Advanced Algorithmic Techniques (COMP523)

Dynamic Programming 2

Recap and plan

- **Last lecture:**
  - Dynamic Programming
  - Weighted Interval Scheduling
- **This lecture:**
  - Subset Sum
  - Knapsack

The subset sum problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative weight w_i.
- We are given a bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$

and $\sum_{i \in S} w_i$ is maximised.

Greedy Approaches

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- Sort items in terms of *increasing weight* and put them in S one by one.

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  - $W = 4$, $w_1 = 3$, $w_2 = 2$, $w_3 = 2$.

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  - If yes, ?

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- What information do we get about the other items?
- In weighted interval scheduling, we could remove all intervals overlapping with *n*.
- Can we do something similar here?
  - There is no reason to a-priori exclude any remaining item, unless adding it would exceed the weight.
  - The only information that we really get is that we now have weight $W - w_n$ left.

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- How many subproblems do we need?

What we really need

- To find the optimal value of OPT(n), we need
  - The optimal value of OPT($n-1$) if n is not in O.
  - The optimal value of the solution on input $\{1, 2, \ldots, n-1\}$ and $w = W - w_n$.
- How many subproblems do we need?
  - One for each initial set $\{1, 2, \ldots, i\}$ of items and each possible value for the remaining weight w.

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- Assumptions:
  - W is an integer.
  - Every w_i is an integer.
- We will have one subproblem for each $i=0,1, \dots ,n$ and each integer $0 \leq w \leq W$.
- Let OPT(i,w) be the value of the optimal solution on subset {1, 2, … , i} and maximum allowed weight w.

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Subproblems

- Using this notation, what are we looking for?
  - $OPT(n, W)$
- Should item n be in the optimal solution O or not?
  - If no, then $OPT(n,W) = OPT(n-1,W)$.
  - If yes, then $OPT(n,W) = w_n + OPT(n-1,W-w_n)$.

Subproblems

Is j in **O** ?

