Advanced Algorithmic Techniques (COMP523)

Network Flows

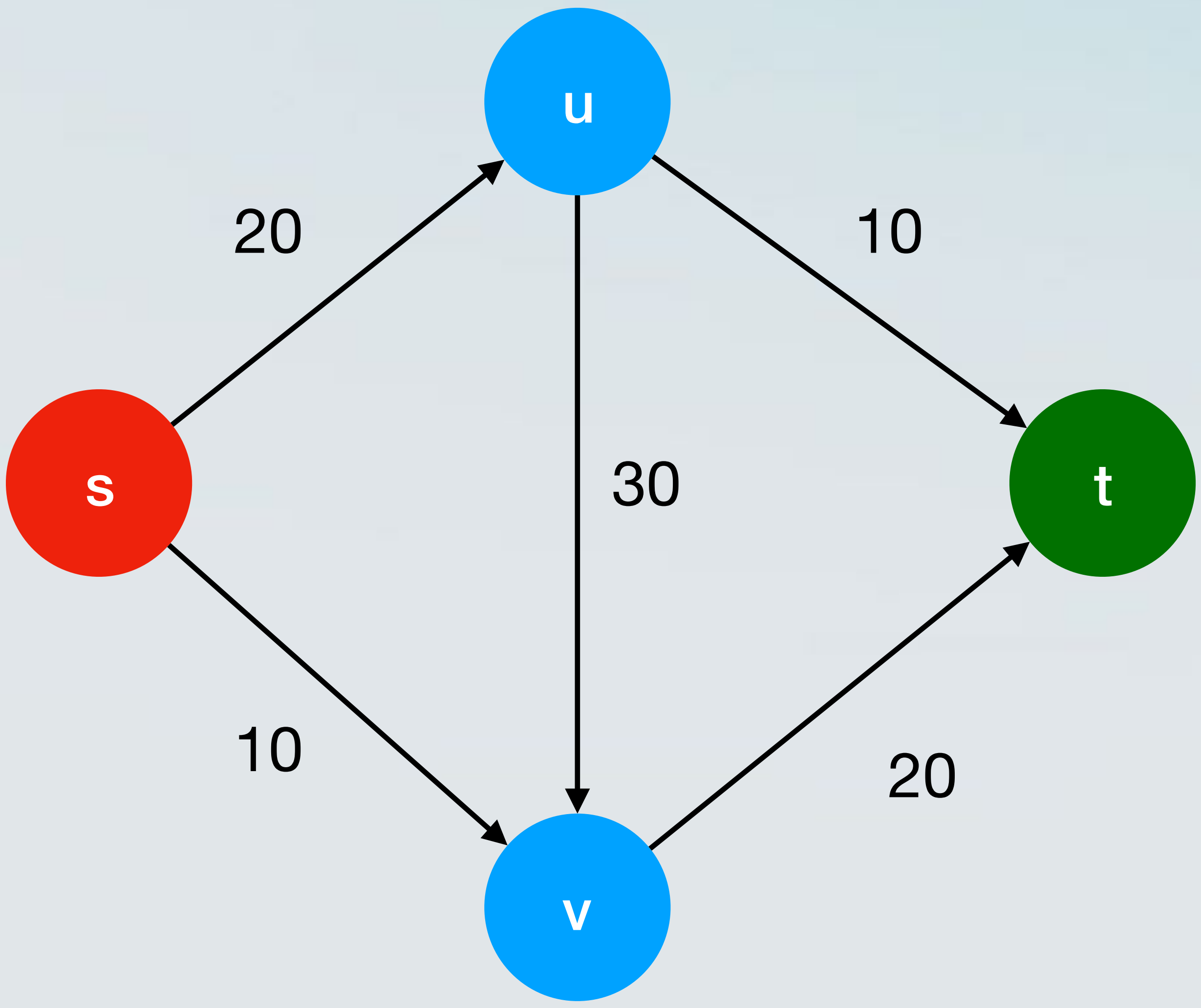
Recap and plan

- **Last 2 lectures:**
  - Dynamic Programming
  - Weighted Interval Scheduling, Subset Sum, Knapsack
- **This lecture:**
  - Network Flows
  - Maximum Flow
  - The Ford-Fulkerson Algorithm
  - Max-Flow - Min-Cut

Flow Networks

- A flow network is a directed graph $G=(V, E)$ with the following properties:
  - Each edge e in E has a nonnegative *capacity* c_e.
  - There is a single *source* node s in V.
  - There is a single *sink* node t in V.
  - All other nodes in $V - \{s, t\}$ are called *internal* nodes.

Example



Flow Networks

- Further assumptions:
  - The *source* s does not have any incoming edges.
  - The *sink* t does not have any outgoing edges.
  - There is at least one edge incident to each node.
  - All the capacities are integer numbers.

Flow

- An (s-t) flow is a function $f: E \to \mathbf{R}^+$, mapping each edge e to a nonnegative real number $f(e)$.

- A (feasible) flow must satisfy the following two properties:

  - (Capacity) For each e in E, we have $0 \leq f(e) \leq c_e$

  - (Flow Conservation) For each node v in V - {s, t}, we have that

$$\sum_{e \textbf{ into } v} f(e) = \sum_{e \textbf{ out of } v} f(e)$$

Flow

- The source s generates flow.
- The source t absorbs flow.
- Value of flow f, denoted val(f):
  - Total flow out of s.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

- Generally, define $f^{out}(v)$ and $f^{in}(v)$ for the flow going out of (resp. going into) node v.
- Similarly, define $f^{out}(S)$ and $f^{in}(S)$ for sets of nodes S.

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Let's try to design an algorithm for that.

An algorithm for max-flow

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- Let’s start with a feasible solution.

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  - Any suggestions?

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  - $f(e) = 0$ for all e in E.
  - This is a feasible flow. But not very good.
  - Let’s try to increase it.

Increasing the flow

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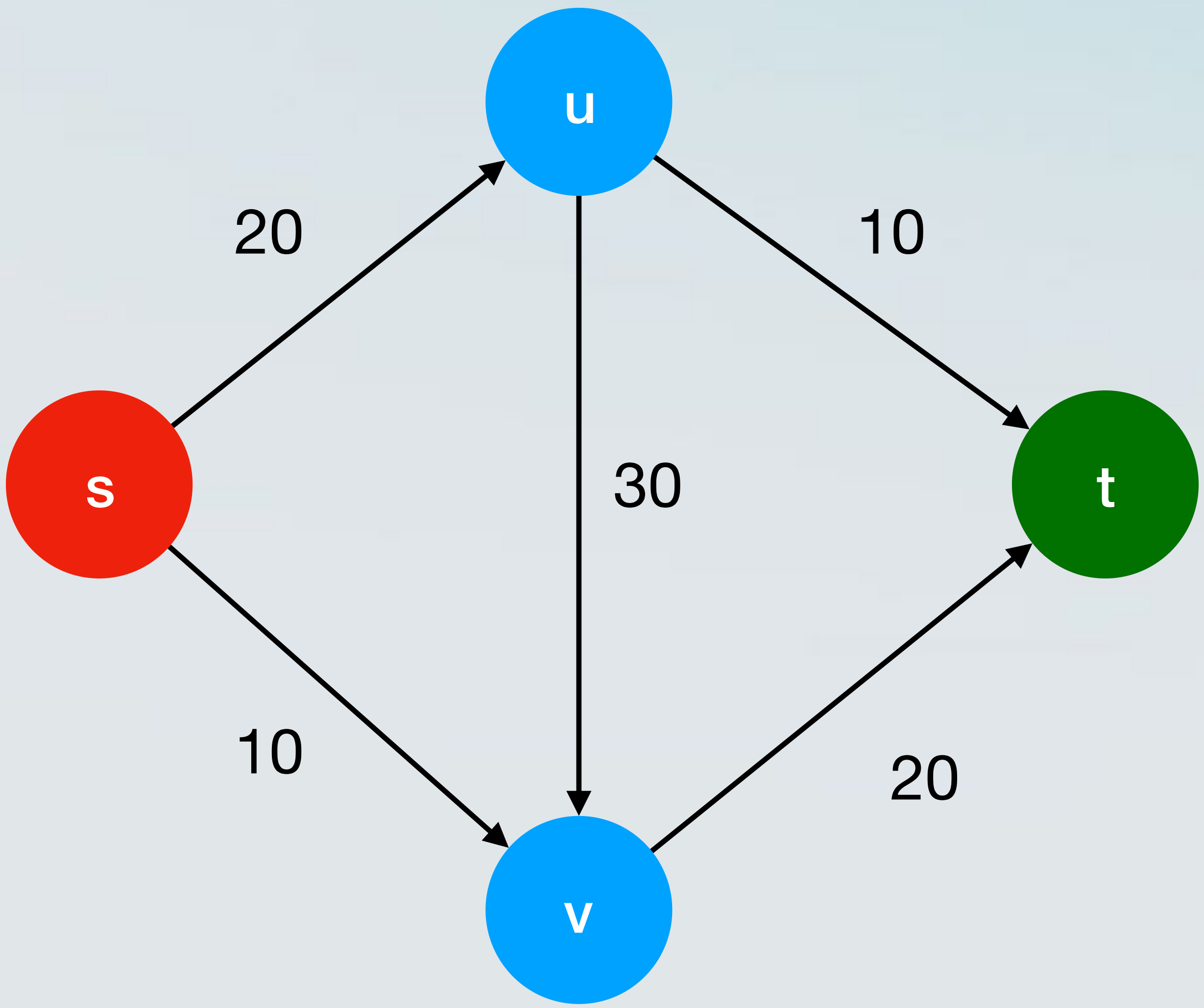
- So we have to find an (s-t) path, and route it via this path.
- How much flow are we allowed to route through this path?
    - As much as the smallest capacity c_e of any edge e on the path.

Example



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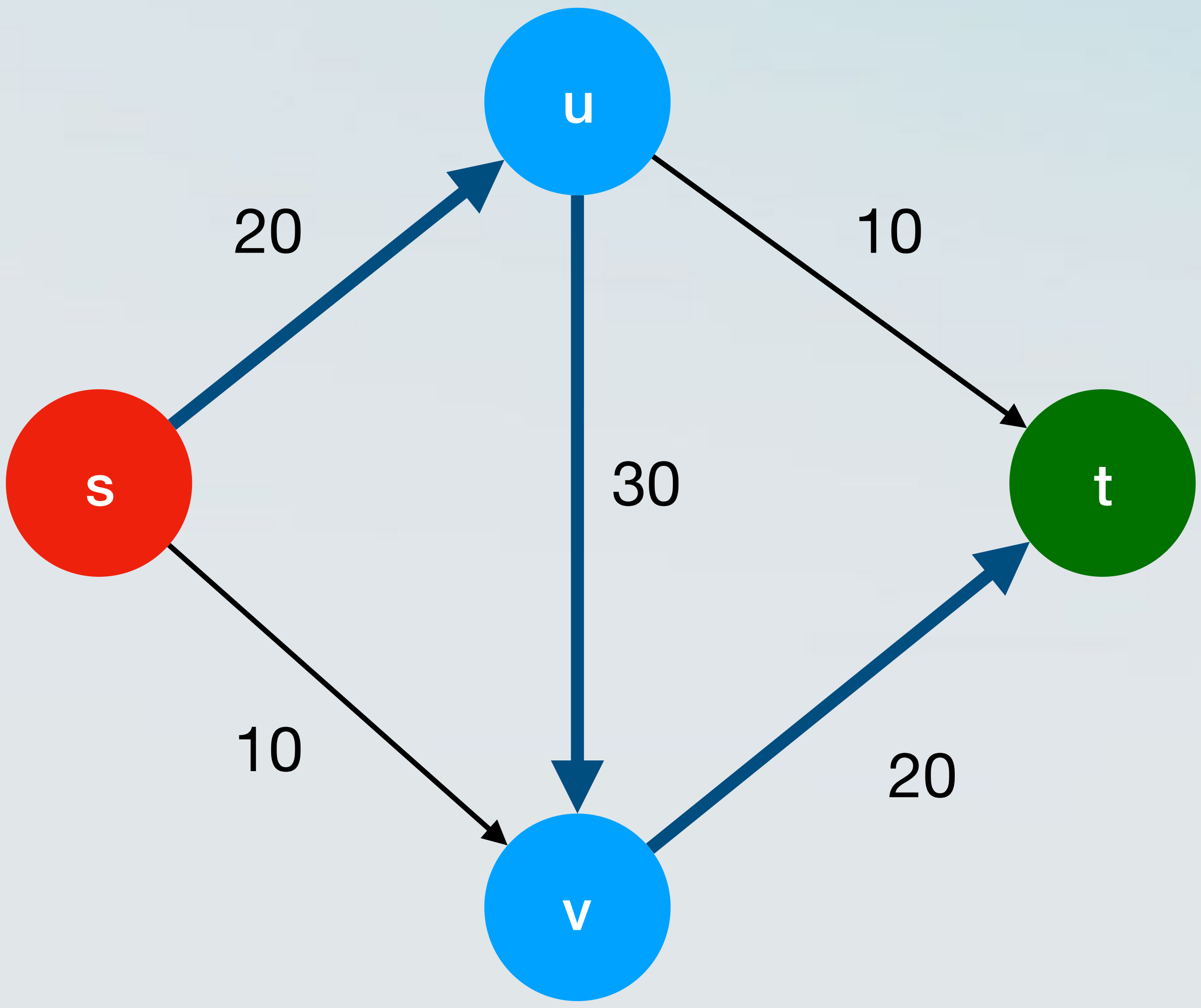

