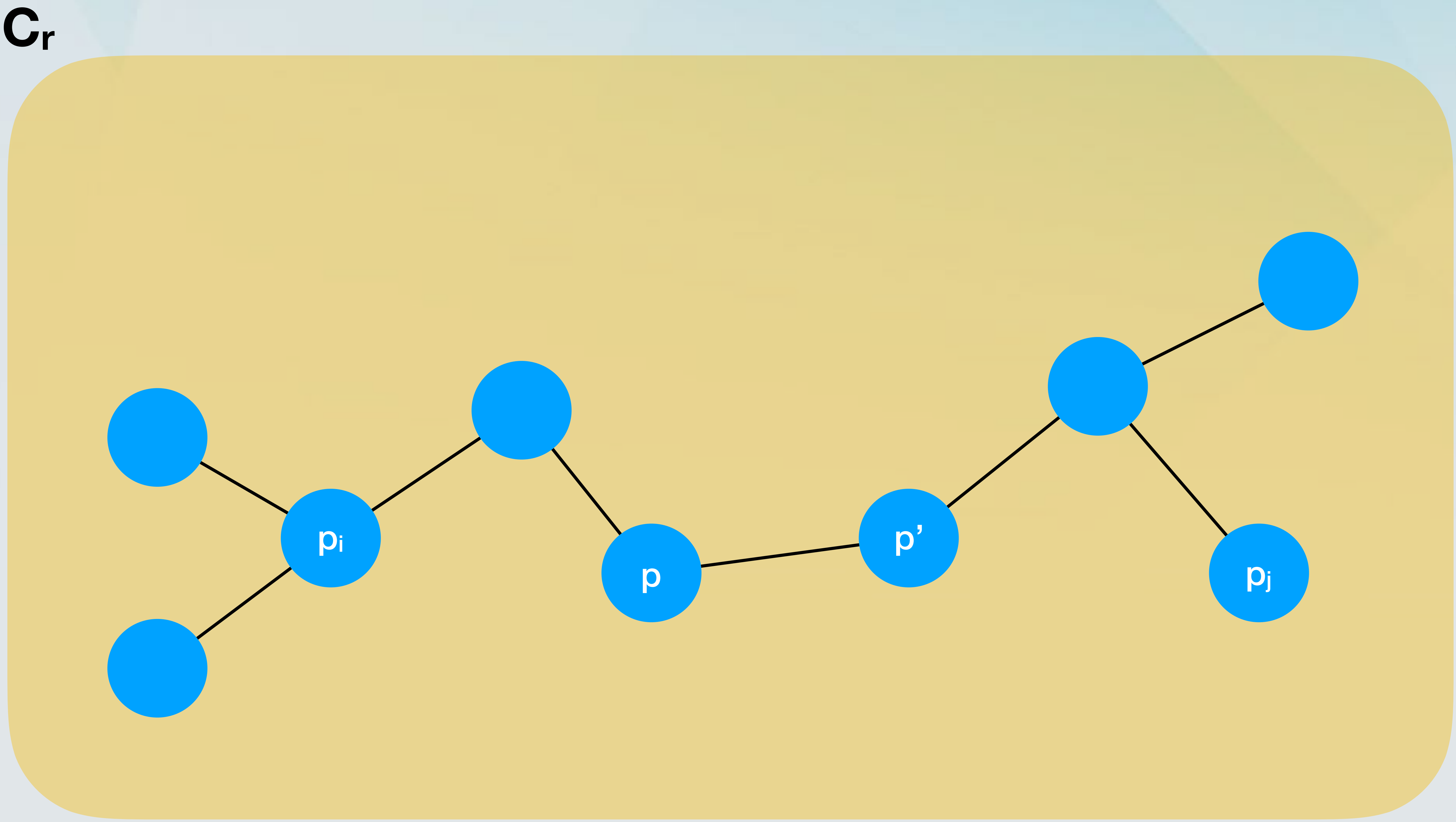