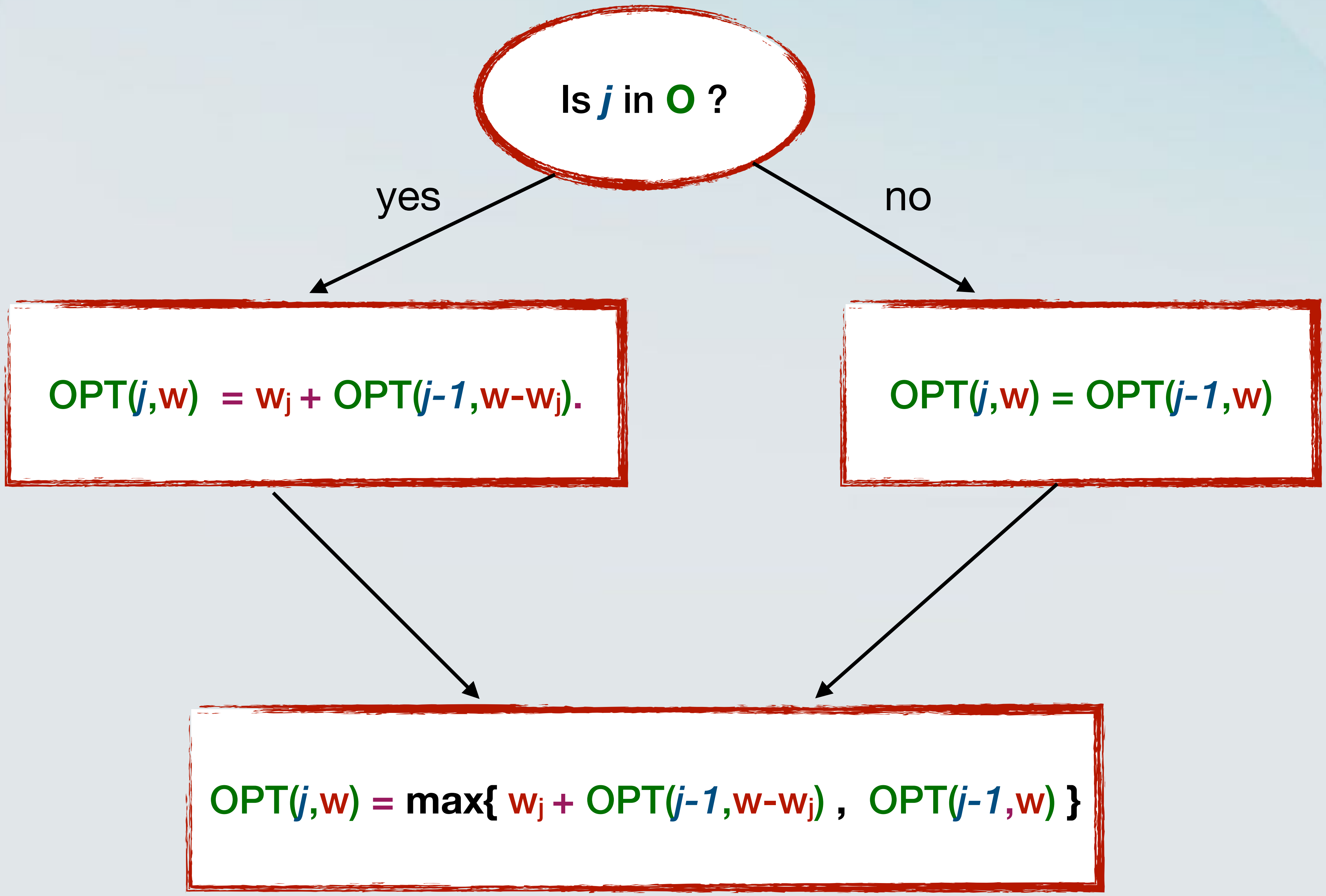
yes

$OPT(j,w) = w_j + OPT(j-1,w-w_j)$.

no

$OPT(j,w) = OPT(j-1,w)$

Subproblems



Subproblems

Is j in **O** ?

Unless $w_j < w$

yes

$\mathrm{OPT}(j,w) = w_j + \mathrm{OPT}(j-1,w-w_j)$.

no

$\mathrm{OPT}(j,w) = \mathrm{OPT}(j-1,w)$

$\mathrm{OPT}(j,w) = \max\{\, w_j + \mathrm{OPT}(j-1,w-w_j)\,,\ \mathrm{OPT}(j-1,w) \,\}$

Algorithm

Algorithm **SubsetSum**(n,W)

Array M=[$0 \ldots n$, 0 … W]
Initialise M[0, w] = 0, for each w = 0, 1 , … , W

For i = 1, 2, … , n
For w = 0 , … , W
If (w_i > w)
M[i, w] = M[i-1, w]
Else
M[i, w] = **max{**M[i-1, w] , w_i + M[i-1, w-w_i]**}**
EndIf

Return M[n, W]

Two dimensional array

n	0										
n-1	0										
…	0										
…	0										
i	0										
i-1	0										
…	0										
2	0										
1	0										
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	…	$w-w_i$			w				W

Example

- *n*=3, W=6, $w_1 = w_2 = 2$ and $w_3 = 3$.

3							
2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

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    If ($w_i > w$)
        $M[i, w] = M[i\text{-}1, w]$
    Else
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3							
$\boxed{2}$	0	0	2	2	4		
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
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2	0	0	2	2	4	4	4
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Optimal value