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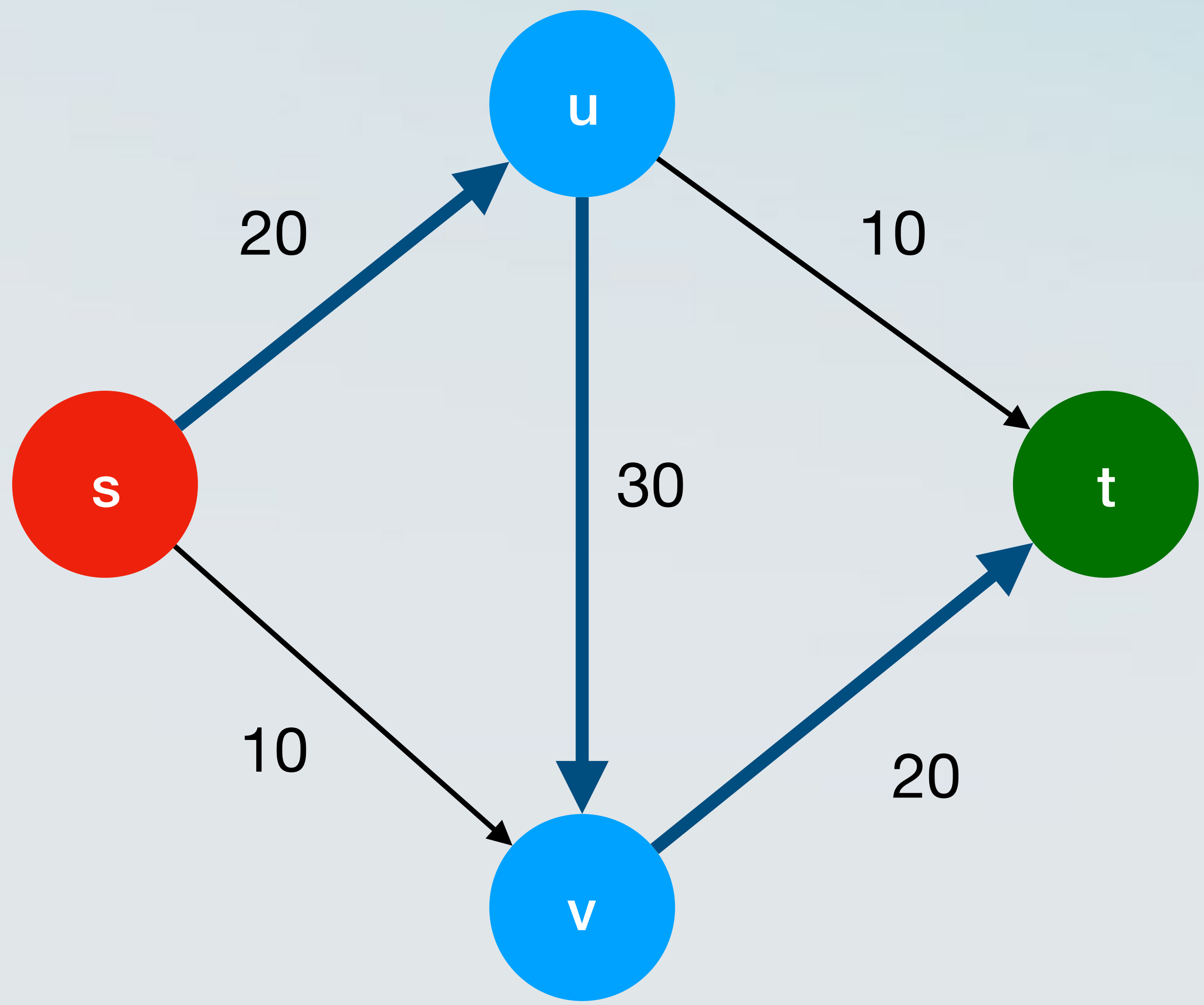
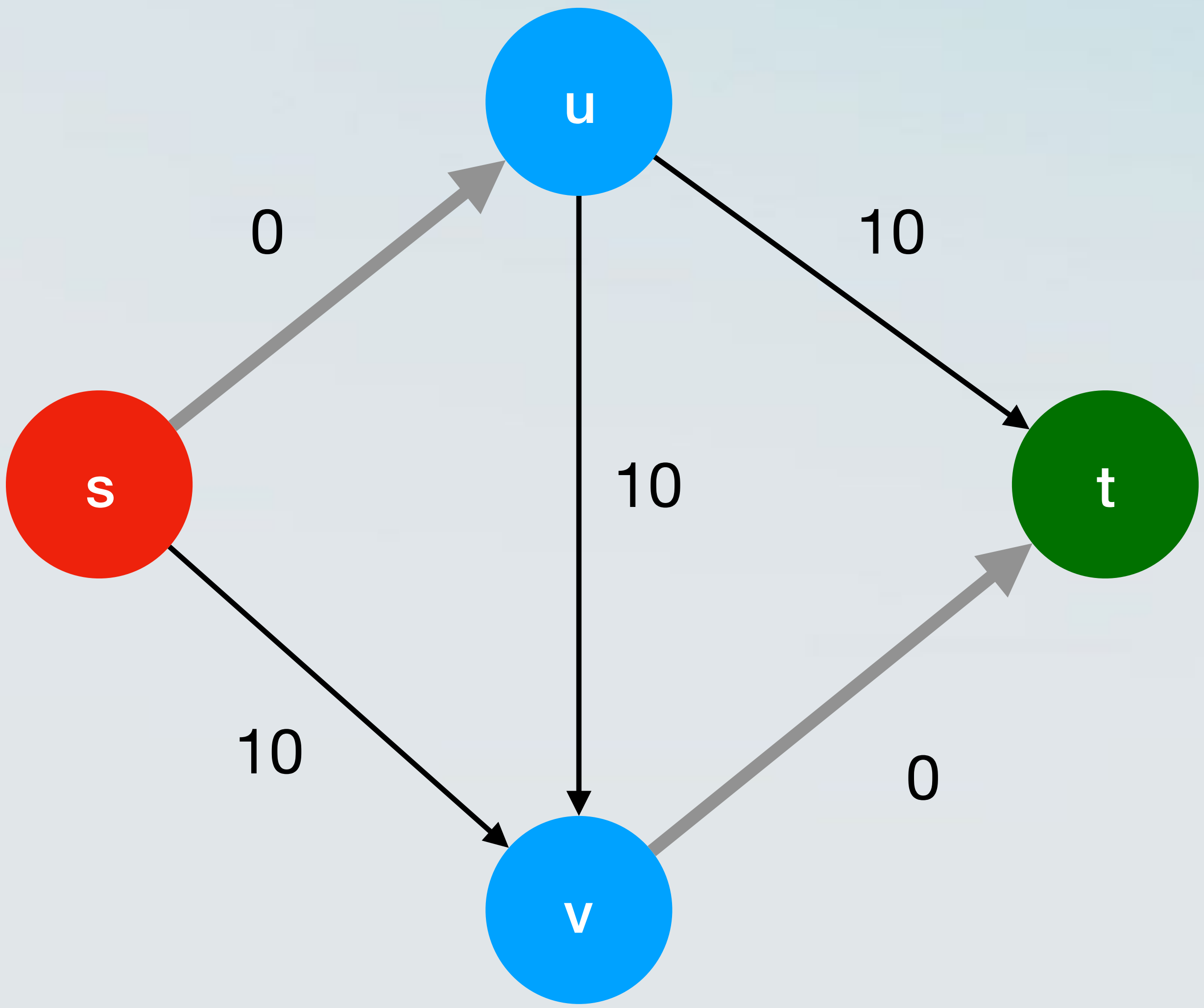


Is 20 the maximum flow?

Example



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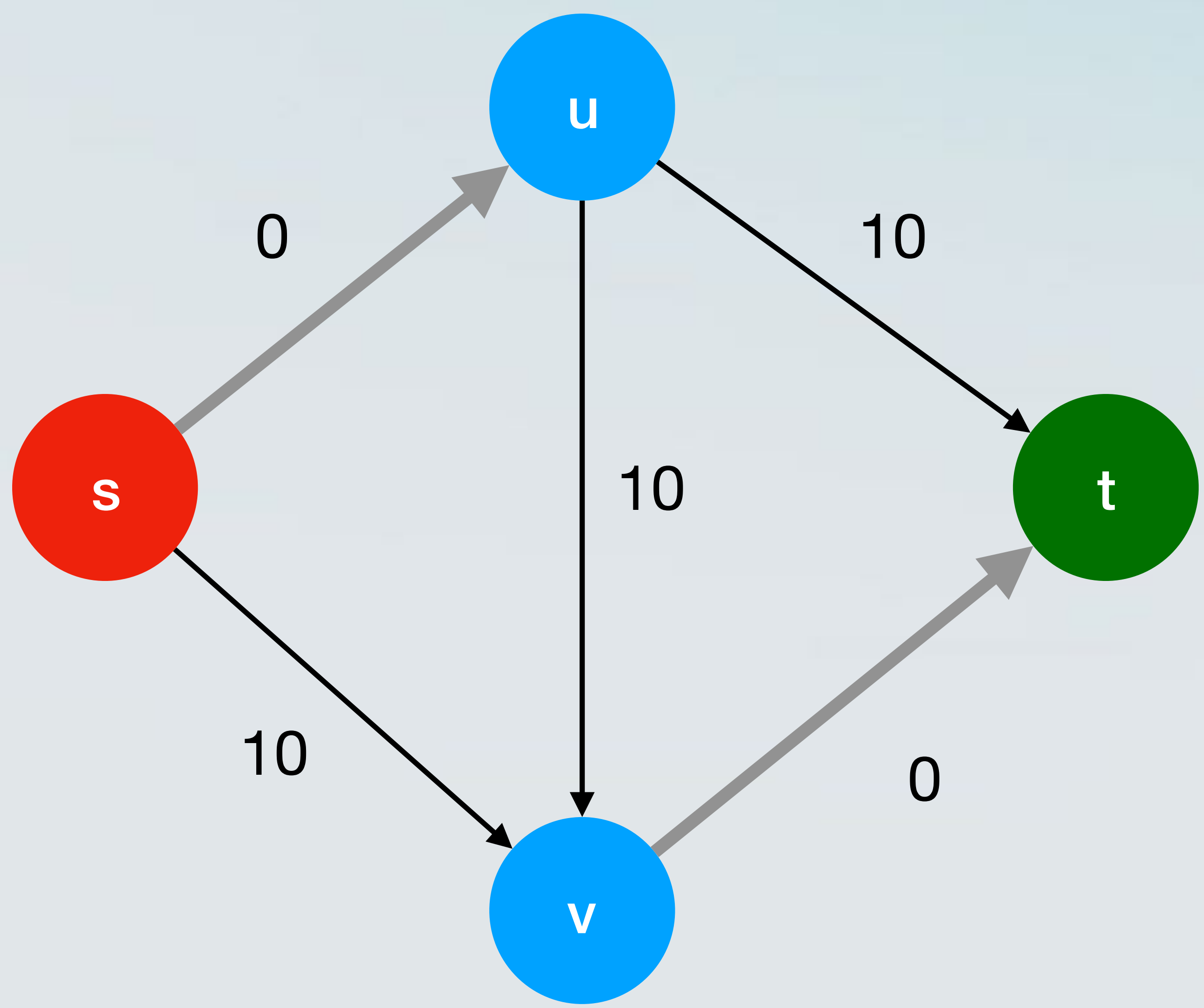
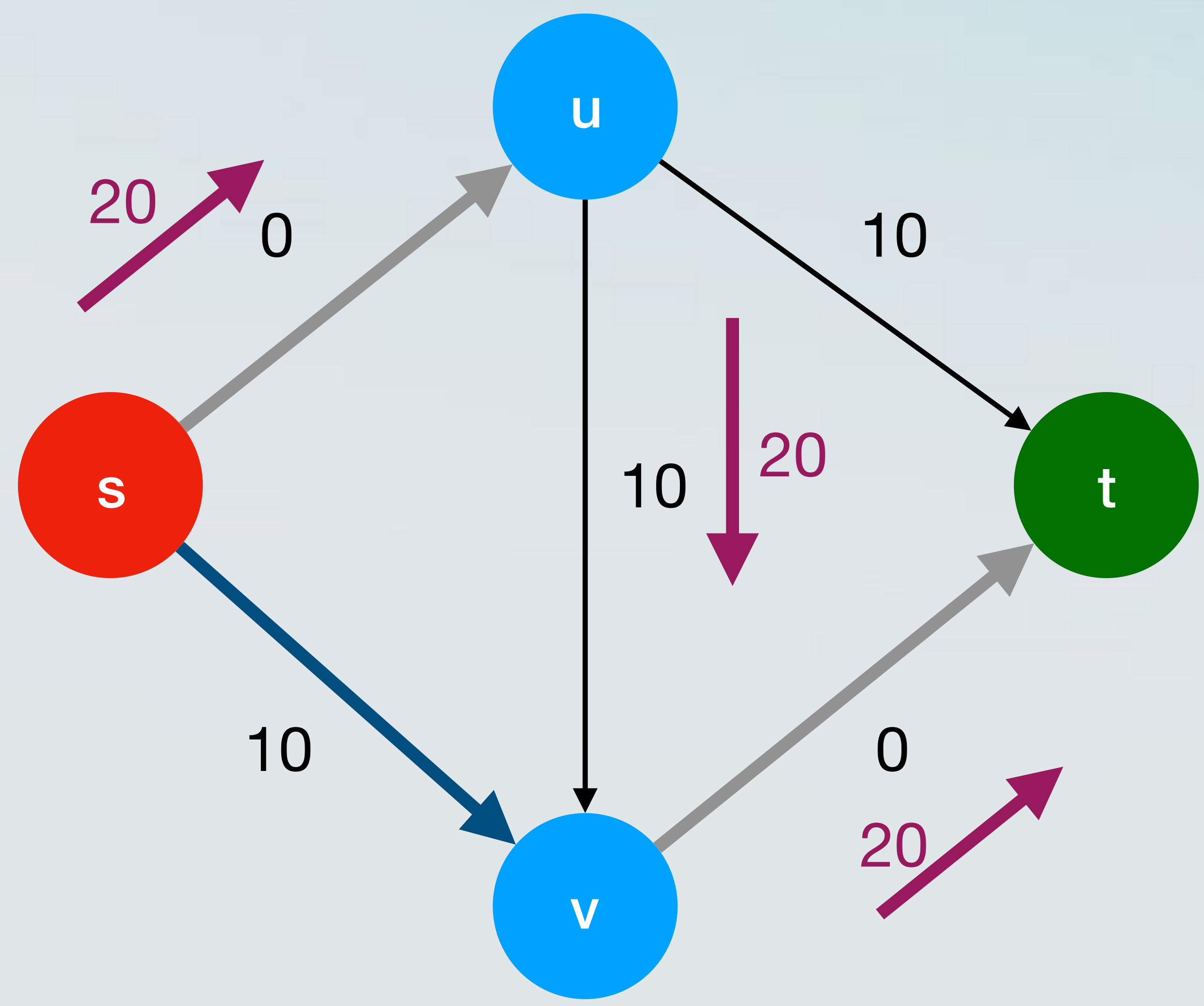


