Advanced Algorithmic Techniques (COMP523)

Network Flows 2

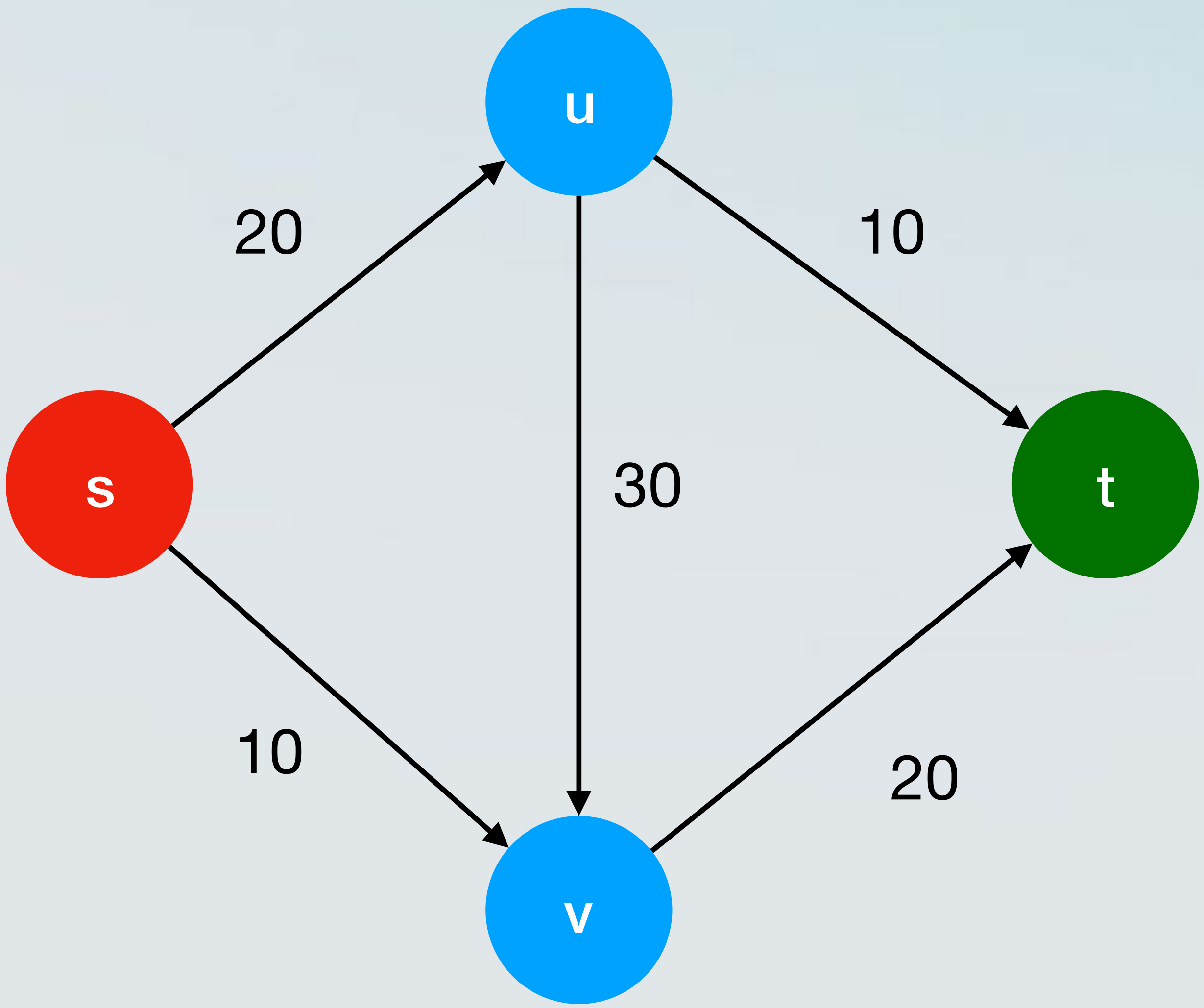
Recap and plan

- **Last lecture:**
  - Network Flows, Maximum Flow
  - Ford - Fulkerson
    - Feasibility, termination, running time
  - Max-Flow - Min-Cut
- **This lecture:**
  - Ford - Fulkerson
    - Optimality / Correctness
  - Better augmenting paths.
  - Maximum Bipartite Matching

Minimum Cut

- A *cut* C is a partition of the nodes of G into two sets S and T, such that s is in S and t is in T.
- The capacity c(S,T) of a cut C is the sum of capacities of all edges "out of S"
  - these are edges (u, v) where u is in S and v is in T.

Example



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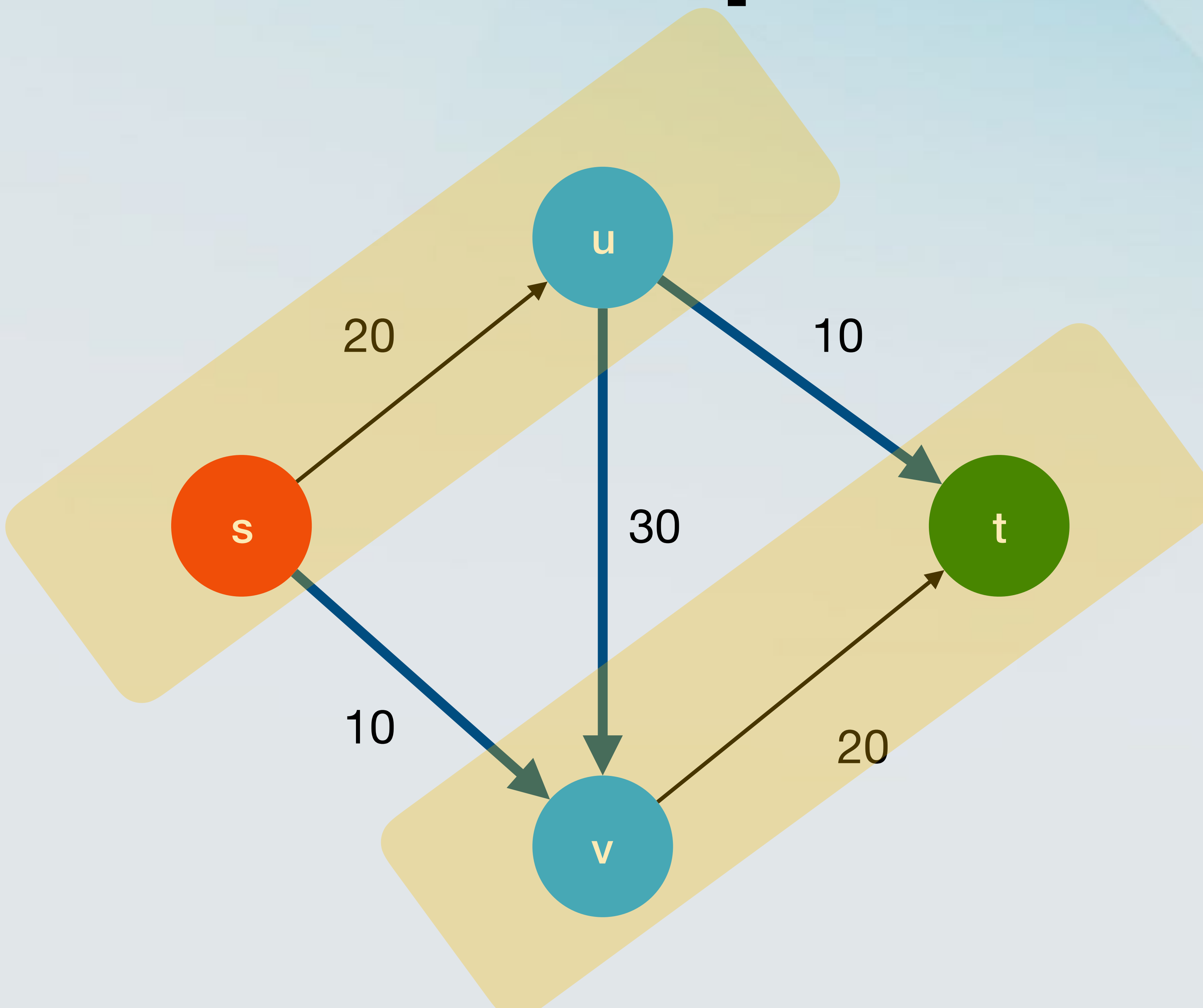