$\boxed{3}$	0	0	2	3	4	5	$\boxed{5}$
2	0	0	2	2	4	4	4
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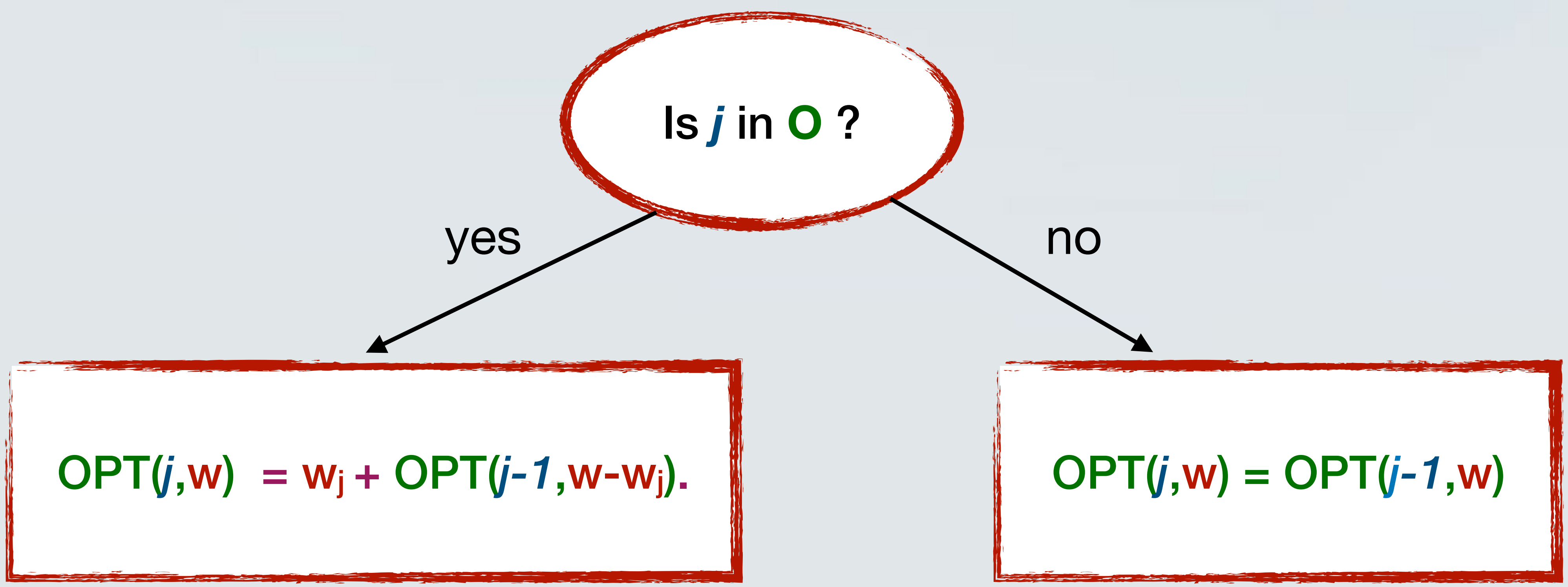
From values to solutions

- Very similar idea to weighted interval scheduling



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- We are building up a table M of solutions (instead of an array).
- We compute each value $M(i, w)$ of the table in $O(1)$ time using the previous values.
- What is the running time overall?
    - How many entries does the table M have?

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- Is this a polynomial time algorithm?
  - No, because it depends on W.
  - It is *pseudopolynomial*, as it runs in time polynomial in n and W.
  - It is fairly efficient, if in the number involved in the input are reasonably small.

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- How hard is it to design a polynomial time algorithm for subset sum?
    - Hard enough to justify a reward of 1 million dollars!
    - Subset sum is NP-hard!
    - More about that later on in the module.

The subset sum problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative weight w_i.
- We are given a bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$

and $\sum_{i \in S} w_i$ is maximised.

The knapsack problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative weight w_i and a non-negative value v_i.
- We are given a bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximised.

The knapsack problem

- The subset sum problem is a specific instance of the knapsack problem (why?)
- You have actually seen this problem before!

The fractional knapsack problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative weight w_i and a non-negative value v_i.
- We are given a bound W.
- Goal: Select a fraction x_i from each item *i* to maximise

$$\sum_{i \in n} x_i \cdot v_i \quad \text{such that} \quad \sum_{i \in n} x_i \cdot w_i \leq W$$

A greedy solution

- Sort the items in terms of their "bang-per-buck" value v_i / w_i : v_1 / w_1, v_2 / w_2, … , v_n / w_n.
- Put as much as possible from the first item in the knapsack, then as much as possible from the second item, … and so on.
- This algorithm solves the fractional knapsack problem optimally.