We are stuck!

What we would like to do



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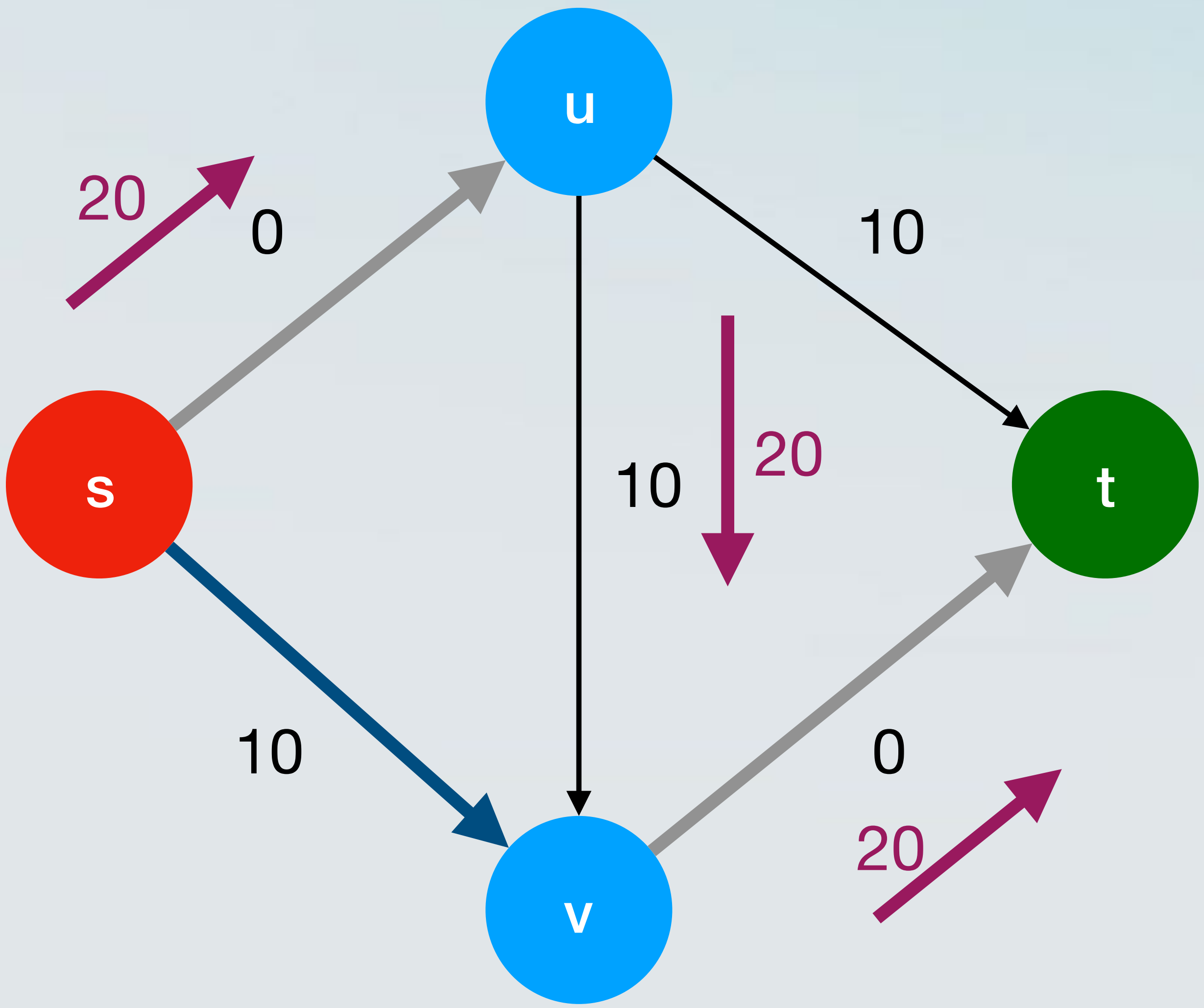

Flow conservation
violated here!

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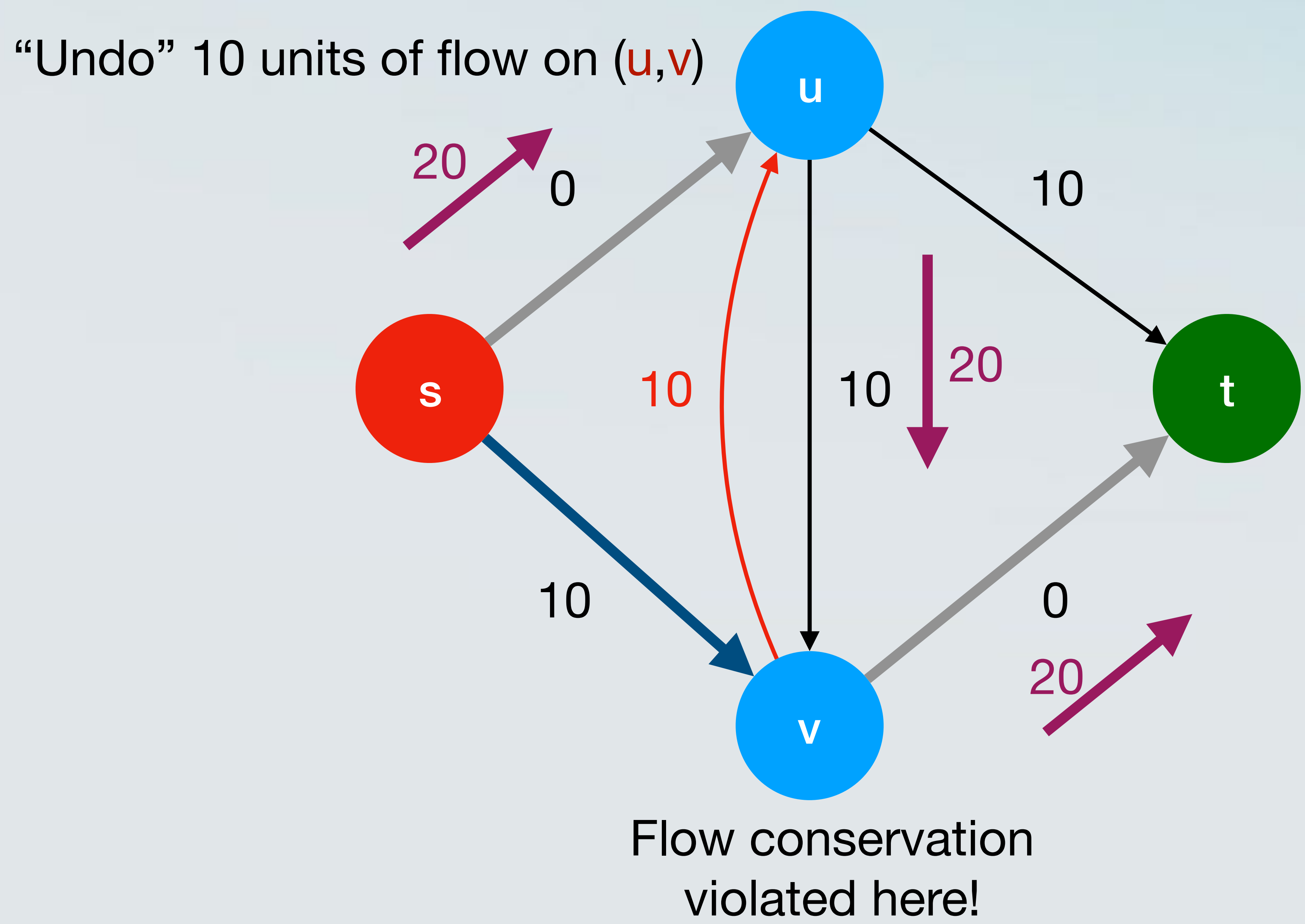

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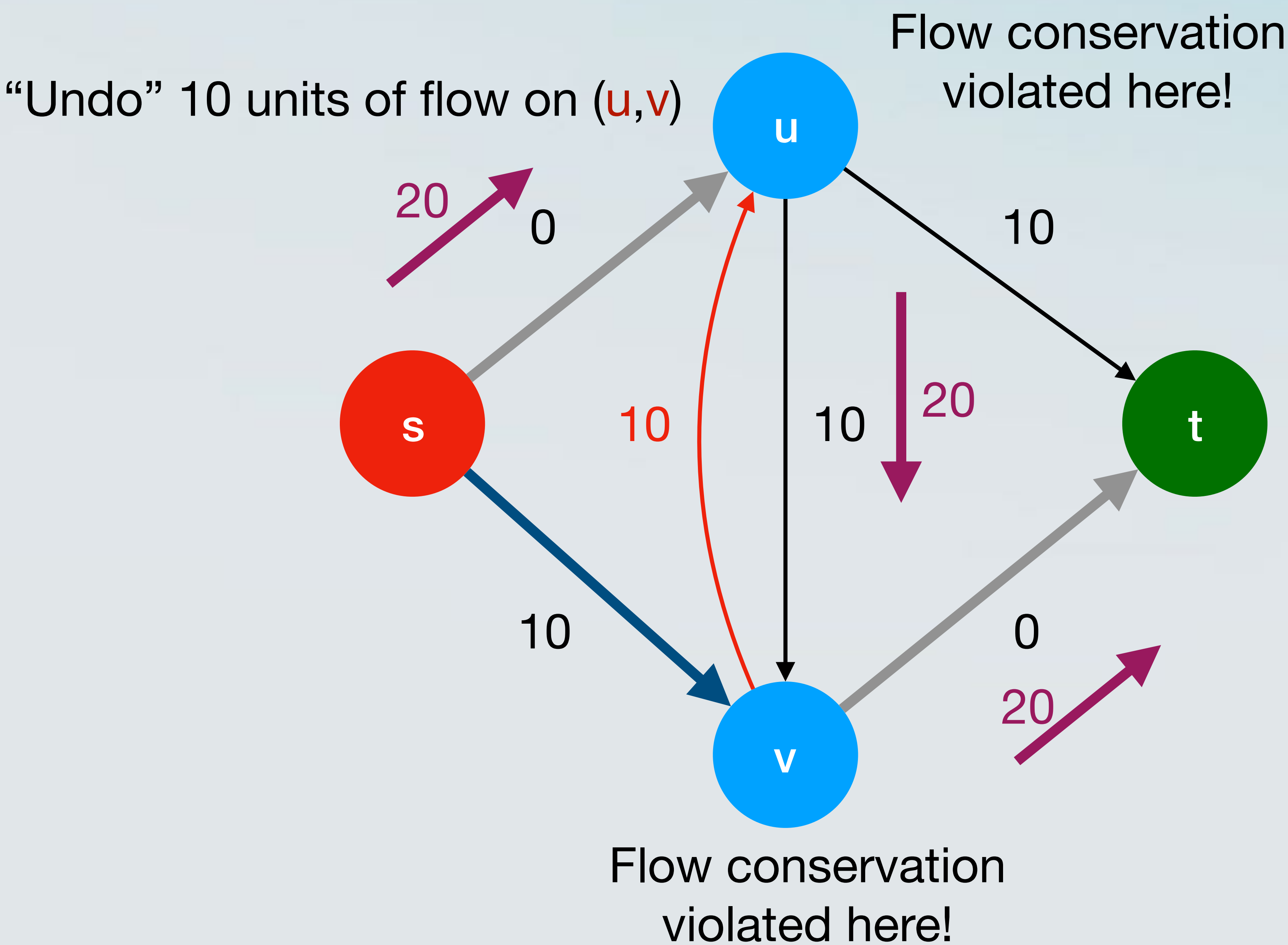