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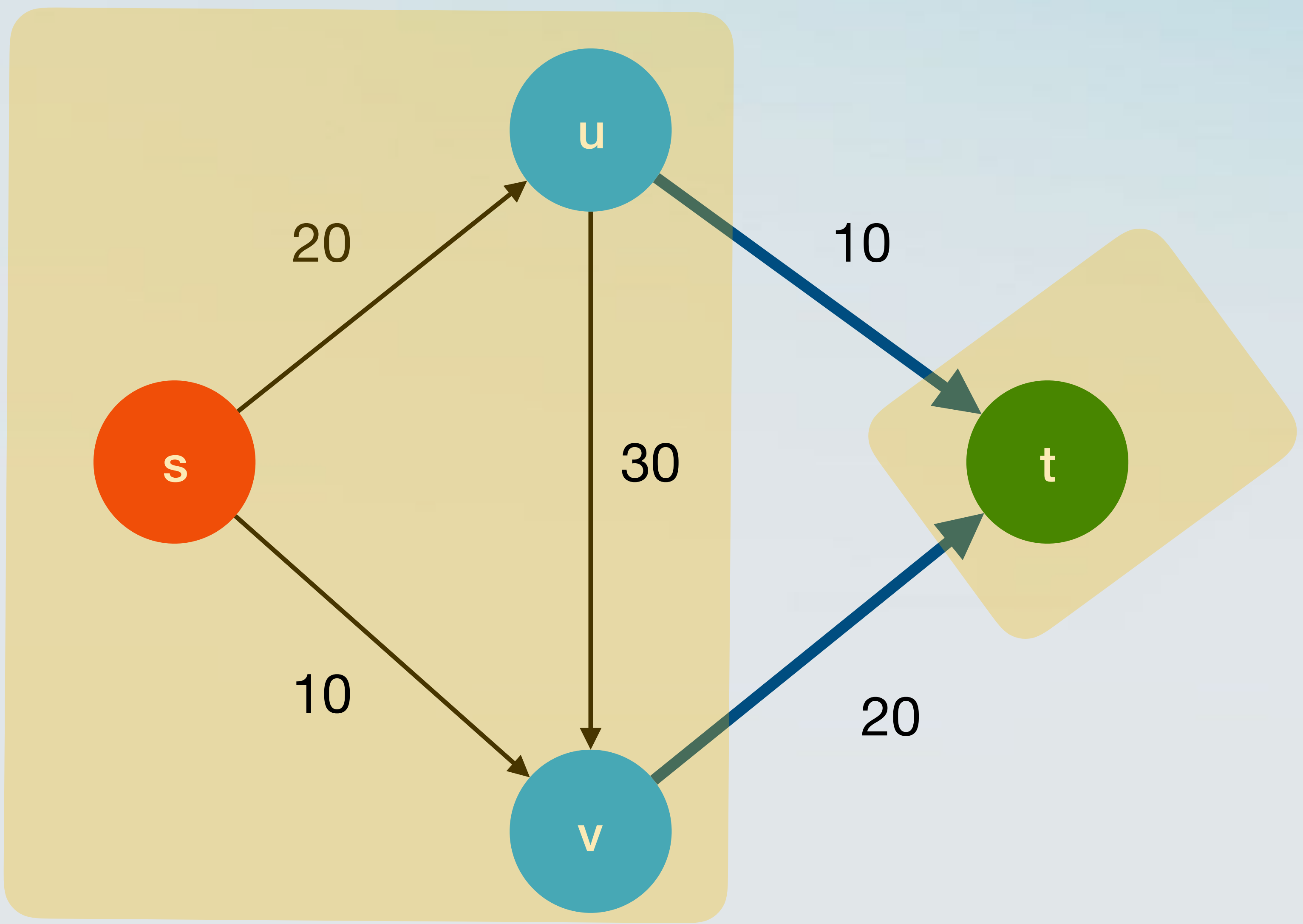
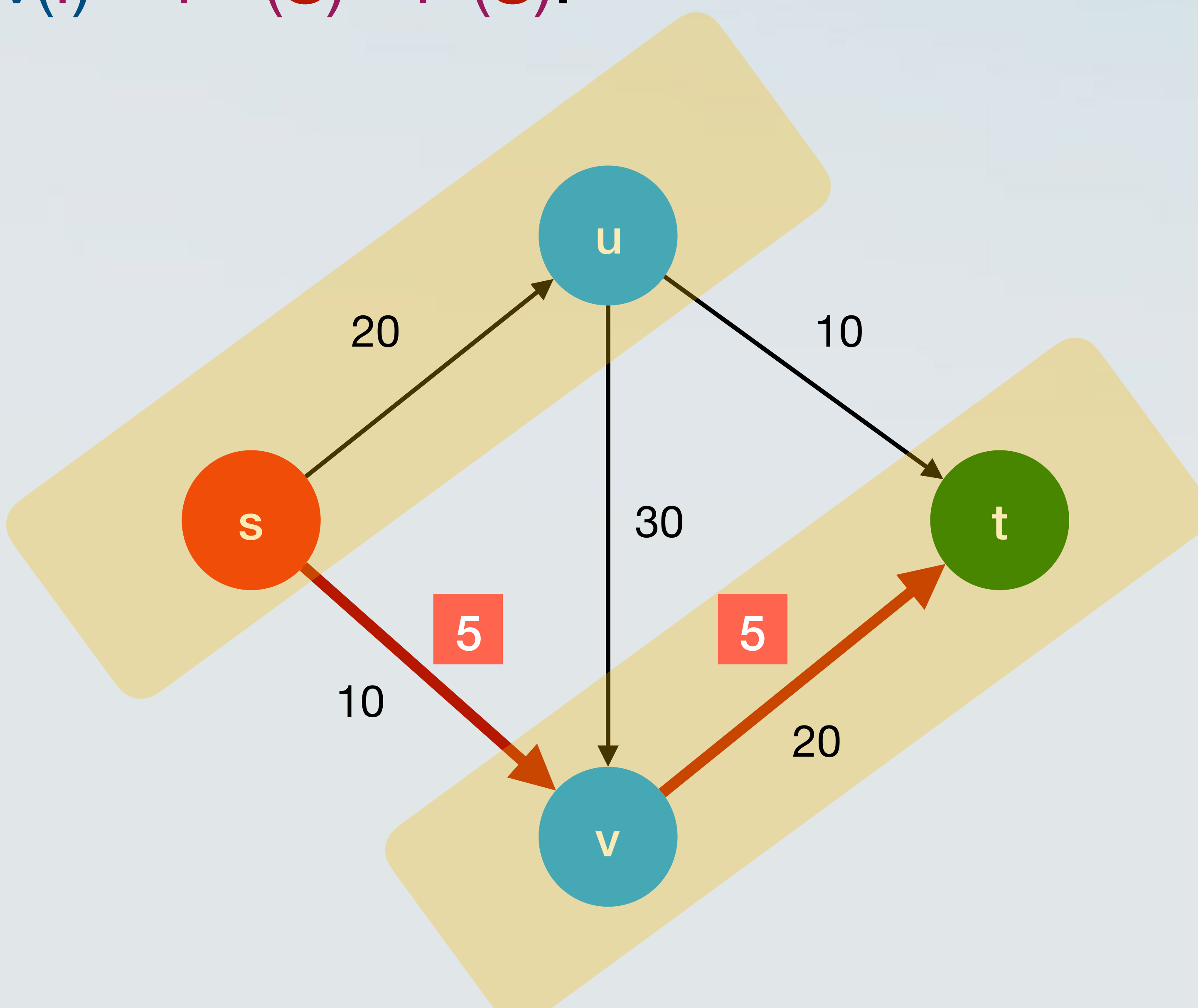


The Max-Flow Min-Cut Theorem

- Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

A series of facts

- Fact 1: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) = f^{out}(S) - f^{in}(S)$.



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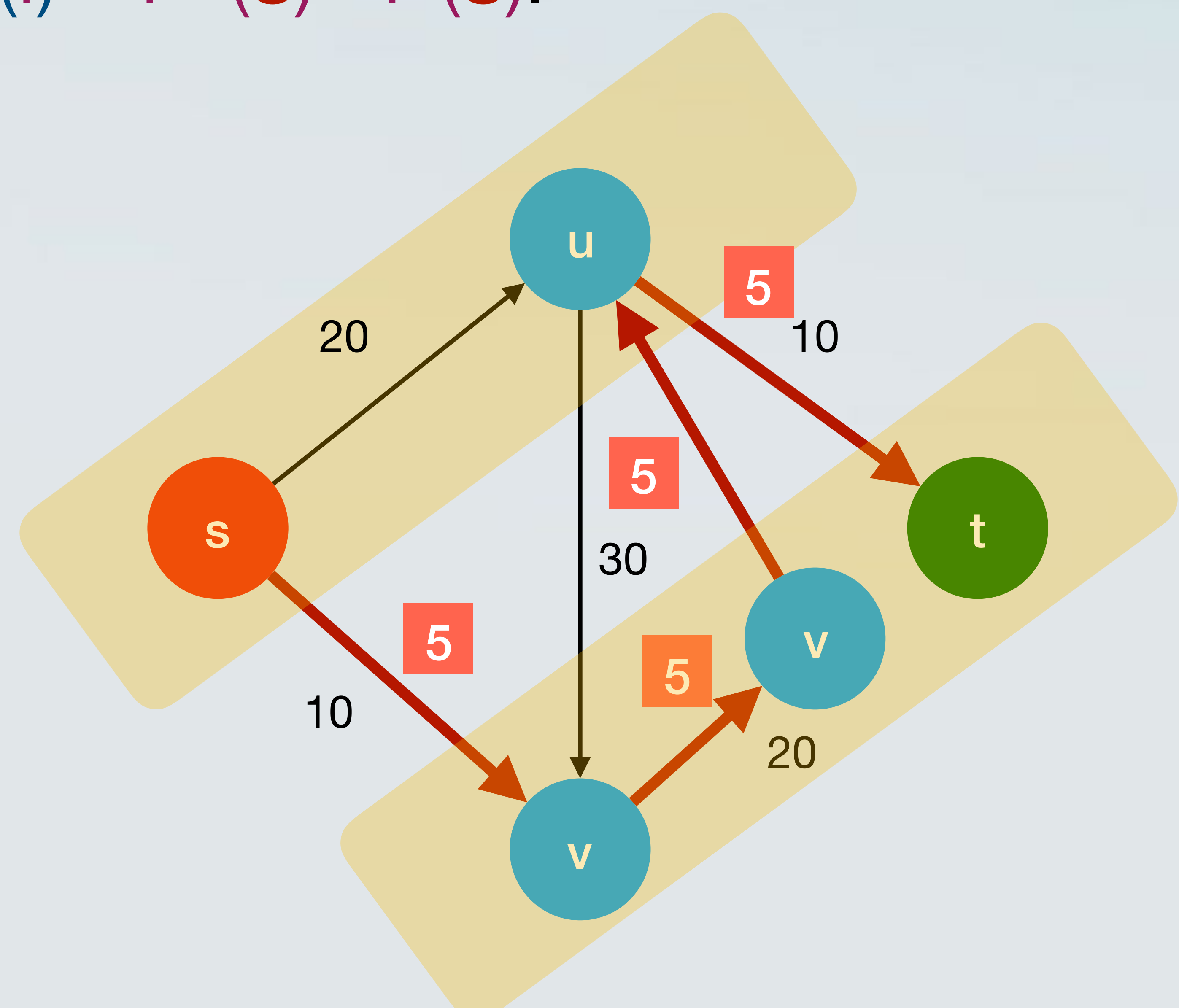

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  - By definition $f^{in}(s) = 0$.