The knapsack problem

The 0/1 knapsack problem

The 0/1 knapsack problem

- Could the greedy algorithm solve the 0/1 knapsack problem optimally?

The 0/1 knapsack problem

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- Counter-example:

The 0/1 knapsack problem

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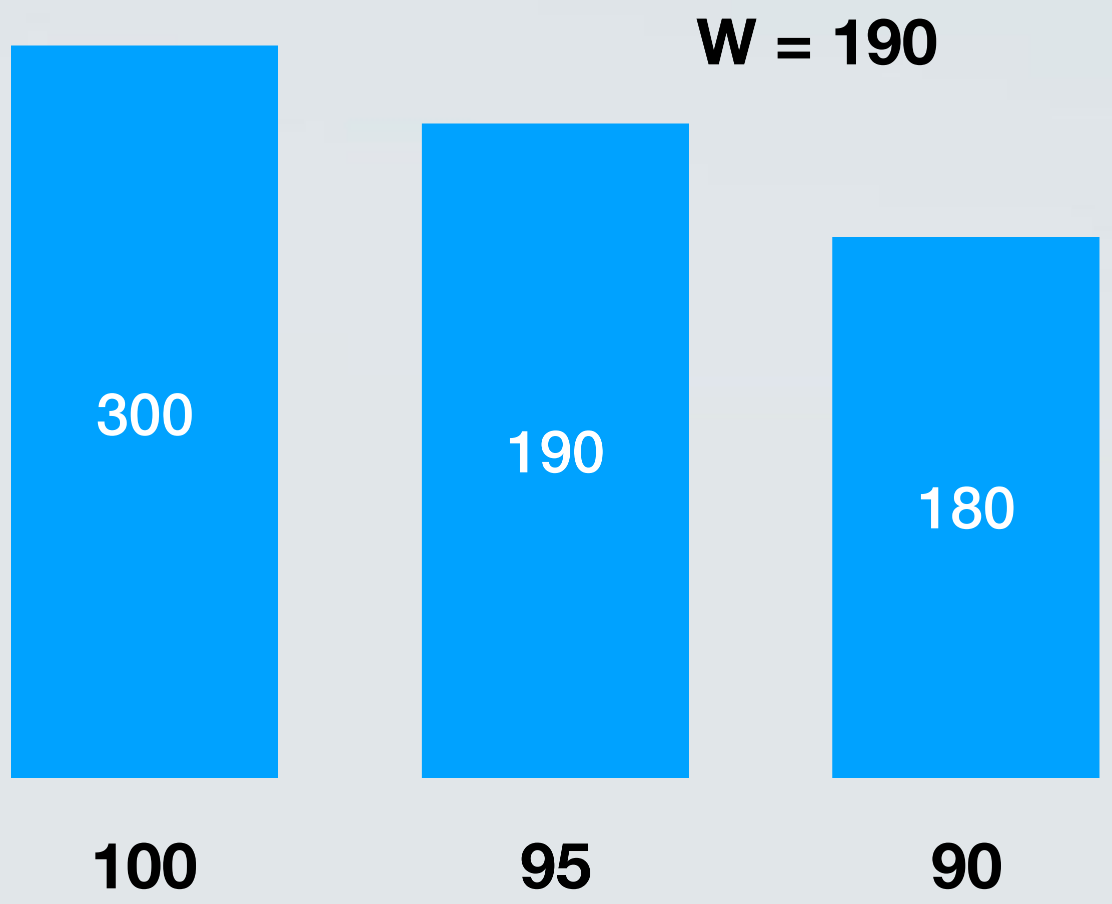

7 minute exercise

Design a dynamic programming algorithm for 0/1 knapsack.

Algorithm **SubsetSum**(n,W)

Array M=[$0 \ldots n$, 0 … W]
Initialise M[0, w] = 0, for each w = 0, 1 , … , W

For i = 1, 2, … , n
For w = 0 , … , W
If (w_i > w)
M[i, w] = M[i-1, w]
Else
M[i, w] = **max{**M[i-1, w] , w_i + M[i-1, w-w_i]**}**
EndIf

Return M[n, W]