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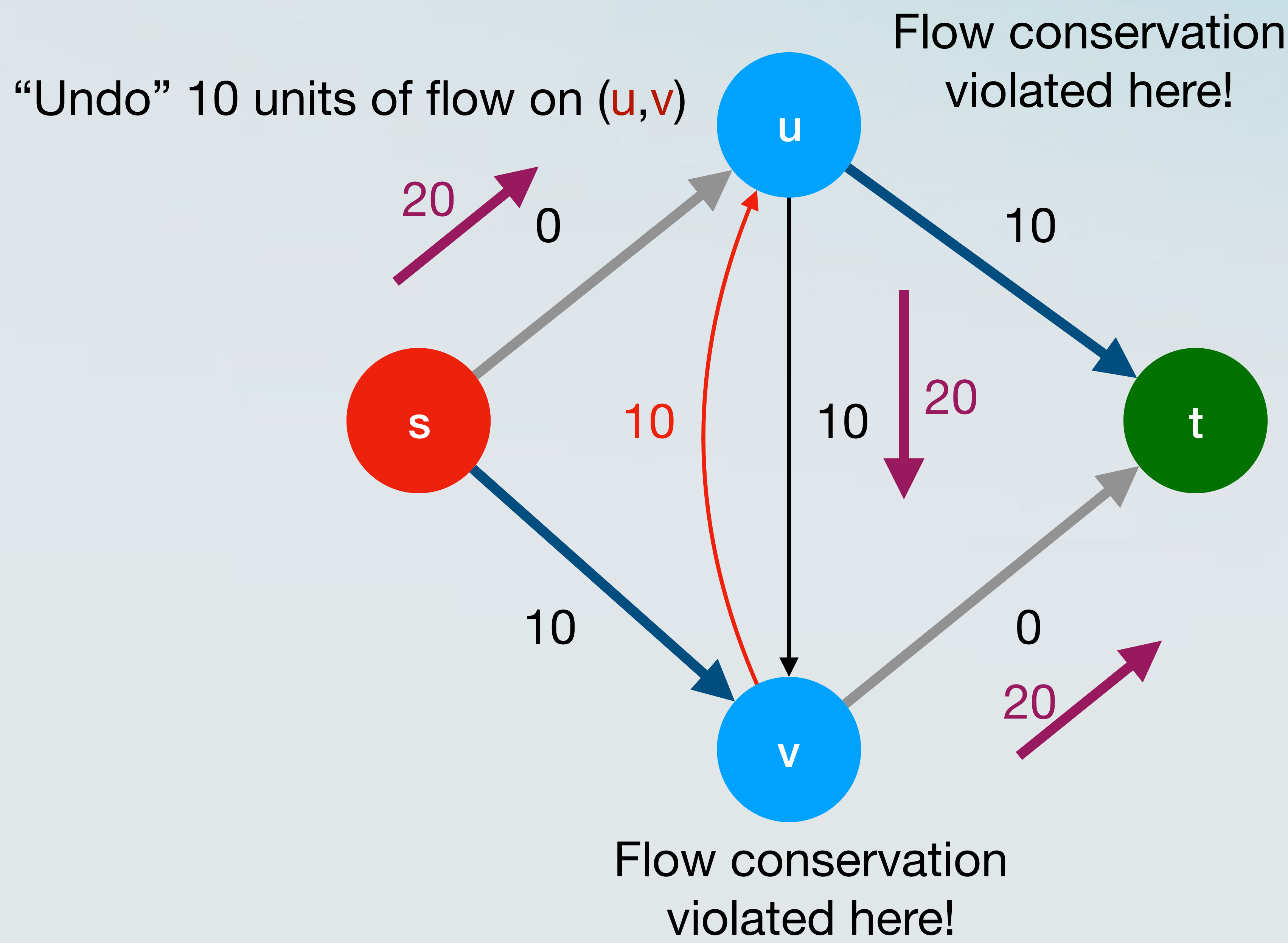

Idea

- We can push flow *forward* on edges with leftover capacity.
- We can push flow *backward* on edges that are already carrying flow.
- How to we do that systematically?

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  - For each edge $e=(u, v)$ of E on which $f(e) > 0$, there is an edge $e'=(u, v)$ in E_f with a capacity of $f(e)$.

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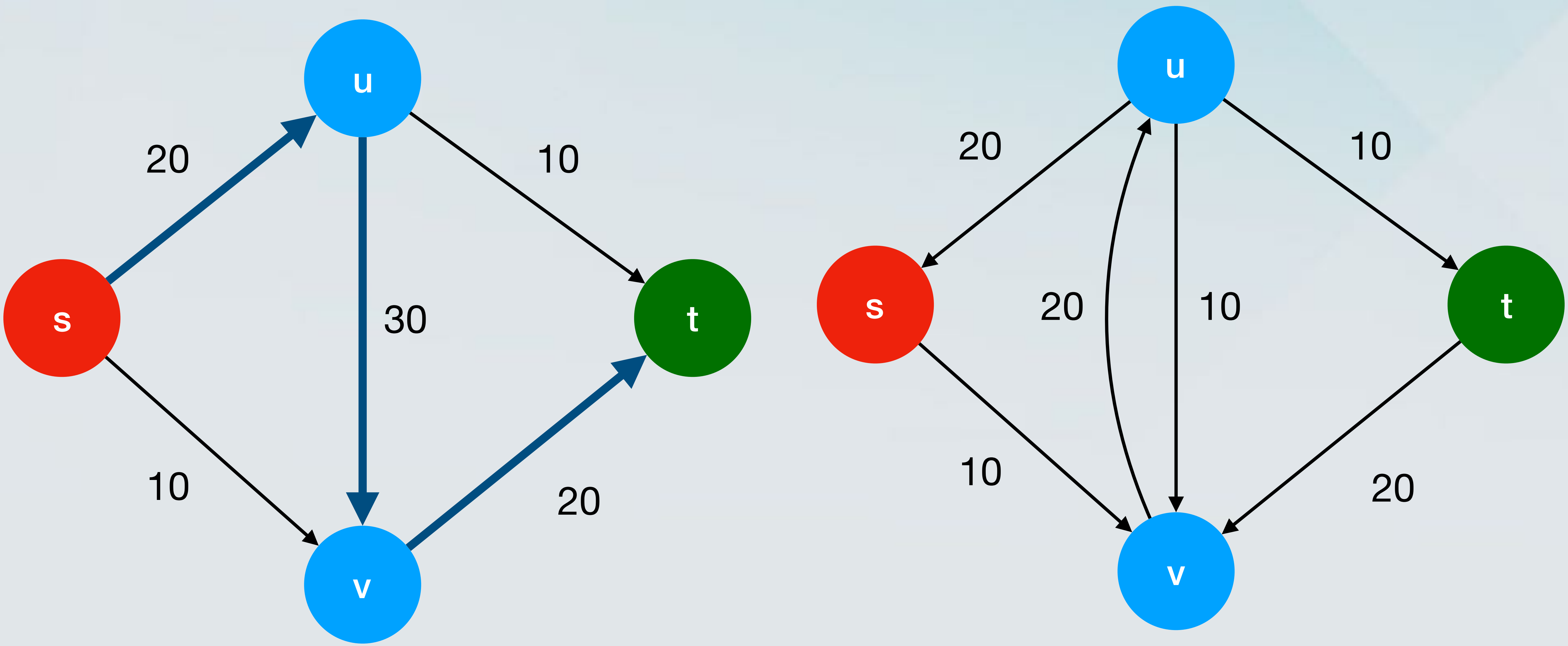

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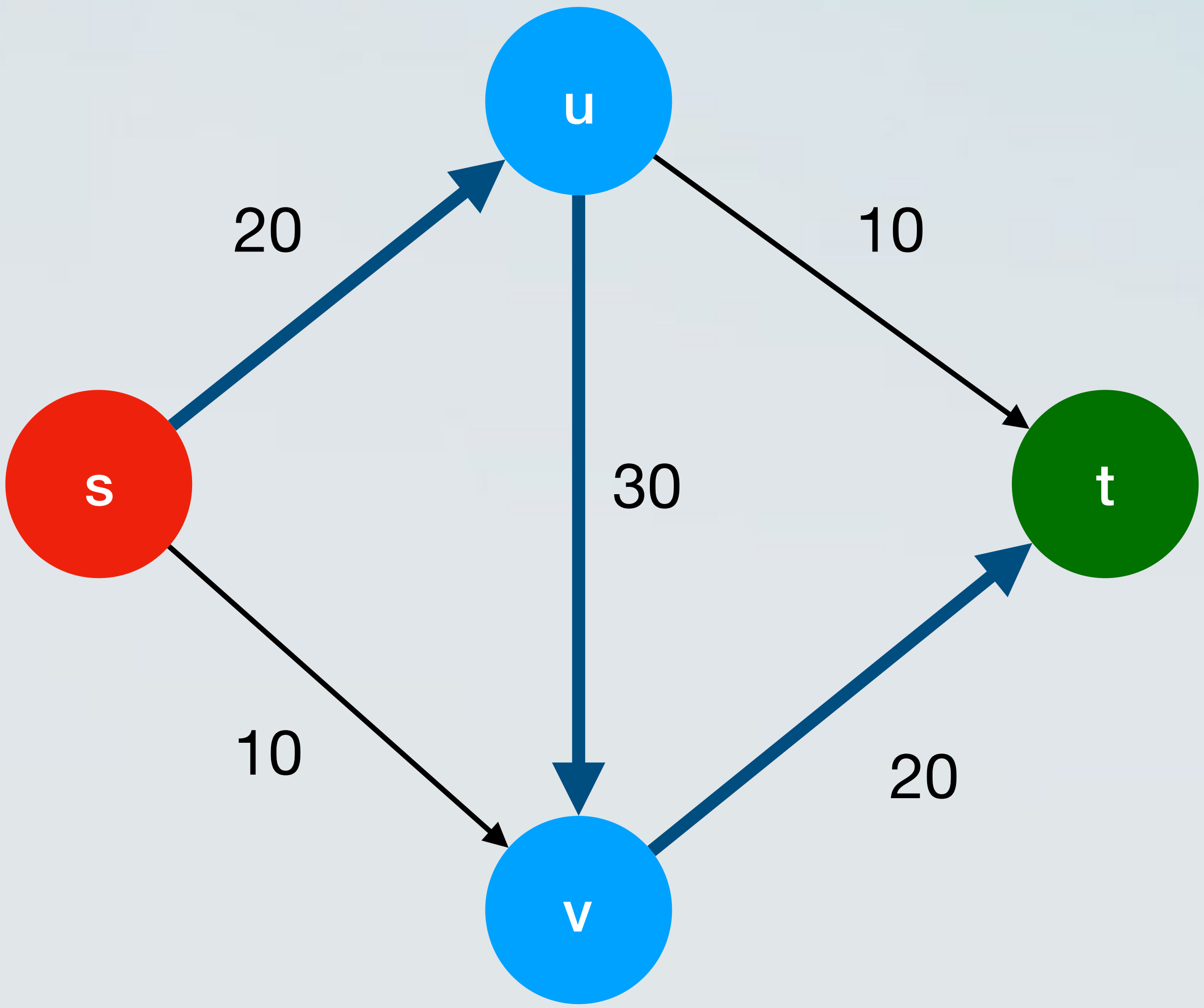