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  - For every other node v, we have that $f^{out}(v) - f^{in}(v) = 0$ (why?)

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    - Therefore we have:

$$v(f) = \sum_{v \in S} \left(f^{\mathbf{out}}(v) - f^{\mathbf{in}}(v) \right)$$

Fact 1 - Rewriting the sums

- Fact 1: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) = f^{out}(S) - f^{in}(S)$.
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    - Otherwise the edge does not appear in the sum.

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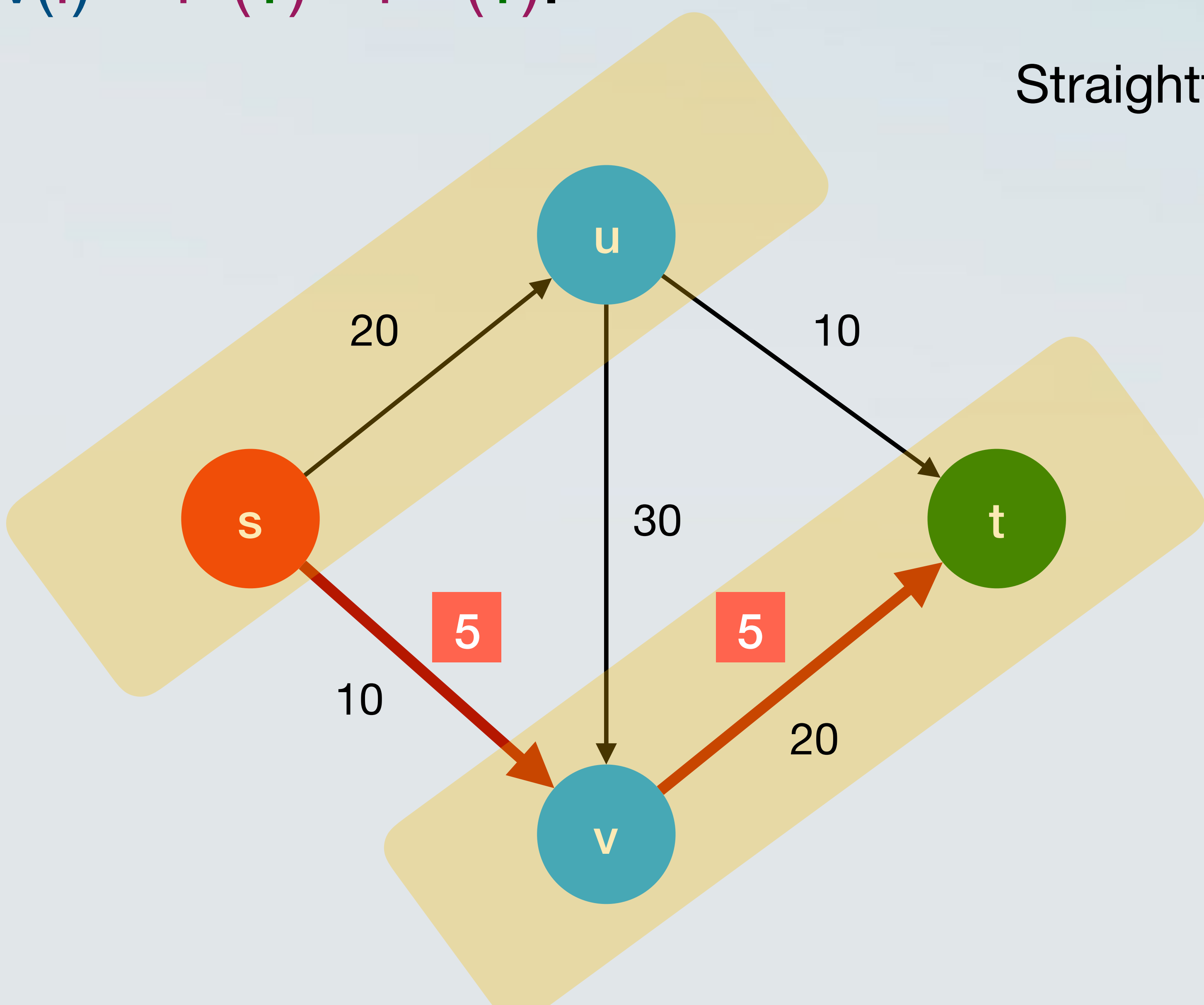
  - We can write

$$v(f) = \sum_{v \in S} \left(f^{\text{out}}(v) - f^{\text{in}}(v) \right) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = f^{\text{out}}(S) - f^{\text{in}}(S)$$

A series of facts

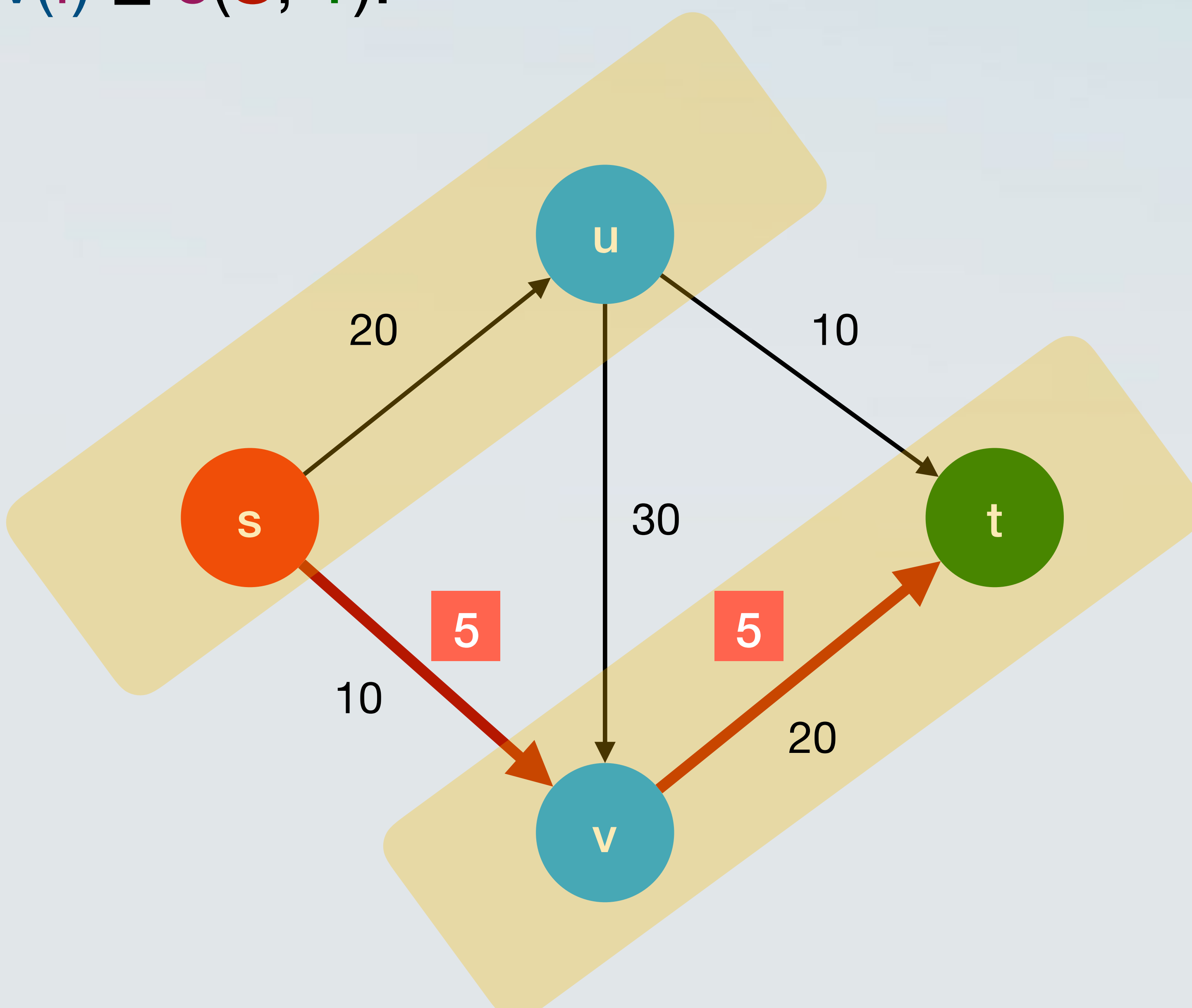
- Fact 2: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) = f^{in}(T) - f^{out}(T)$.

Straightforward by Fact 1.