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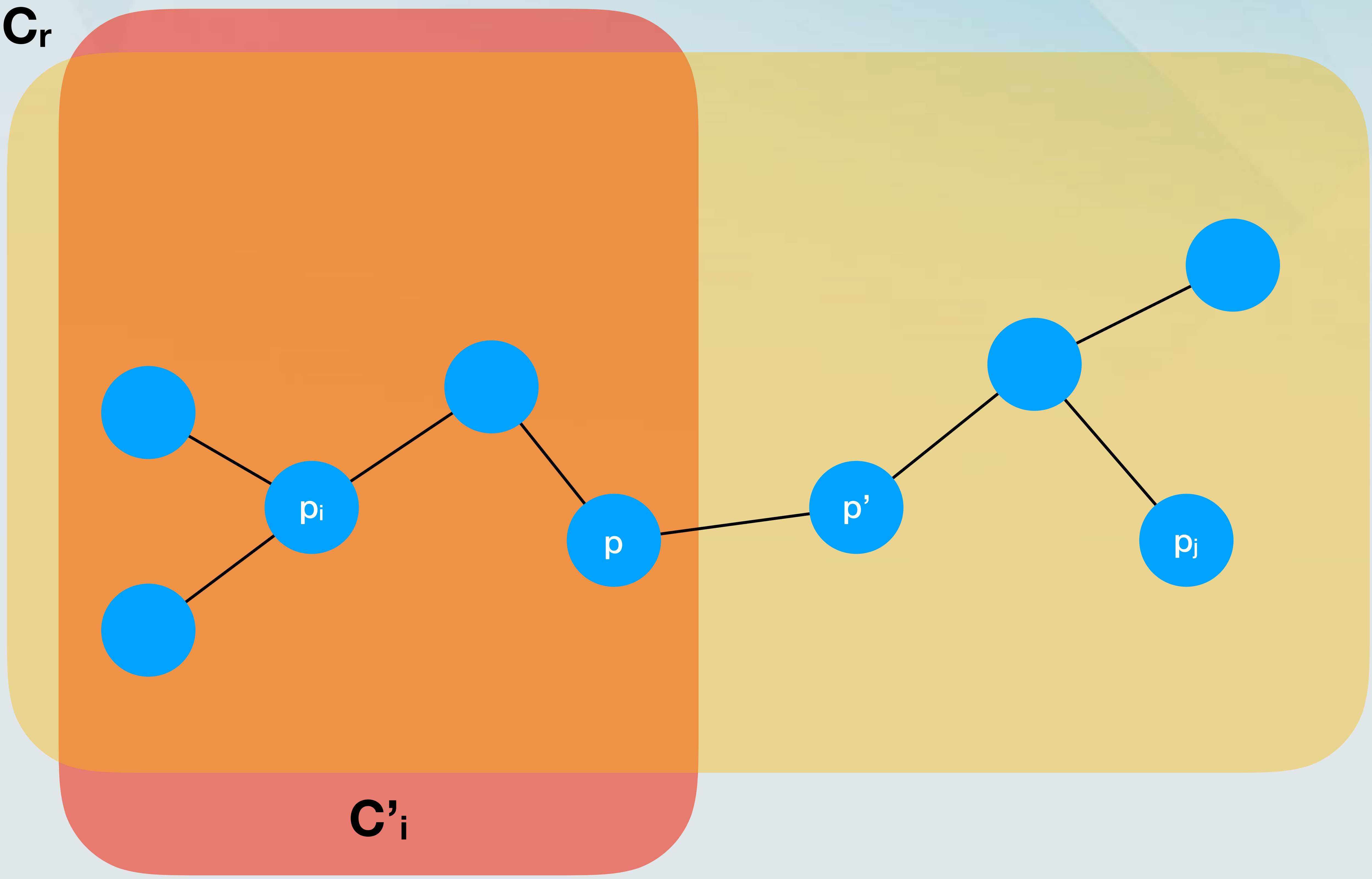