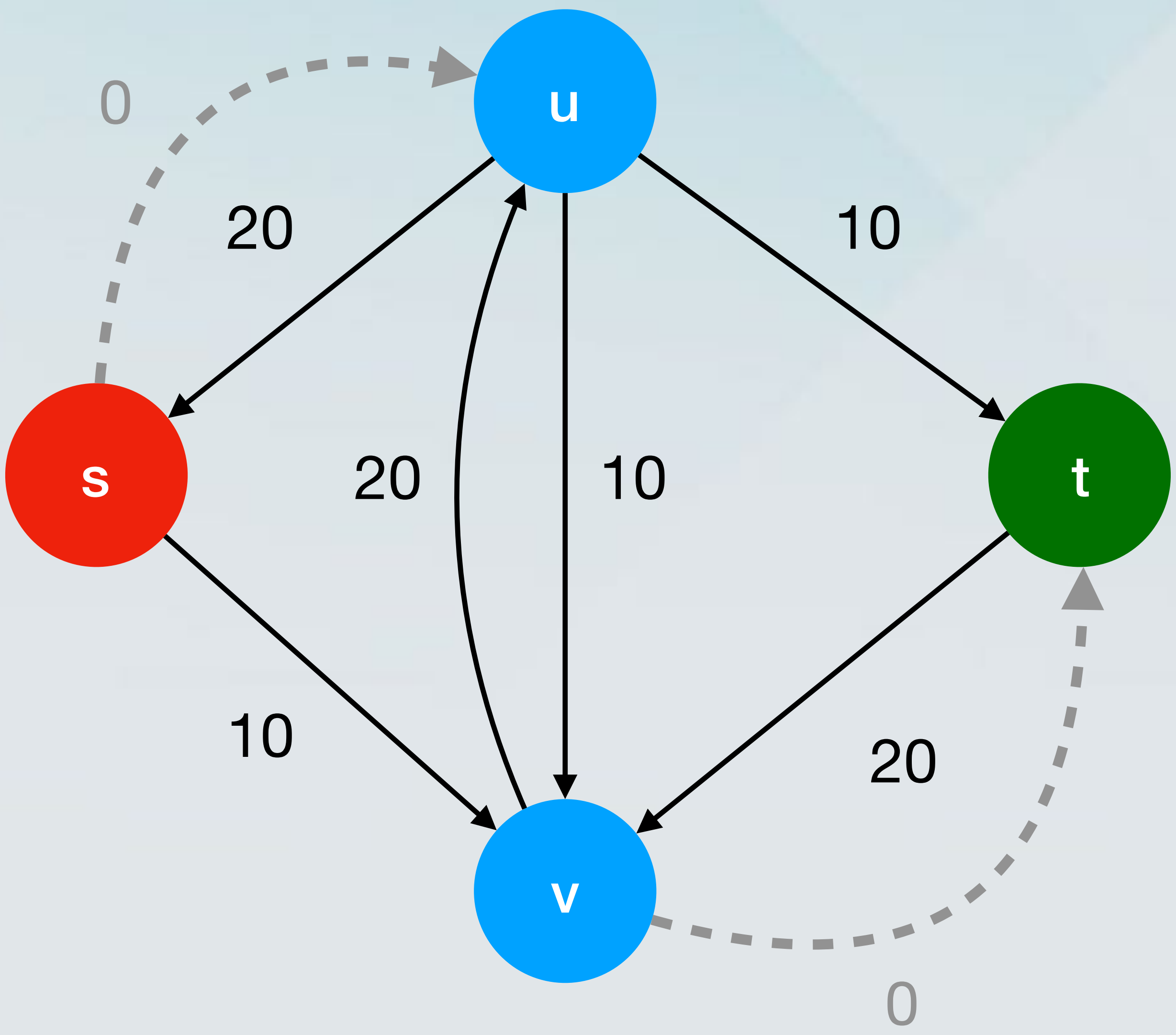

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- We will use paths in the residual graph G_f.

Working with the residual graph

- Find an (s-t) path P in the residual graph.
  - We will call this an ***augmenting path***.
- Define the *bottleneck* of P,
  - denoted **bottleneck**(P, f)
  - to be the minimum residual capacity on any edge on P.
- Define the augmentation of flow f into flow f'
  - denoted **augment**(f, P)

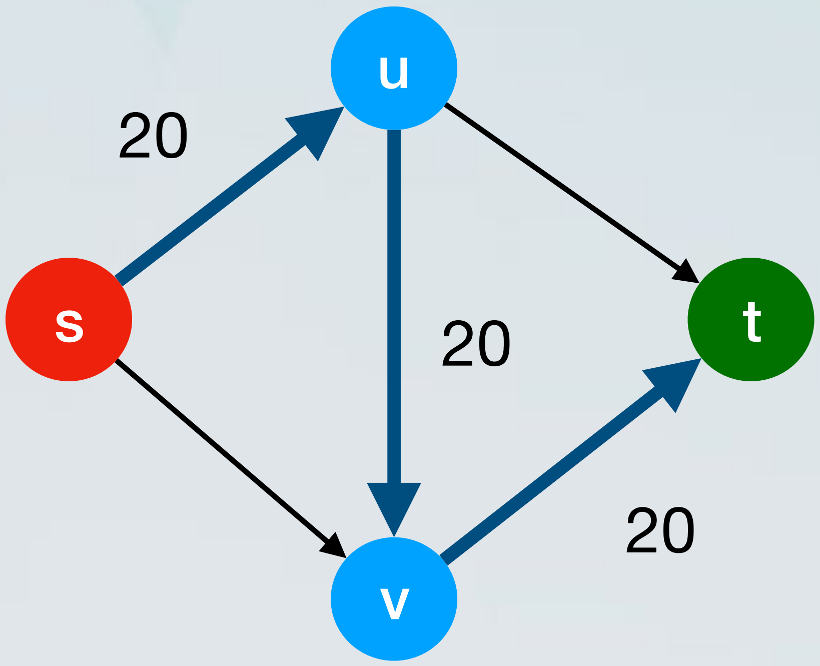
Augmenting the flow

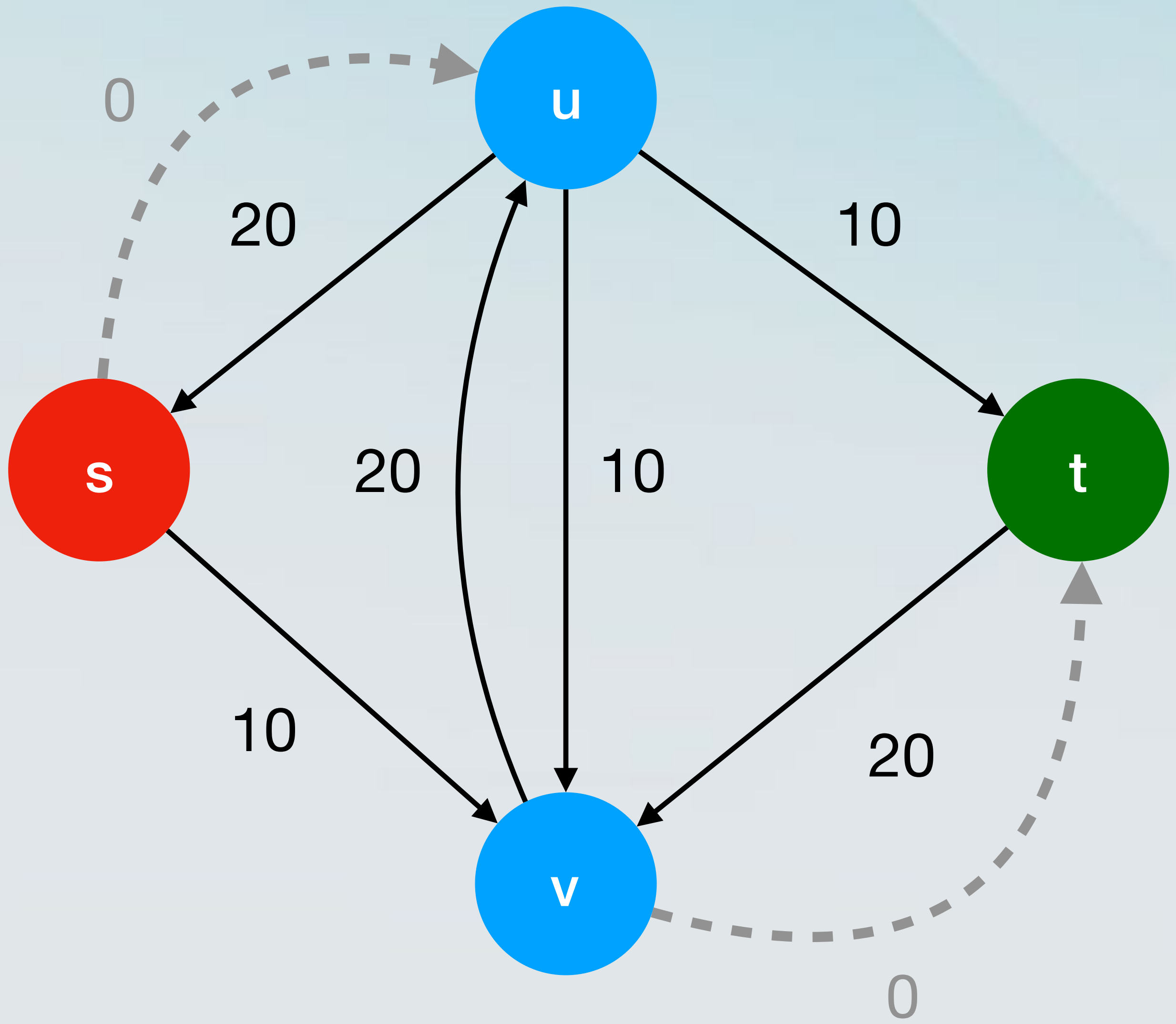
augment(f, P)

```
Let b = bottleneck(P, f)
For each edge e=(u, v) in P
      If e is a forward edge then
          Increase f(e) in G by b
      Else (e is a backward edge, and let e' = (v, u))
          Decrease f(e') in G by b
      EndIf
 EndFor

Return( f );
```

Example





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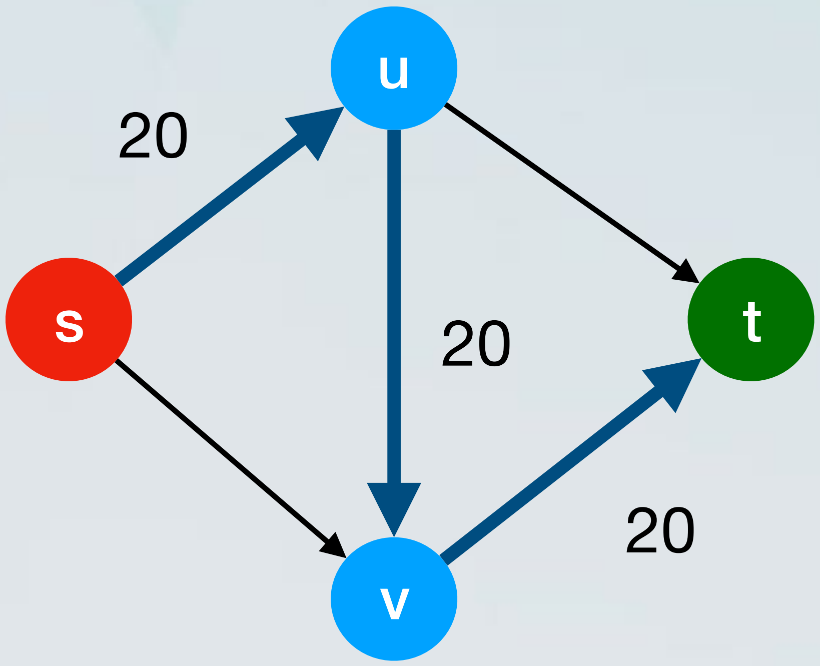