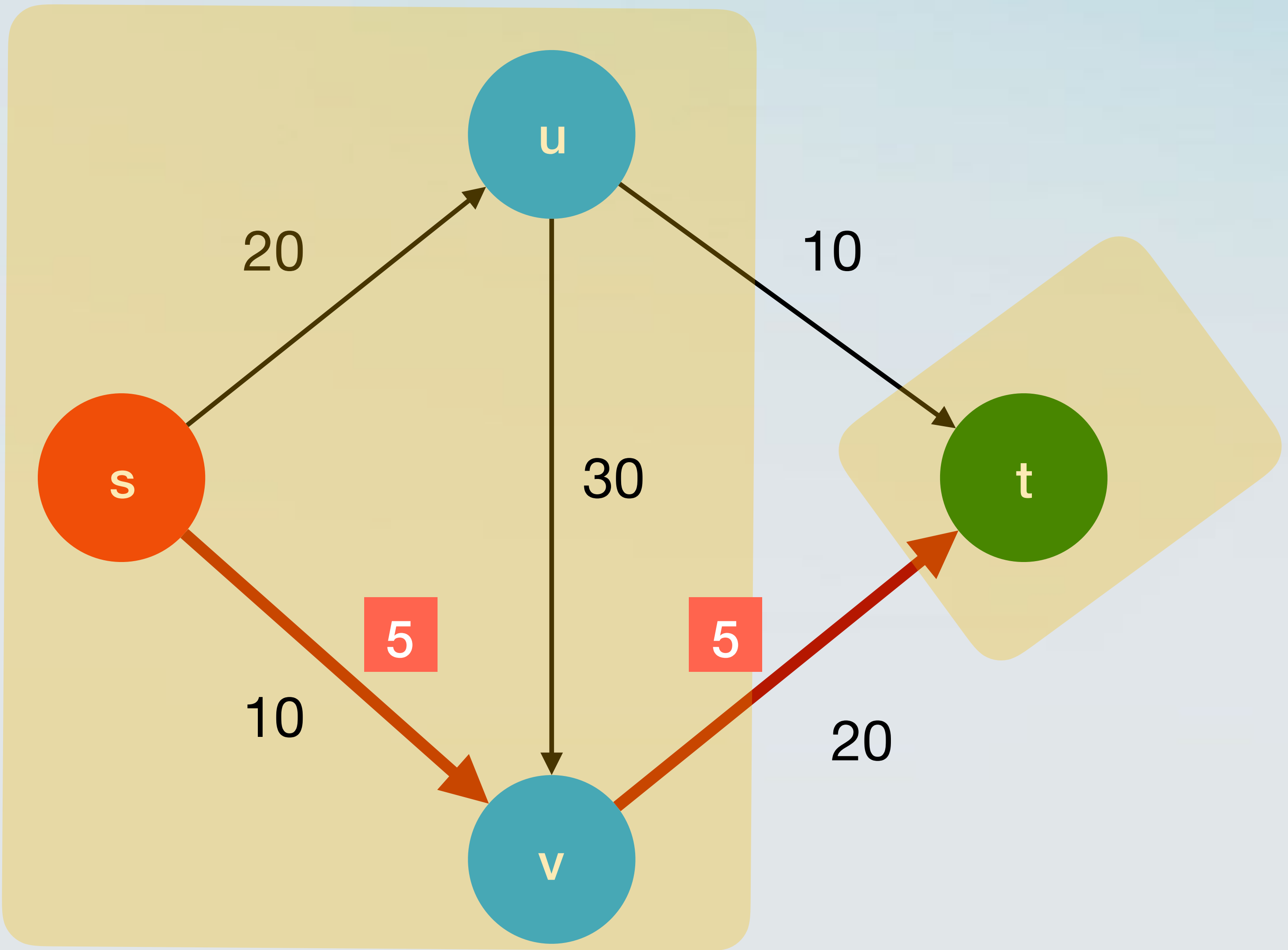


A series of facts

- Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then v(f) ≤ c(S, T).



Another (s-t) cut



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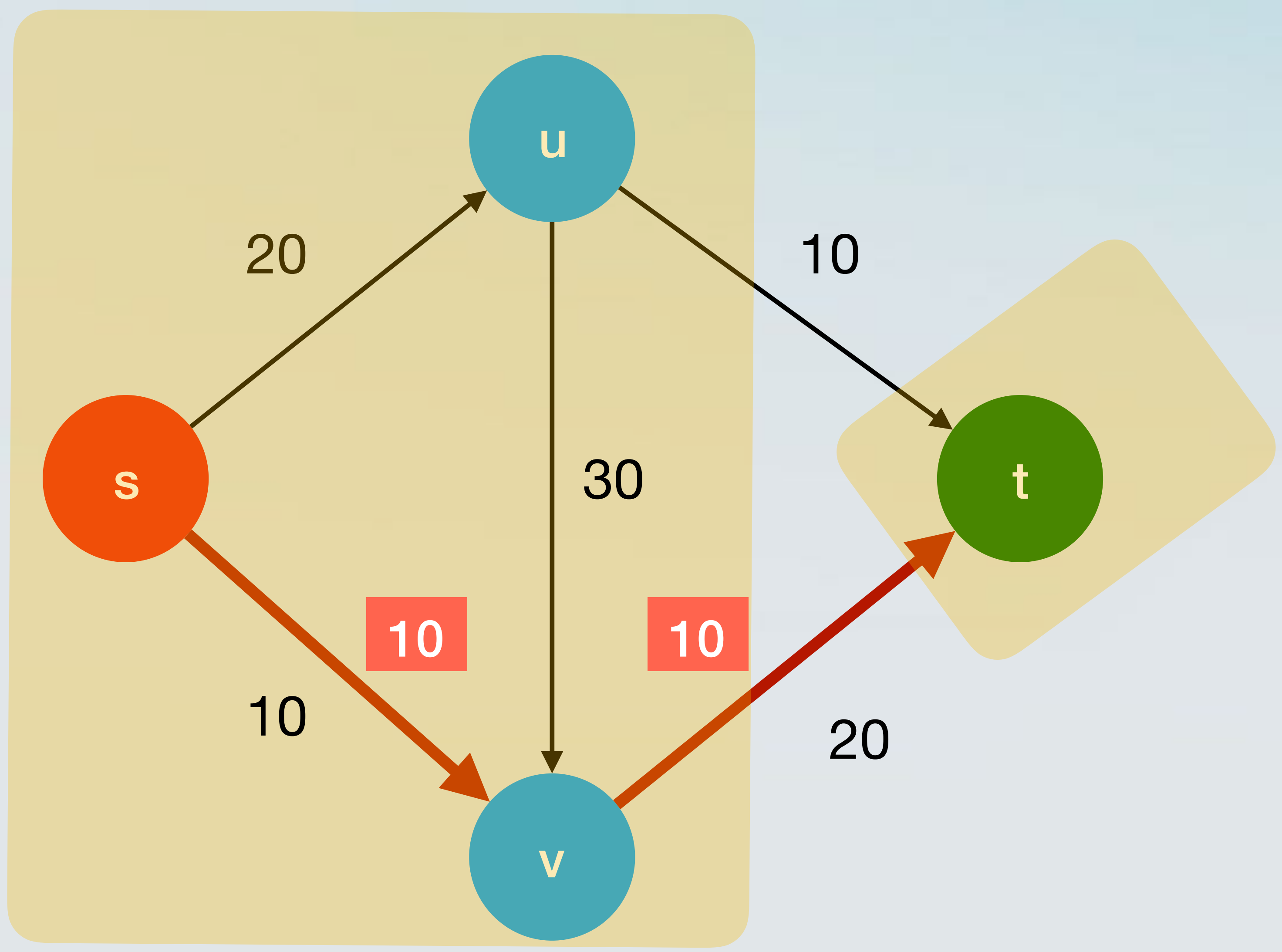

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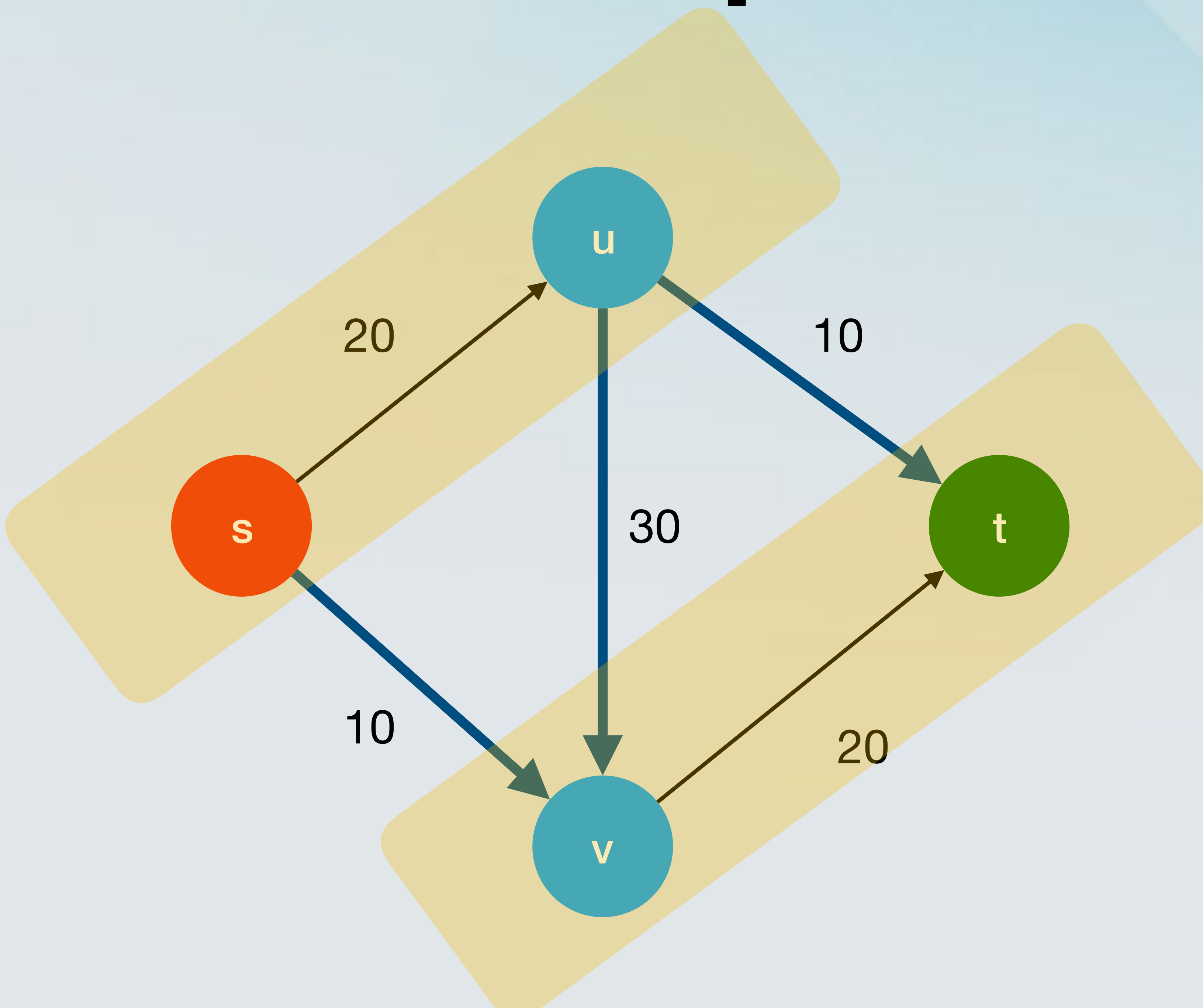
Comparing facts

- Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then $v(f) \leq c(S, T)$.

- Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

Example



What can we safely say about the maximum flow?

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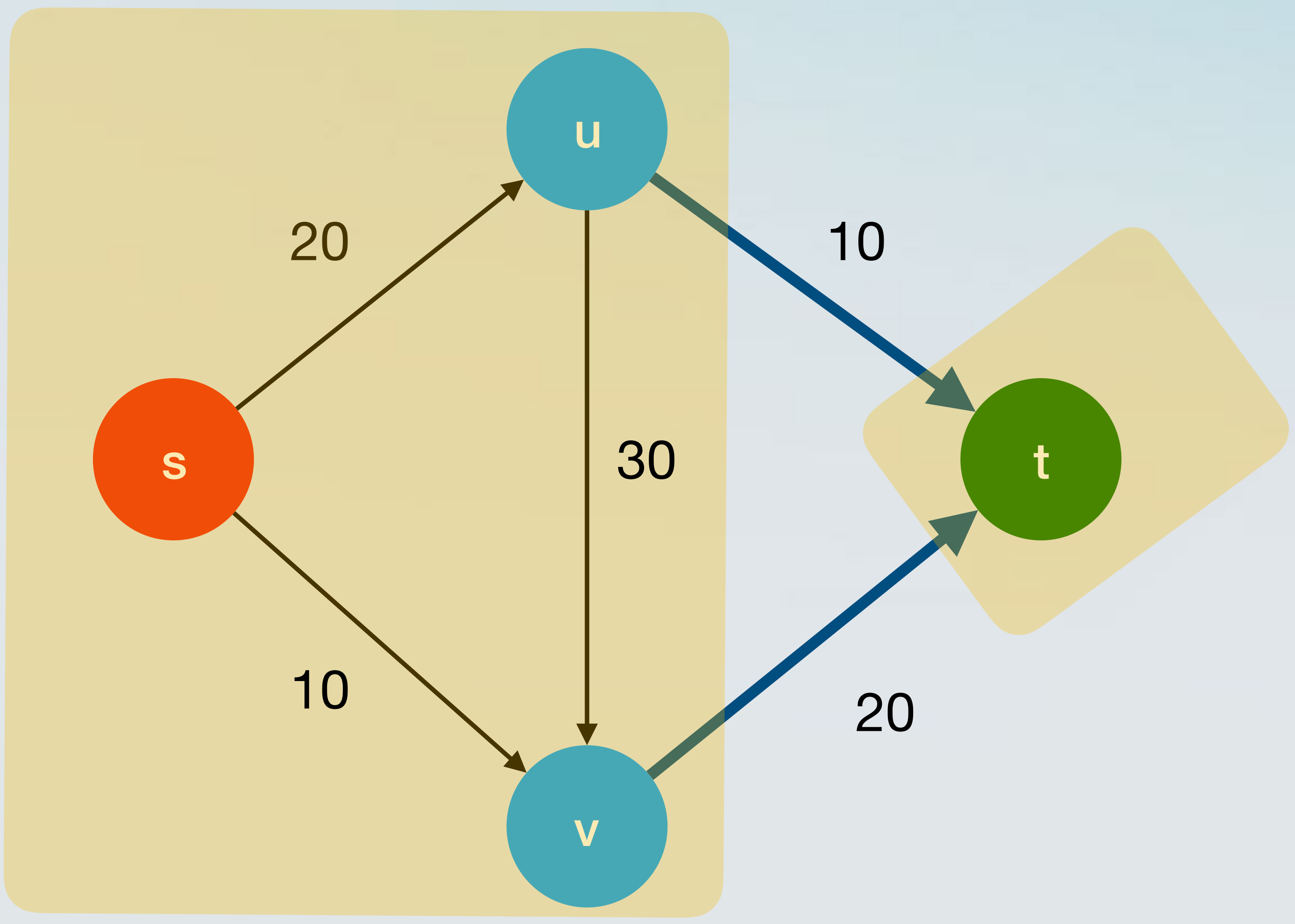

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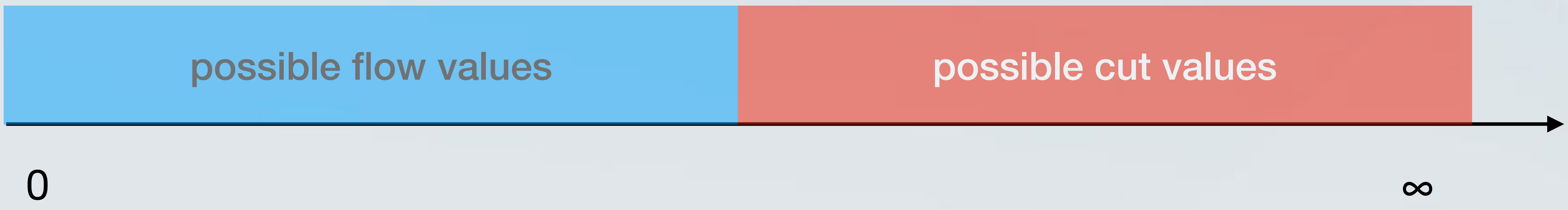
Proof idea



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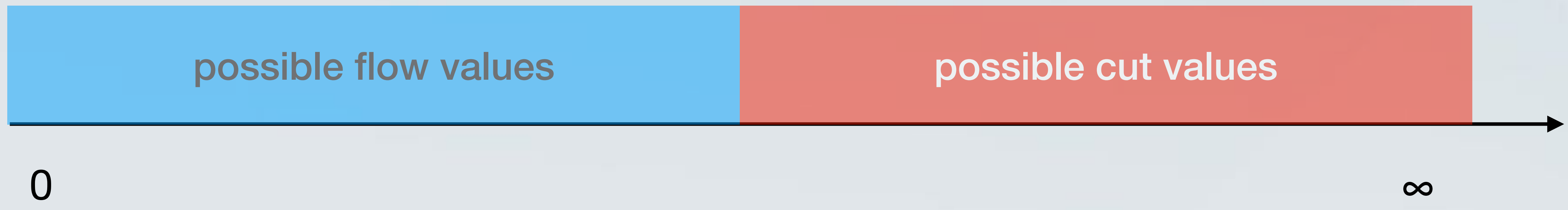

- How can we prove that a flow f* is maximum?

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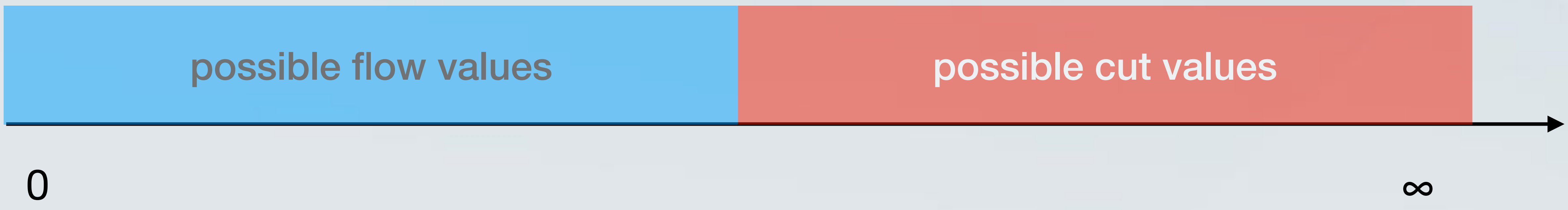

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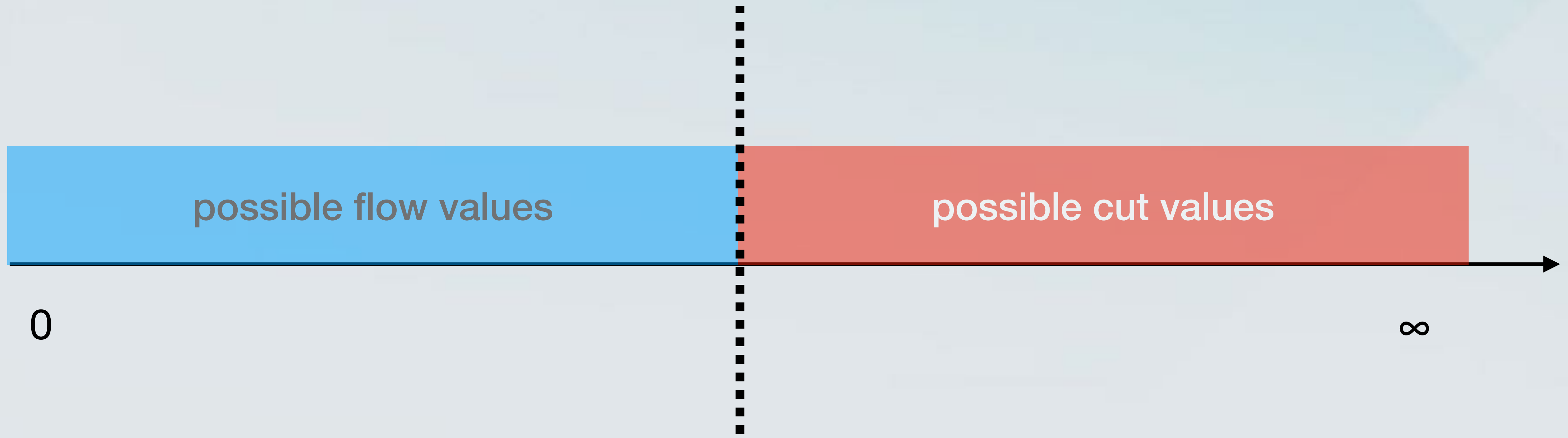