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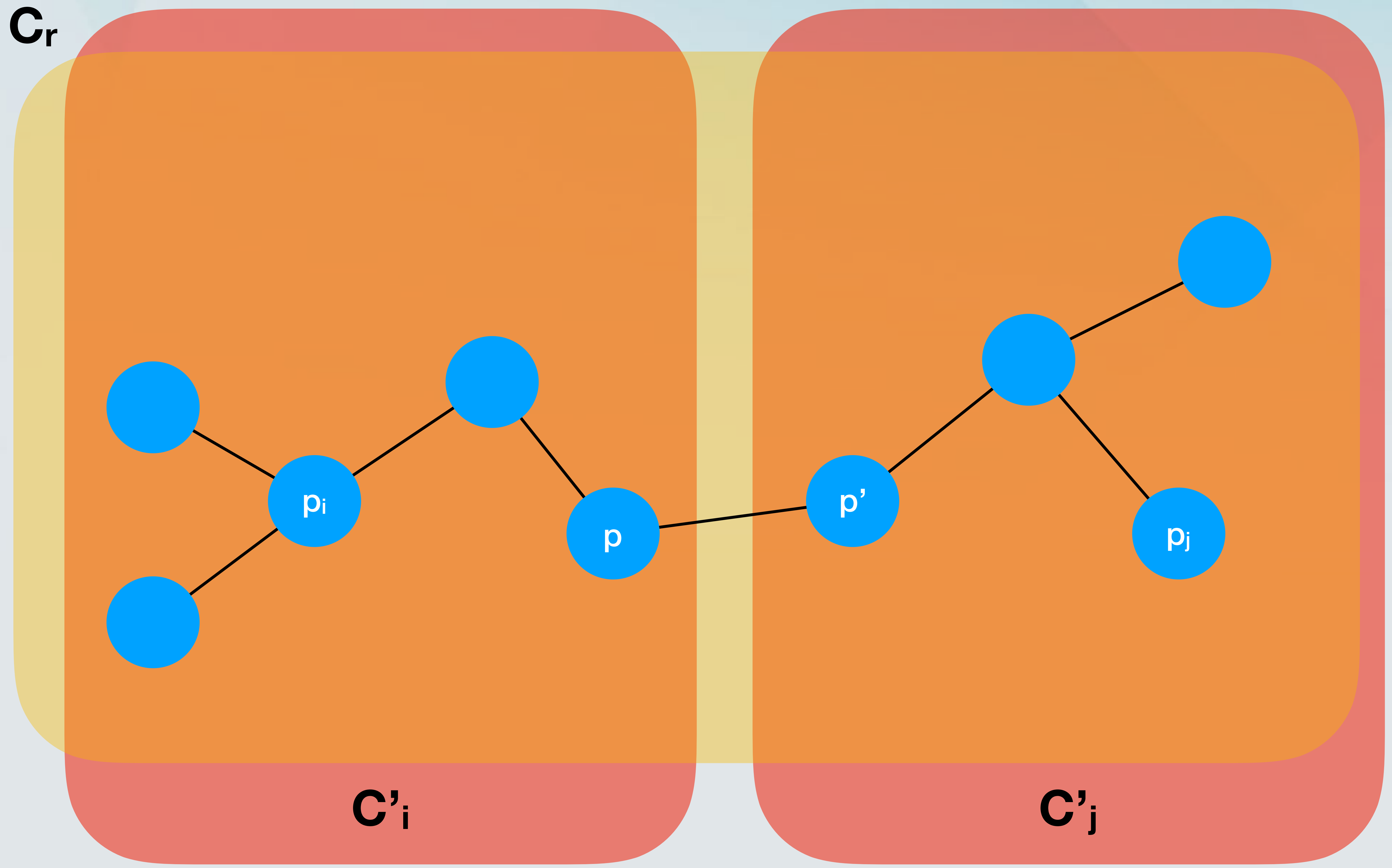

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  - The cost of e_1.

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  - The spacing of C' is not smaller.

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