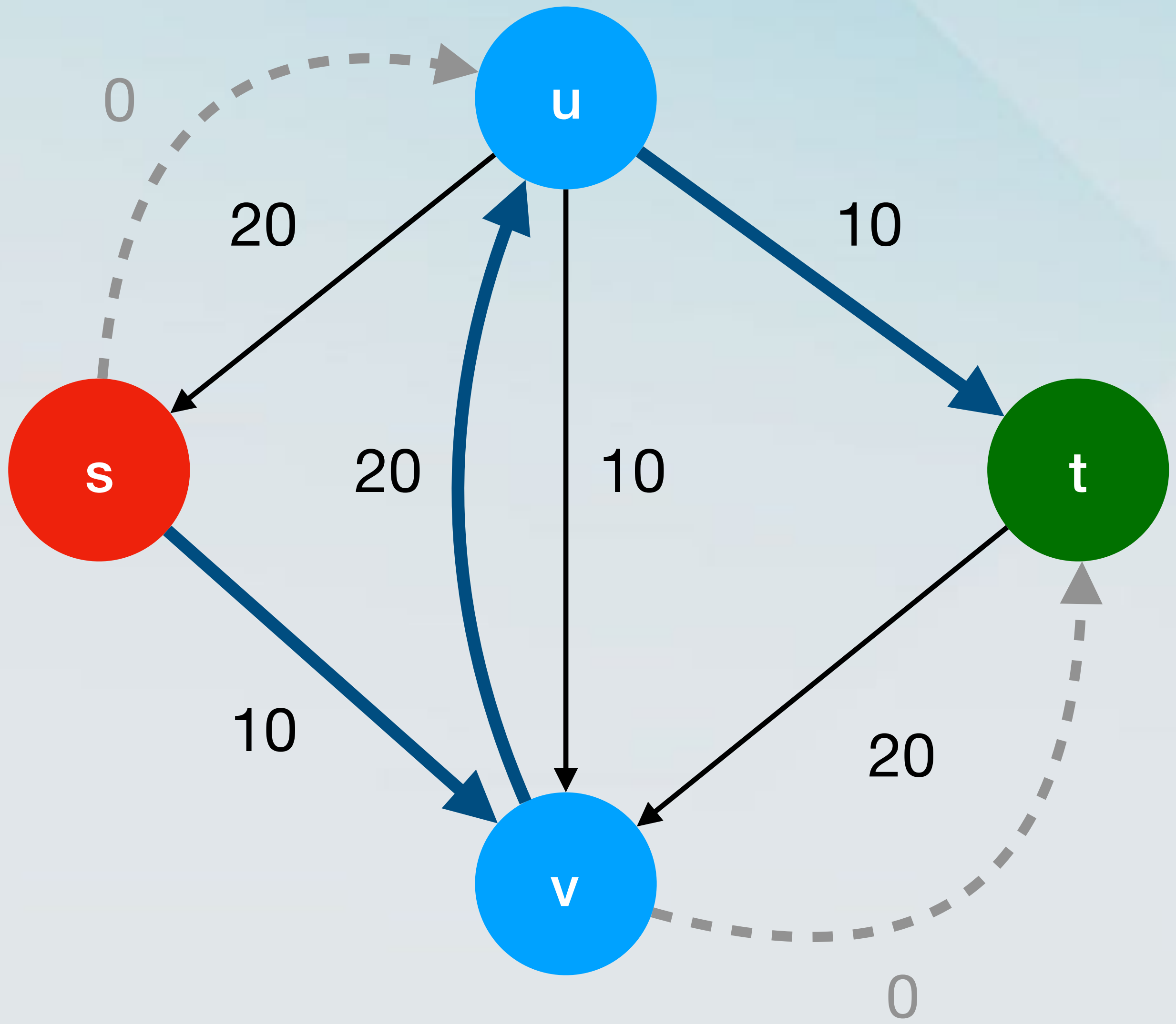

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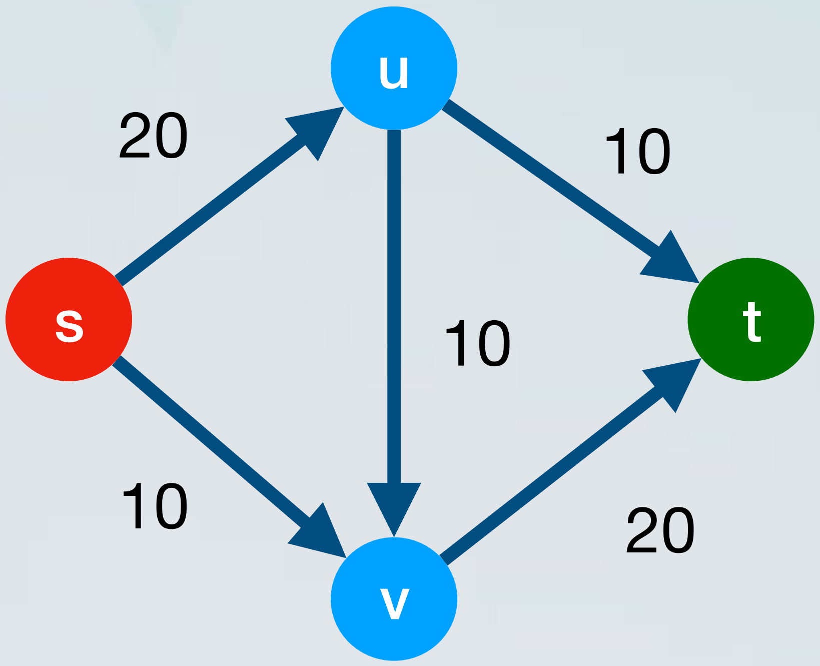

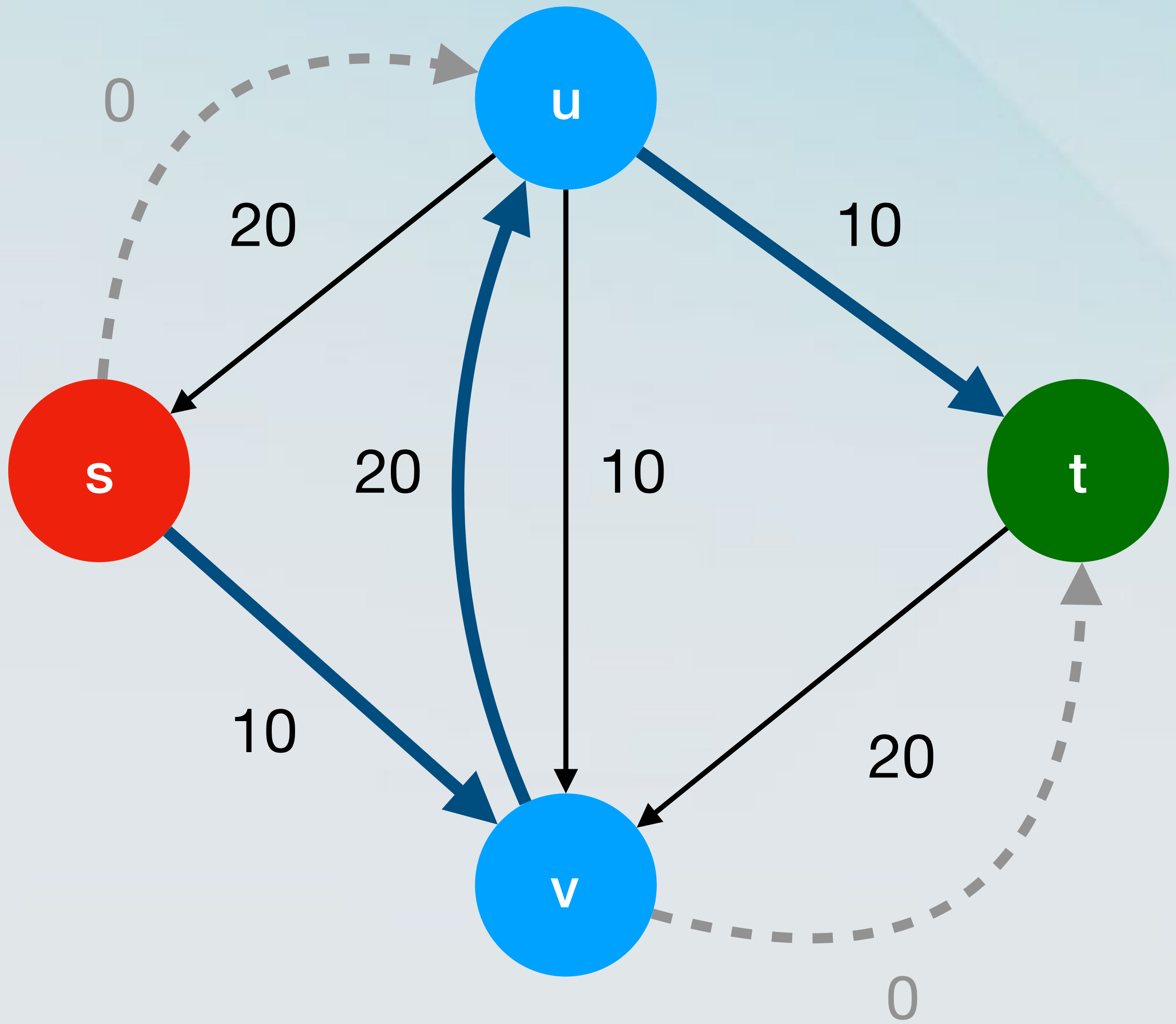

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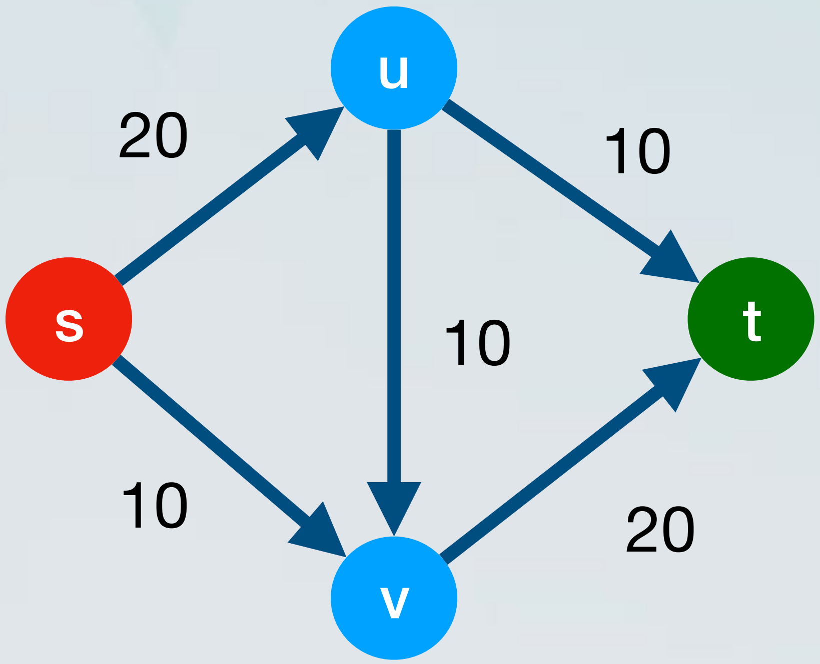

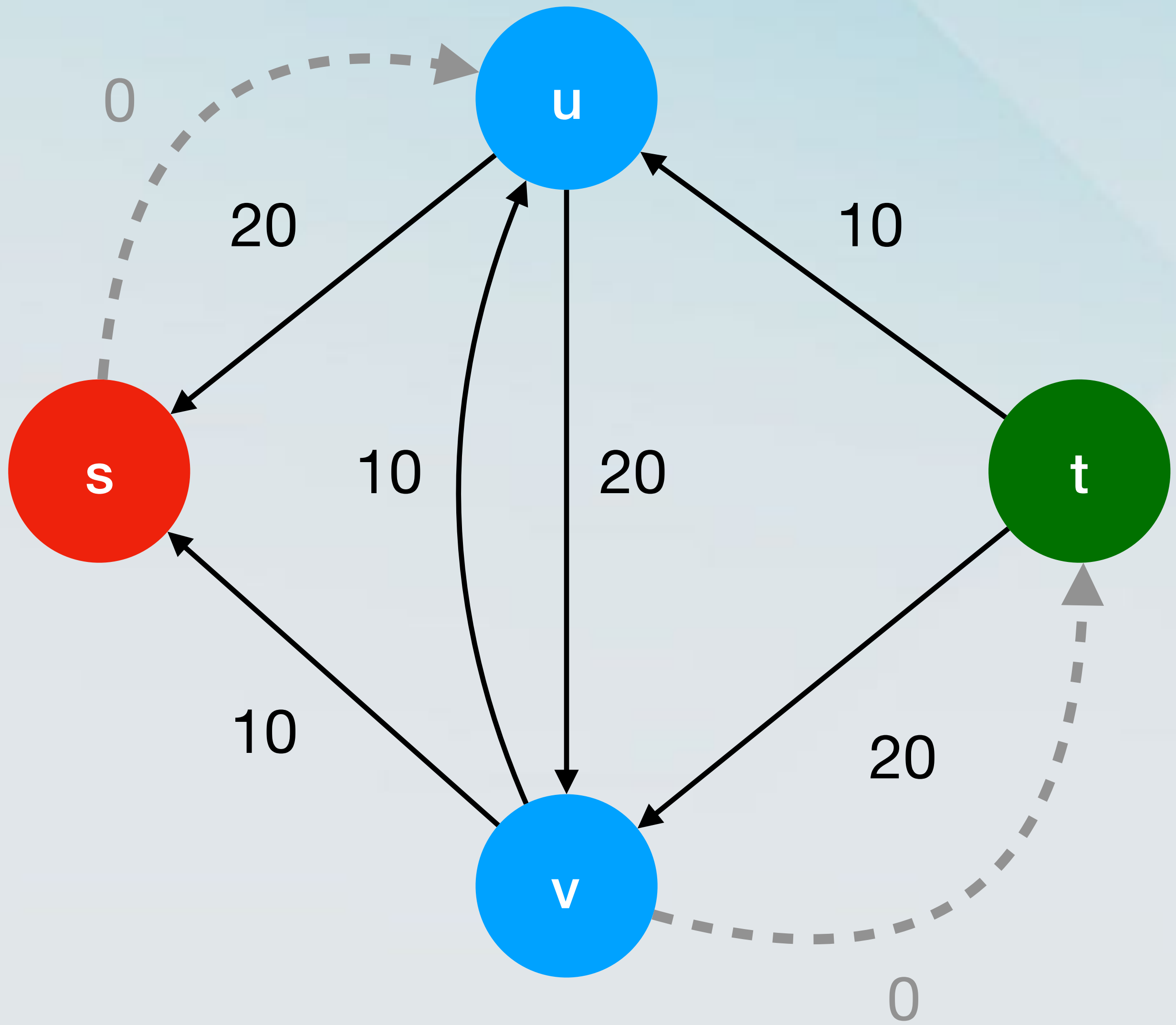

Feasibility

- Let f’ = **augment**(f, P)
- Is f’ a flow?
    - Suffices to only check edges e in P (why?)
    - Consider an arbitrary edge e = (u, v) of P.