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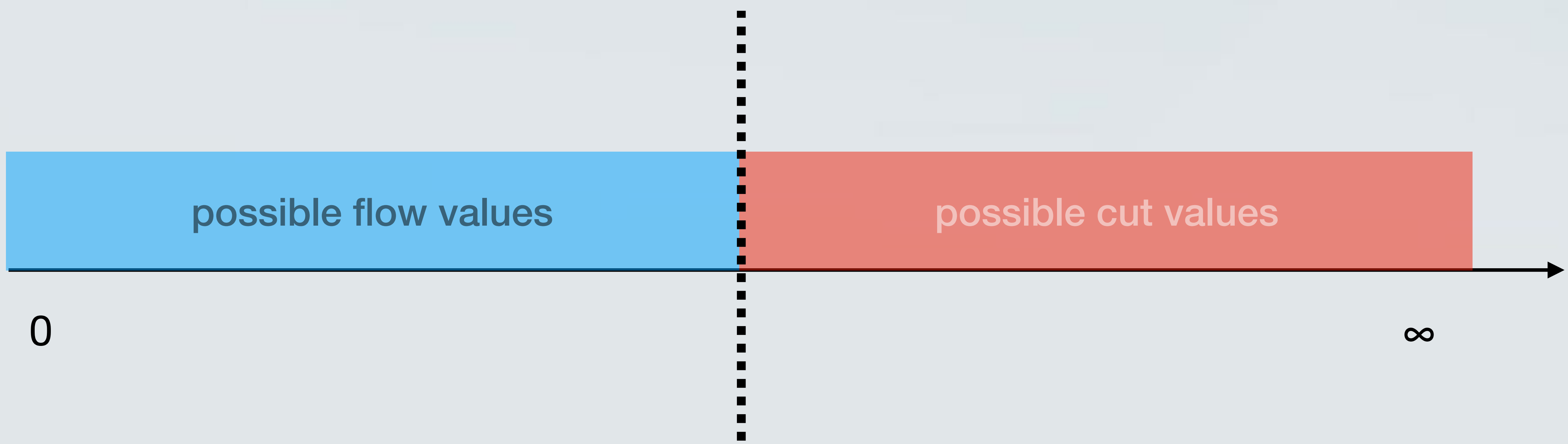

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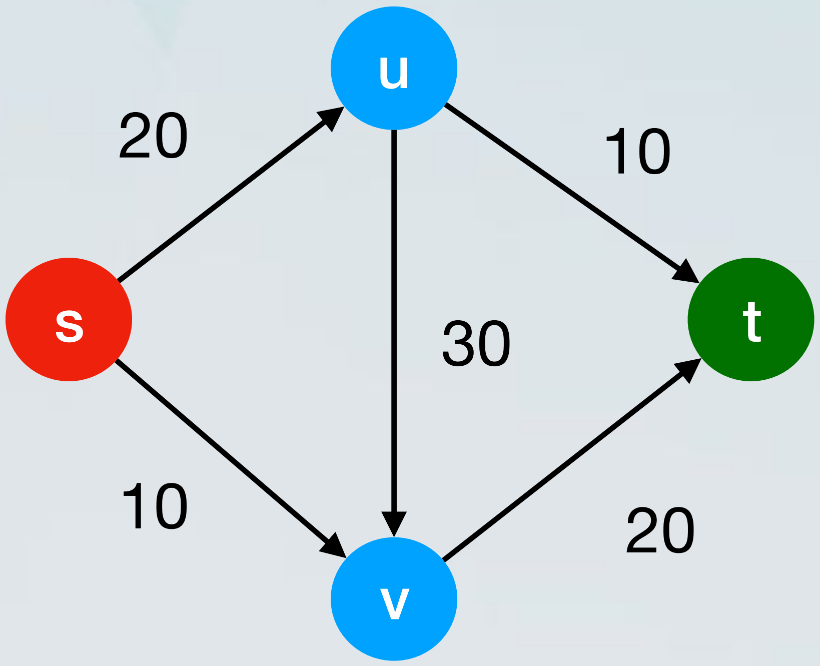

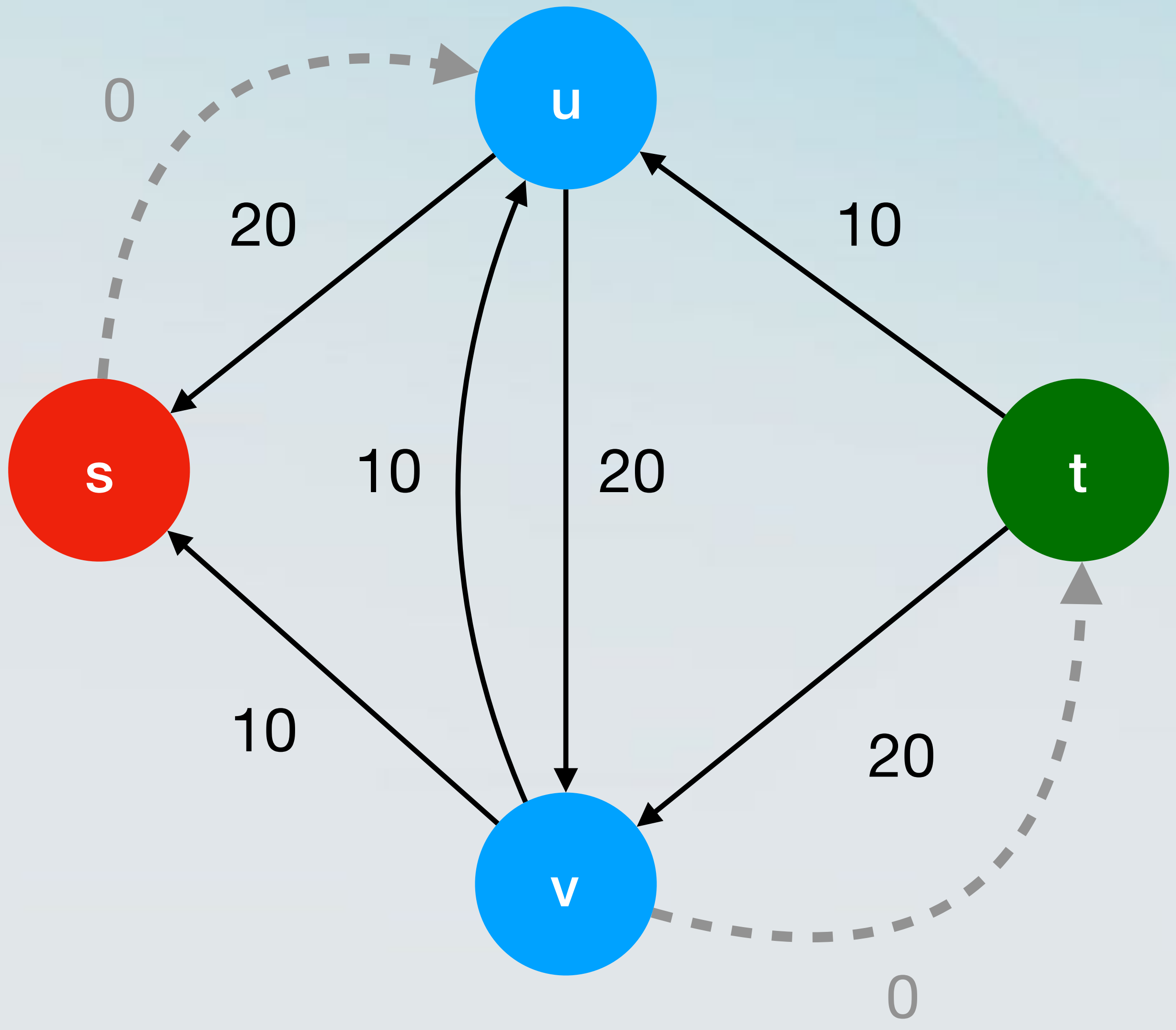

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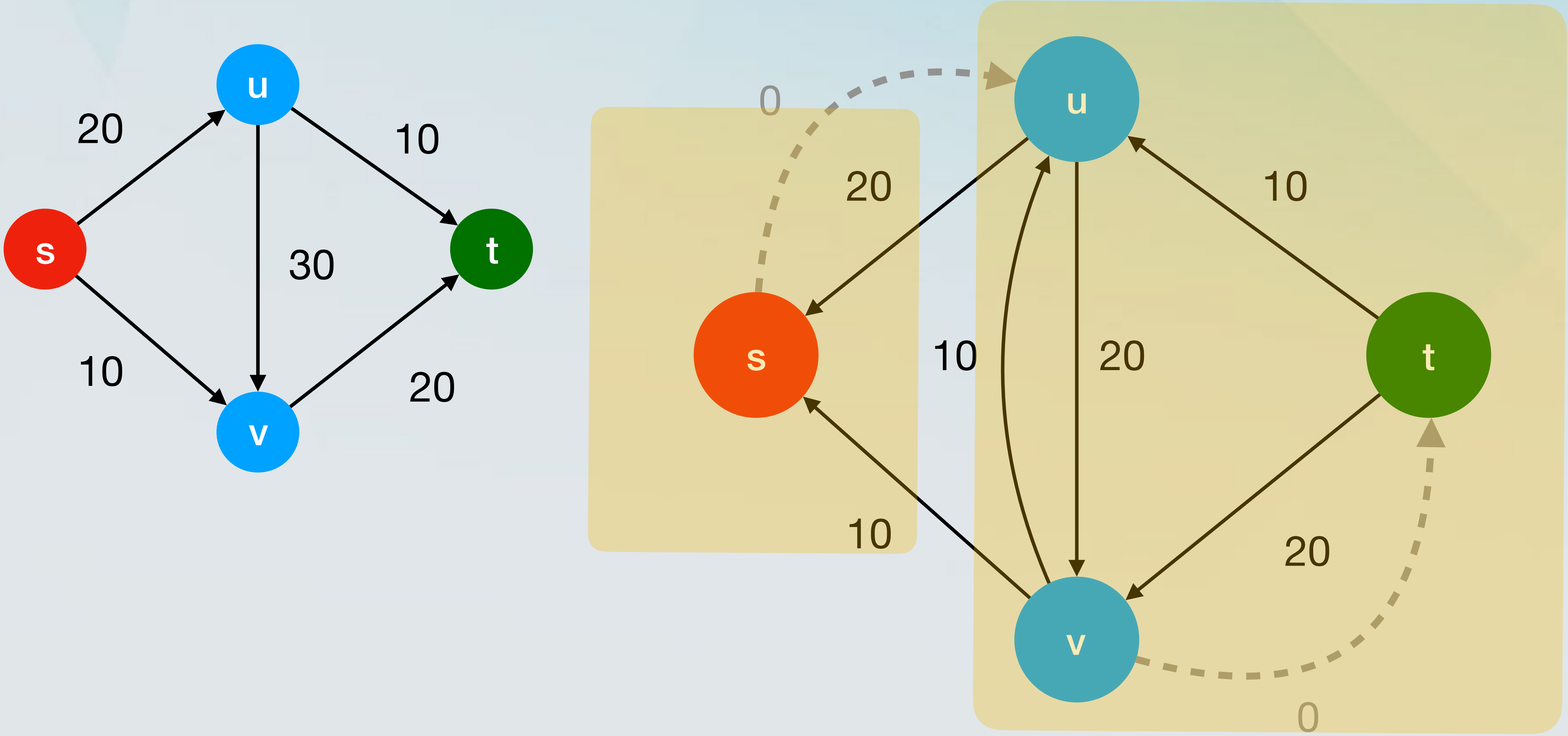

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  - s is in S*.
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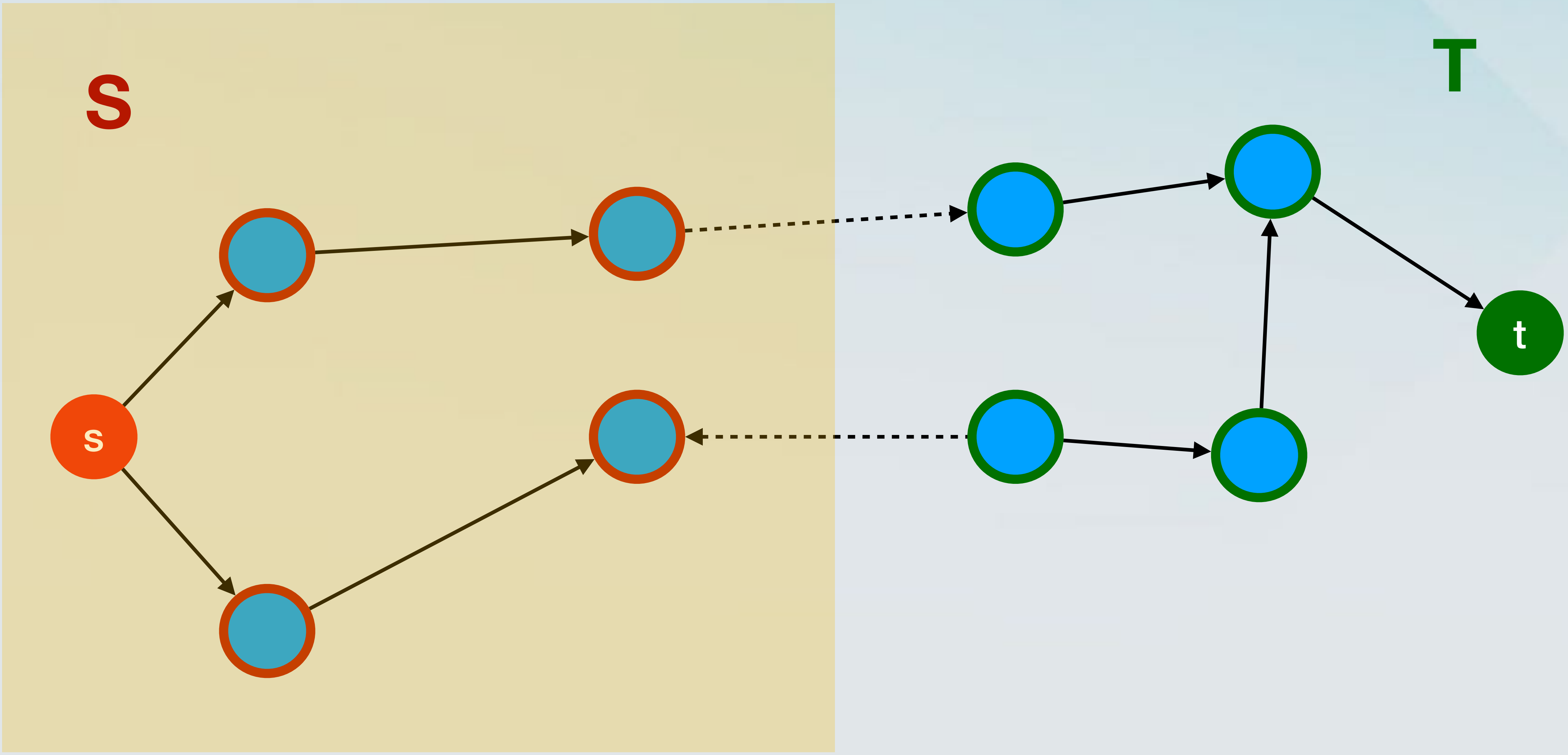

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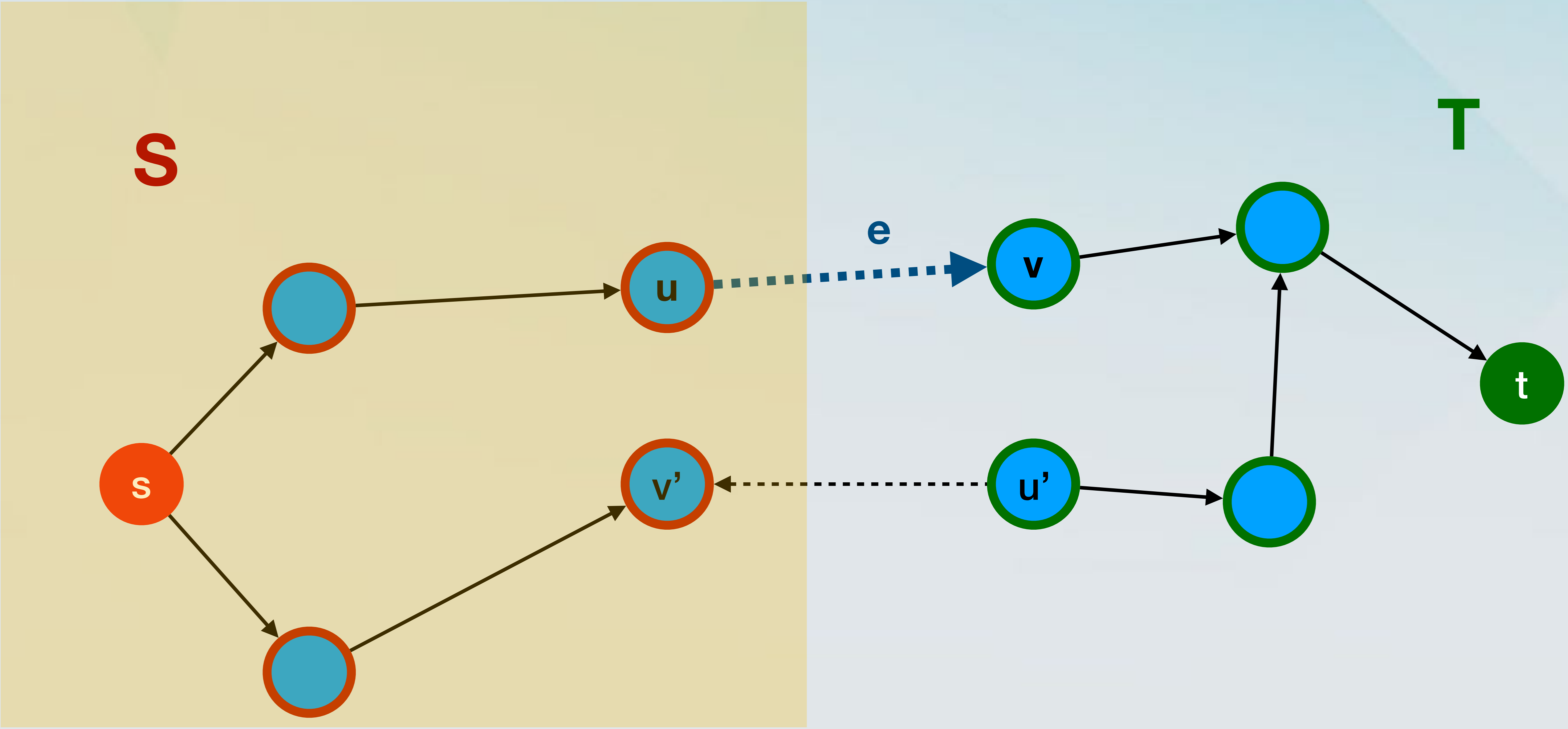