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  - The capacity condition holds.

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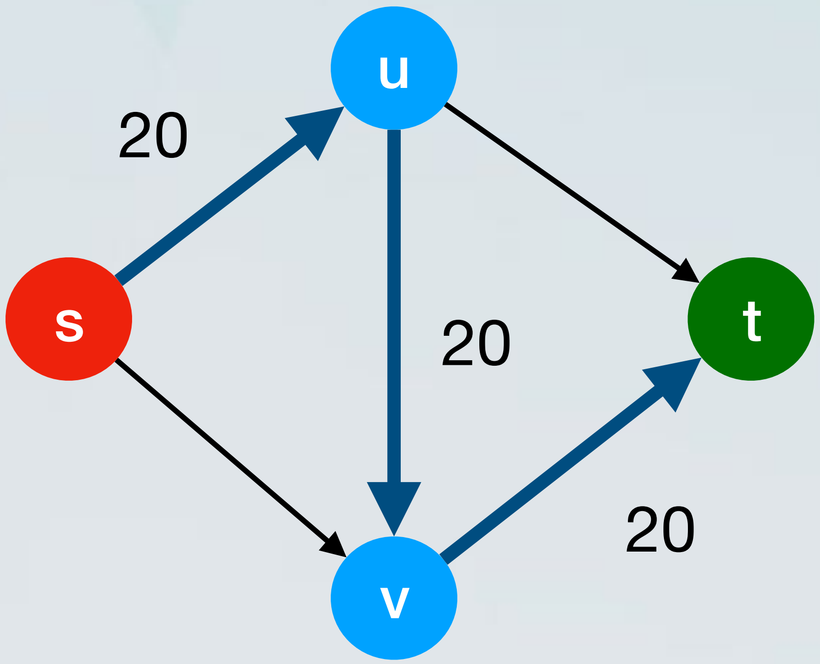
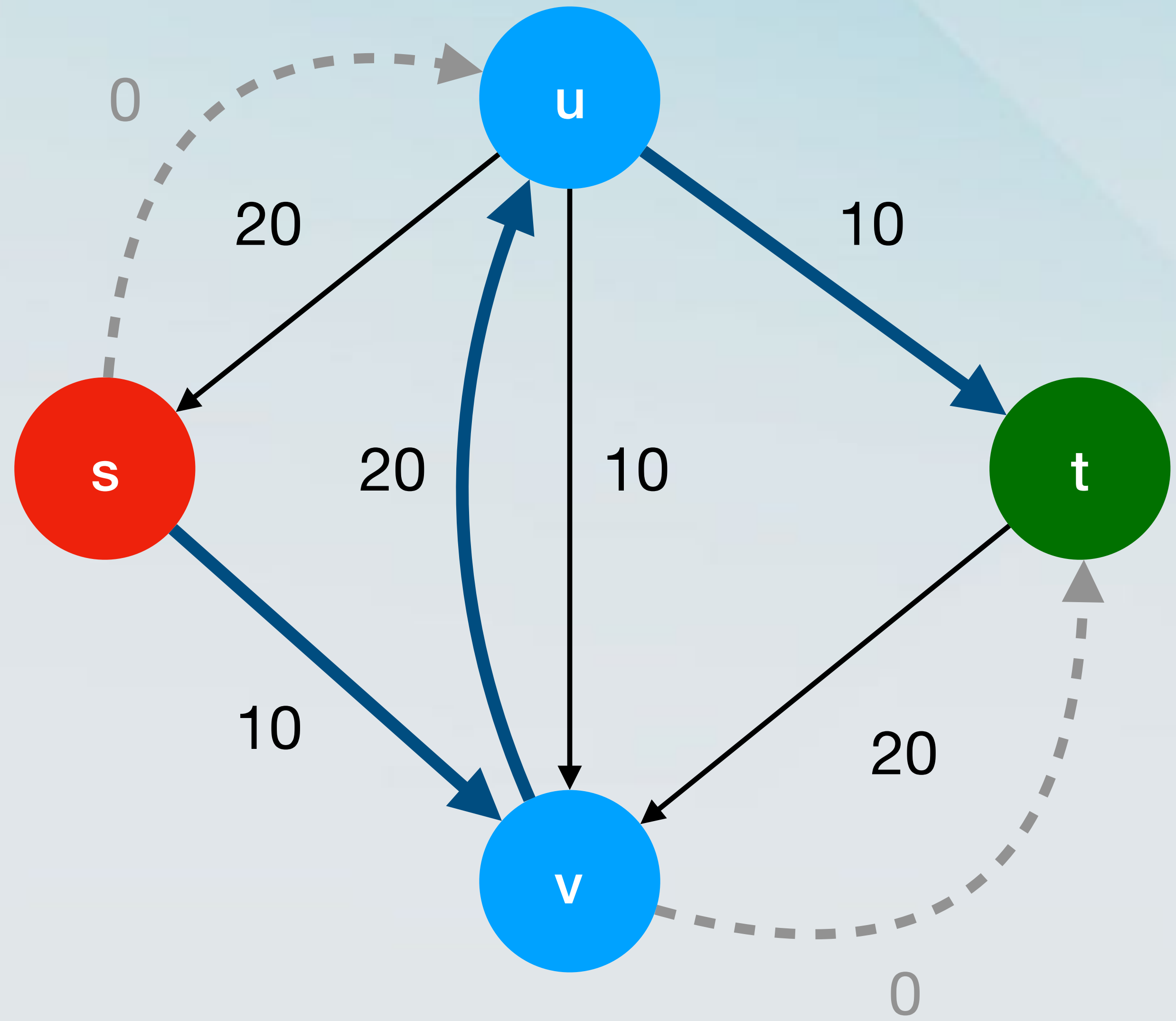
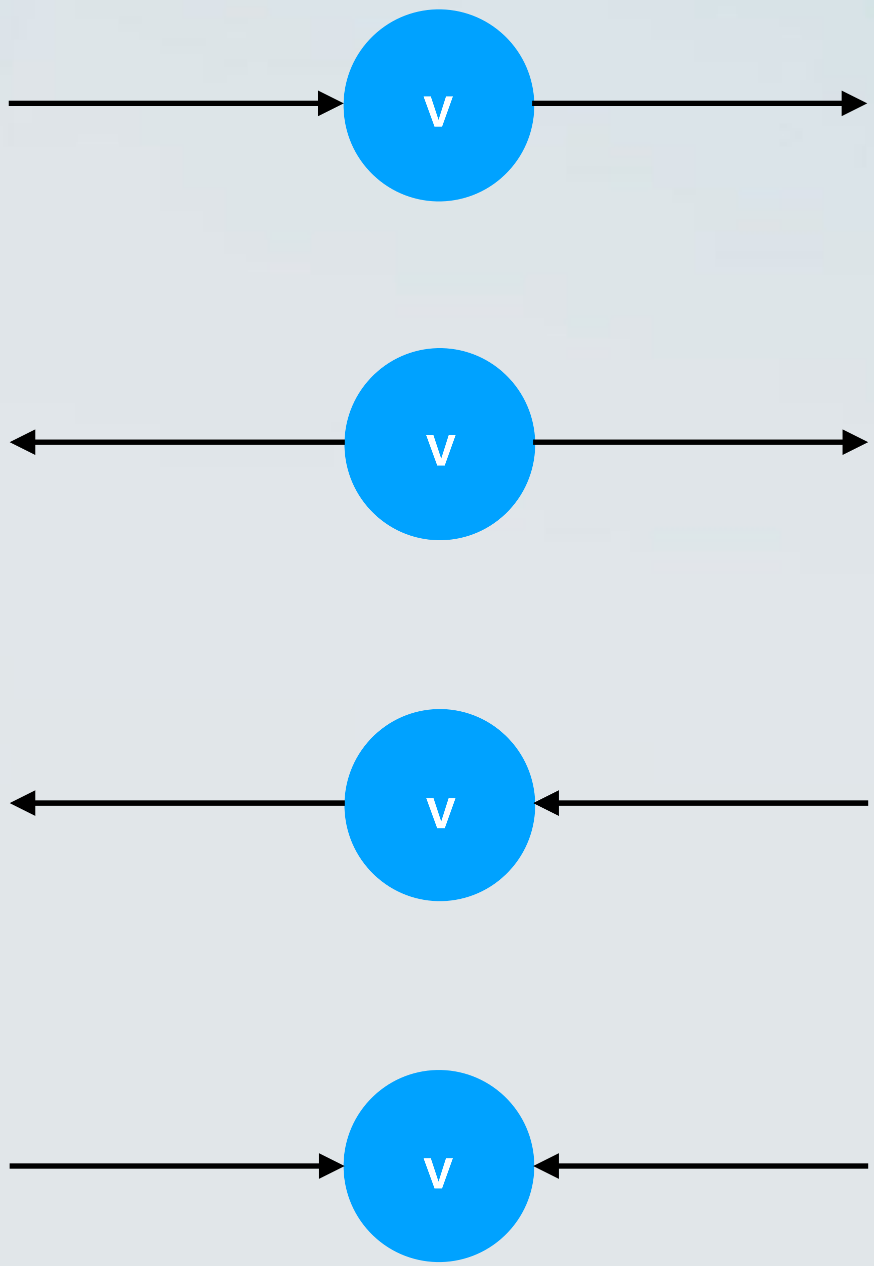




Feasibility (flow conservation)



forward , forward

backward , forward

backward , backward

forward , backward

The Ford-Fulkerson Algorithm

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- Repeat the same process for flow f' and graph G_f'.
- Until the residual graph has no more (s-t) paths.

The Ford-Fulkerson Algorithm

Max-Flow

Initially set f(e) = 0 for all e in E.

While there exists an s-t path in the residual graph Gf

Choose such a path P
f' = **augment**(f, P)
Update f to be f'
Update the residual graph to be Gf'

Endwhile

Return (f)

Ford-Fulkerson analysis

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- In each call f' = **augment**(f, P) , where f is a feasible flow we get a feasible flow f'.
    - This is what we established in the previous slides.
- We never, at any step, produce an infeasible flow.

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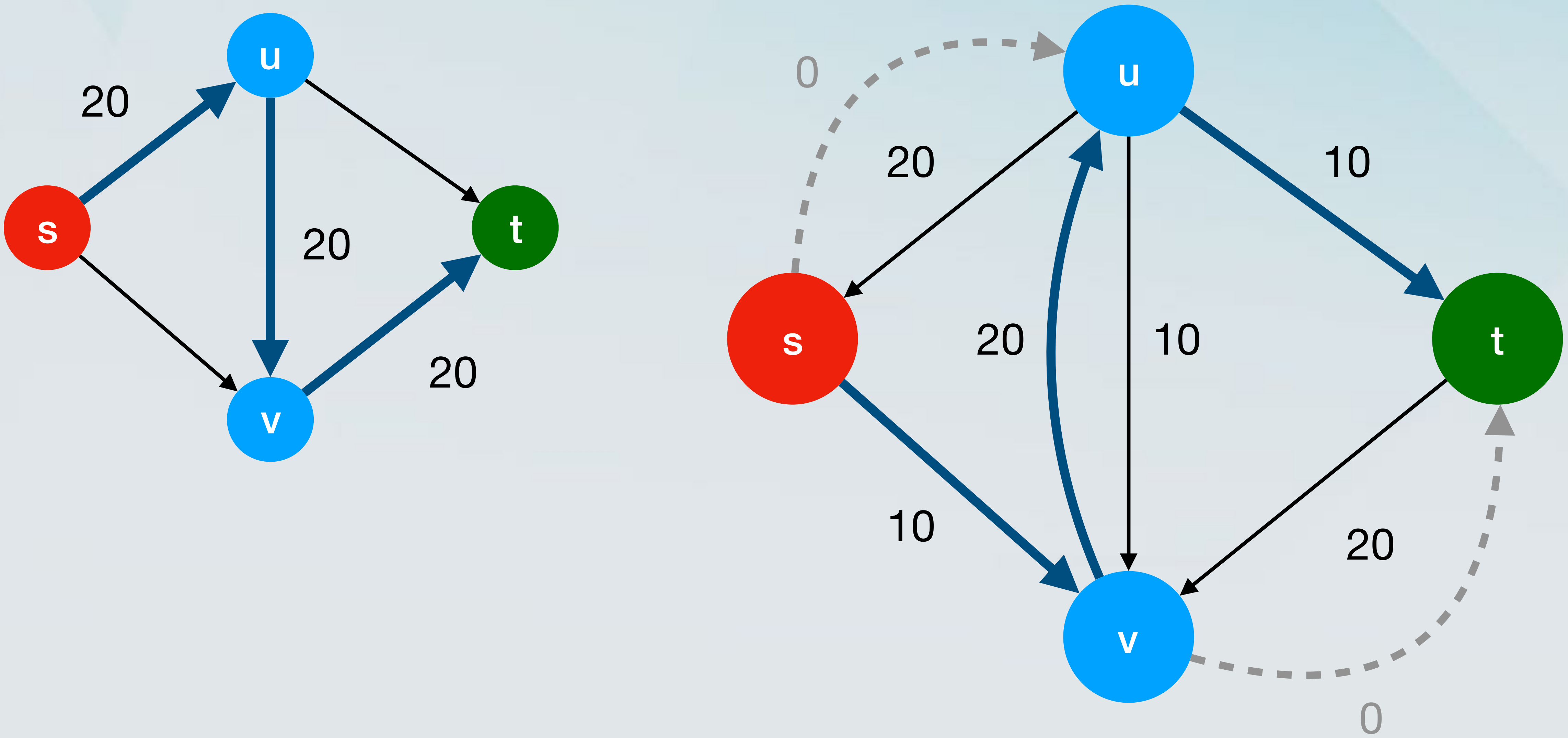
  - Therefore f' is larger than f by bottleneck(P, f), which is strictly positive.

Example



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  - Not necessarily!
  - The maximum flow has to be bounded!

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    - Is this true?

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Integer Capacities

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- So yes, it is true for integer capacities?
- Is it true for capacities which are real numbers?
    - No - tutorial on Friday.

Running Time

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    - This means that O(m+n) = O(m).

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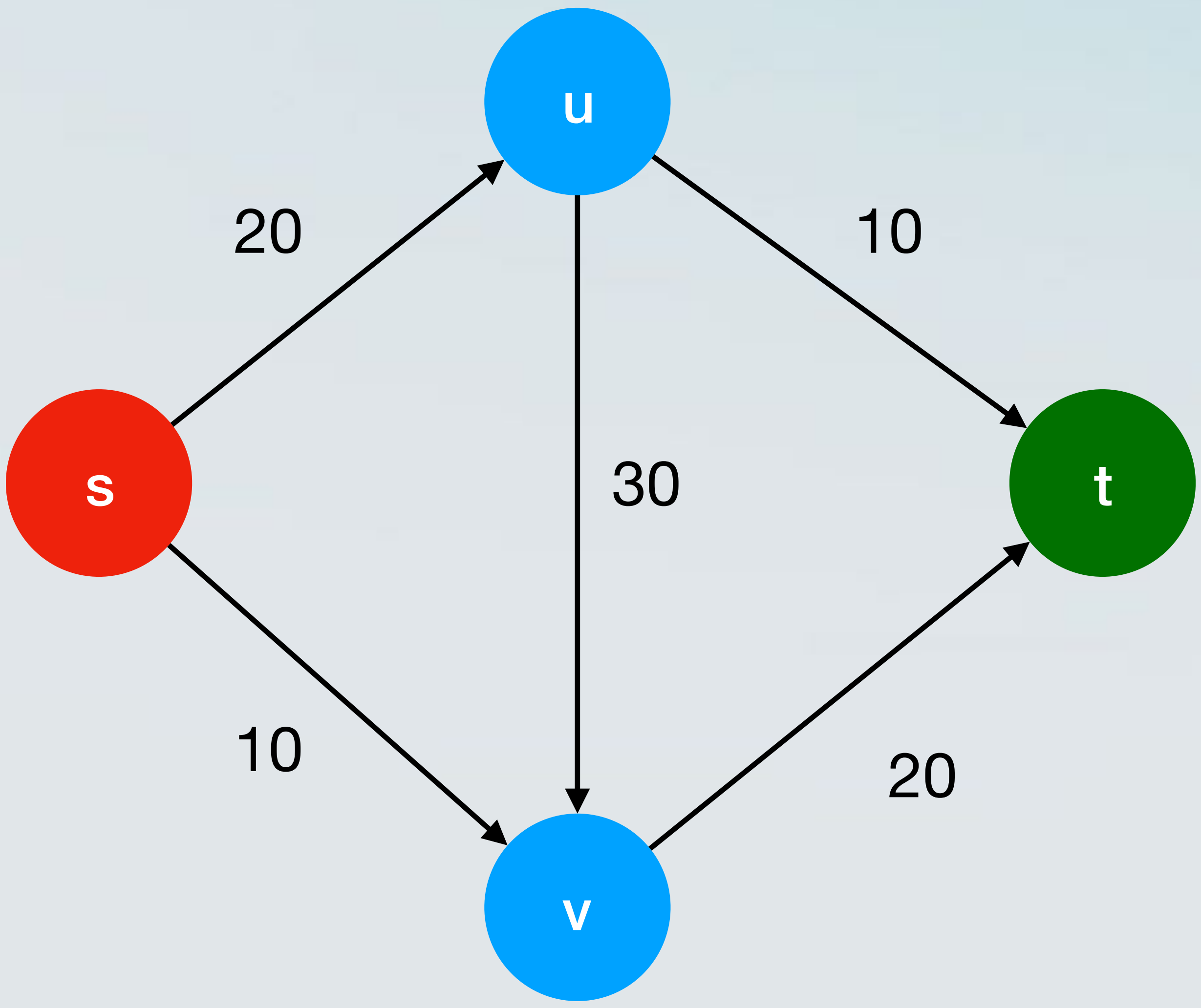
- Is this a polynomial time algorithm?

Minimum Cut

- A *cut* C is a partition of the nodes of G into two sets S and T, such that s is in S and t is in T.
- The capacity c(S,T) of a cut C is the sum of capacities of all edges "out of S"
    - these are edges (u, v) where u is in S and v is in T.

Example



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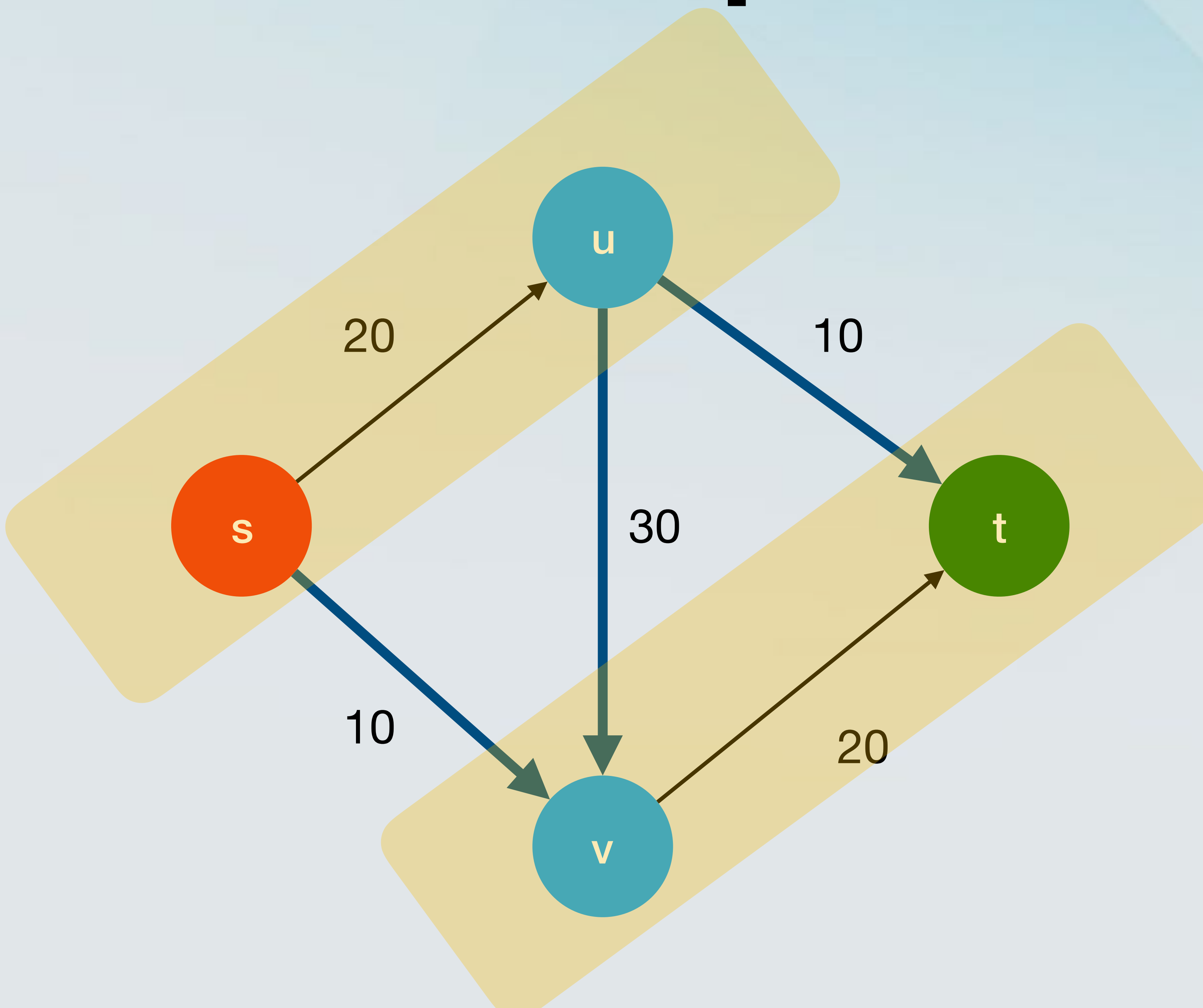

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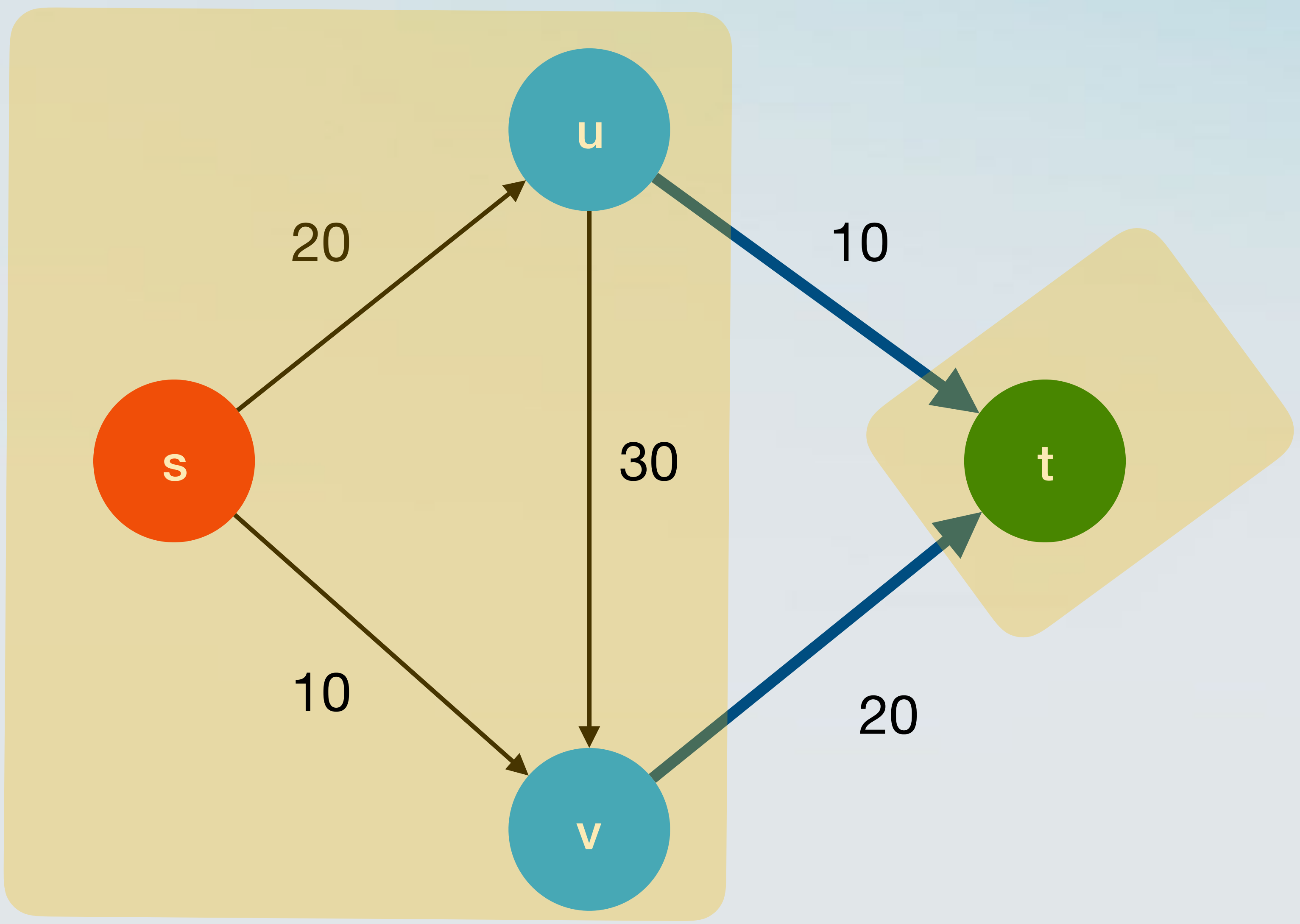

The Max-Flow Min-Cut Theorem

- Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

Optimality / Correctness

- Next Lecture!