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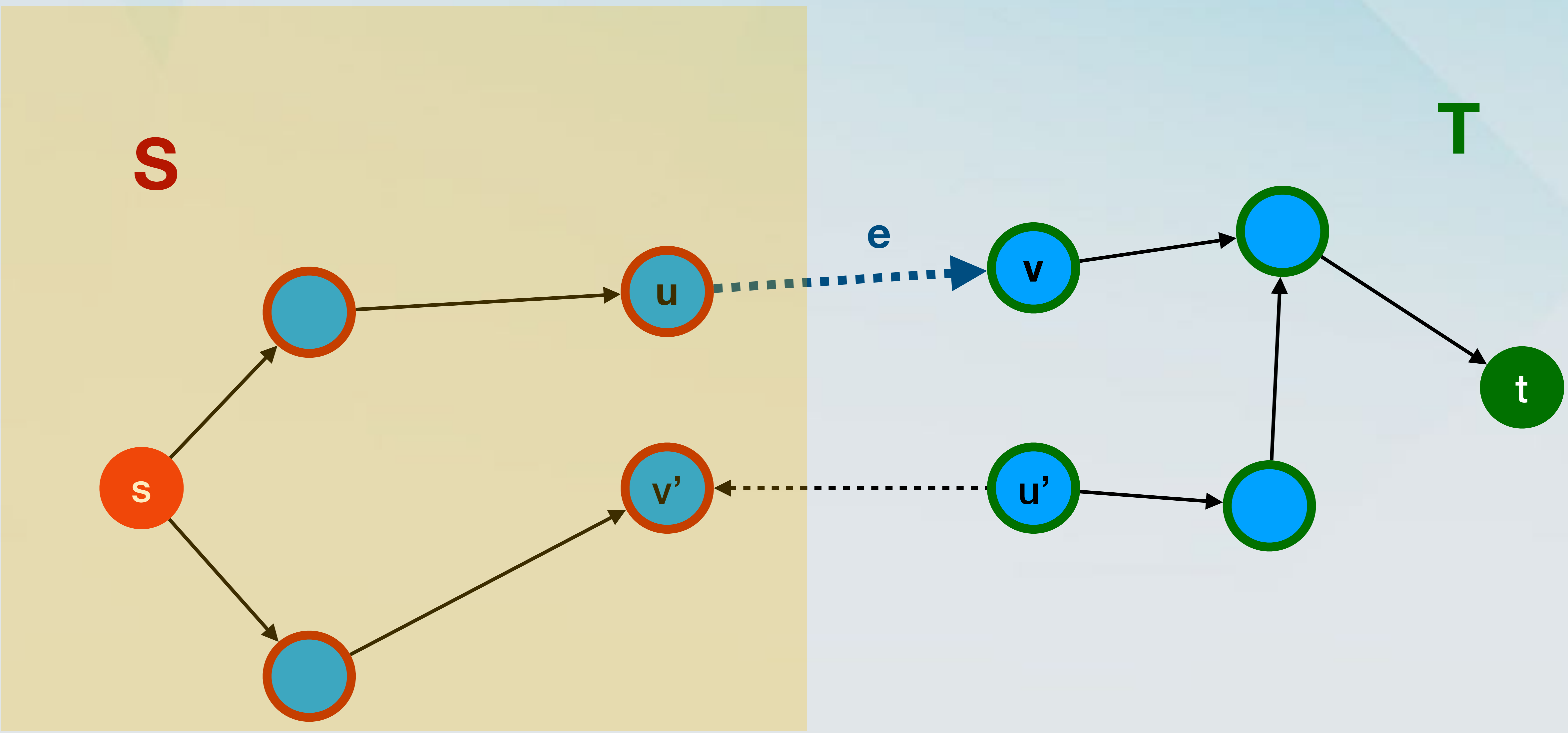

- Claim: **in G , f(e)** = c_e (i.e., e **in G** is *saturated* by the flow f).

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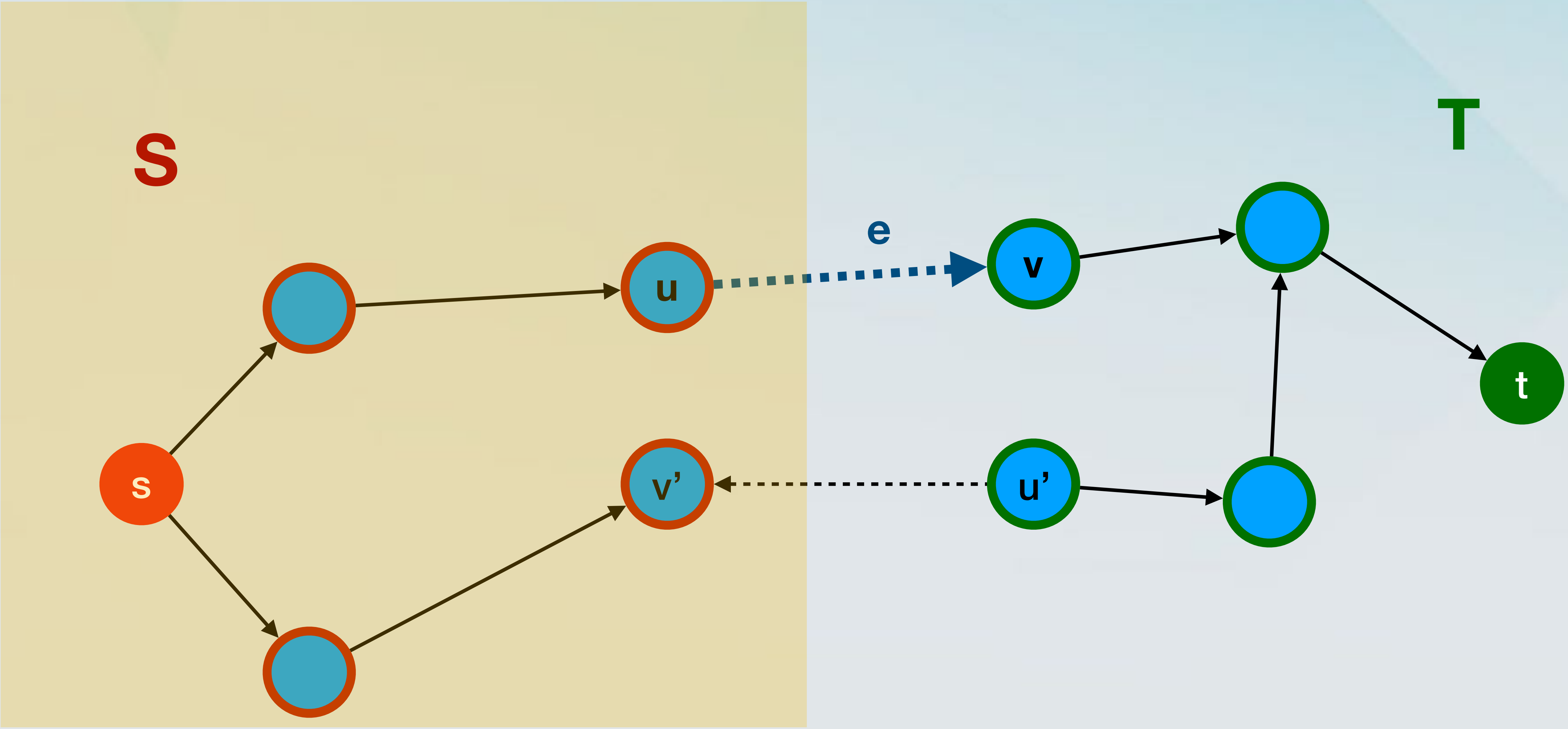

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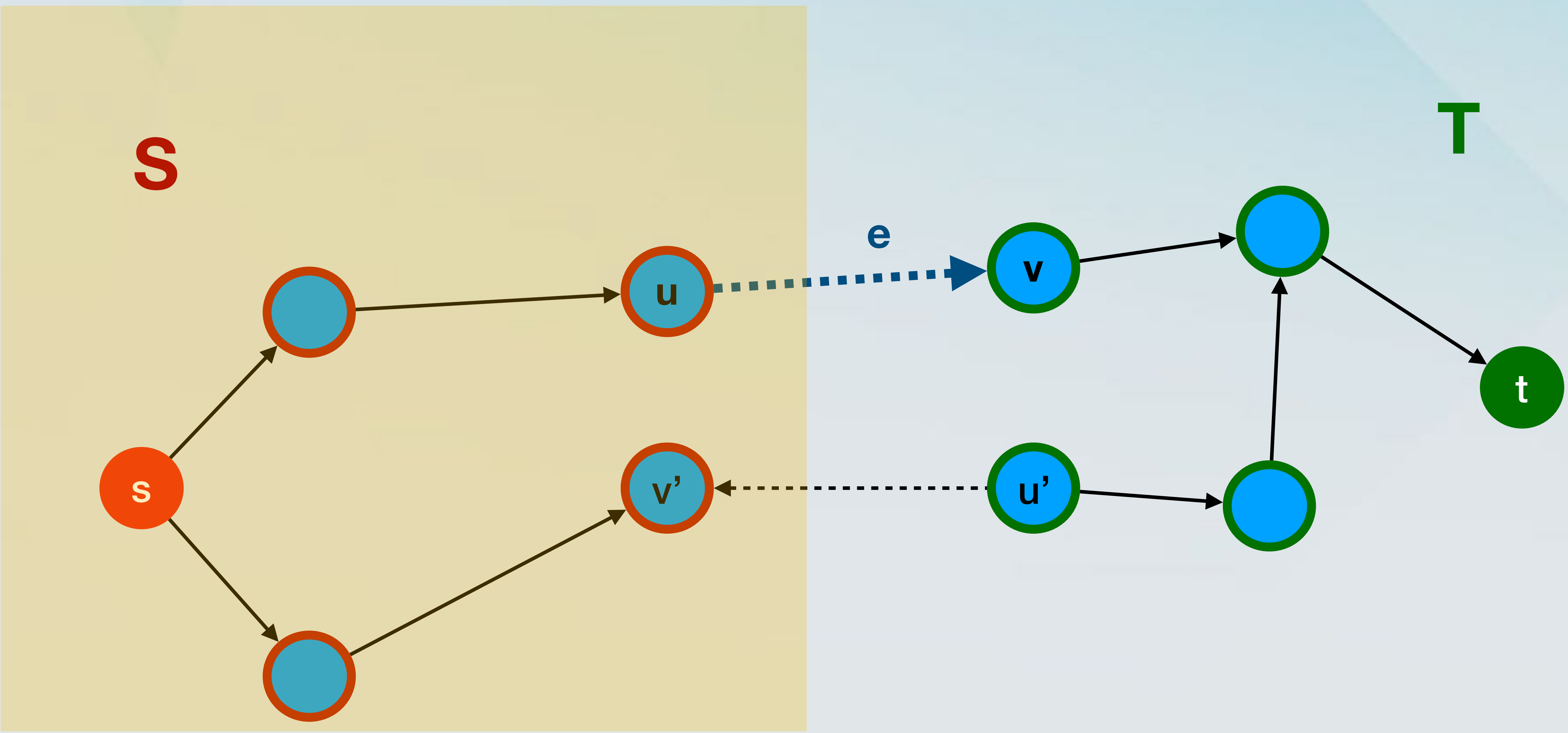

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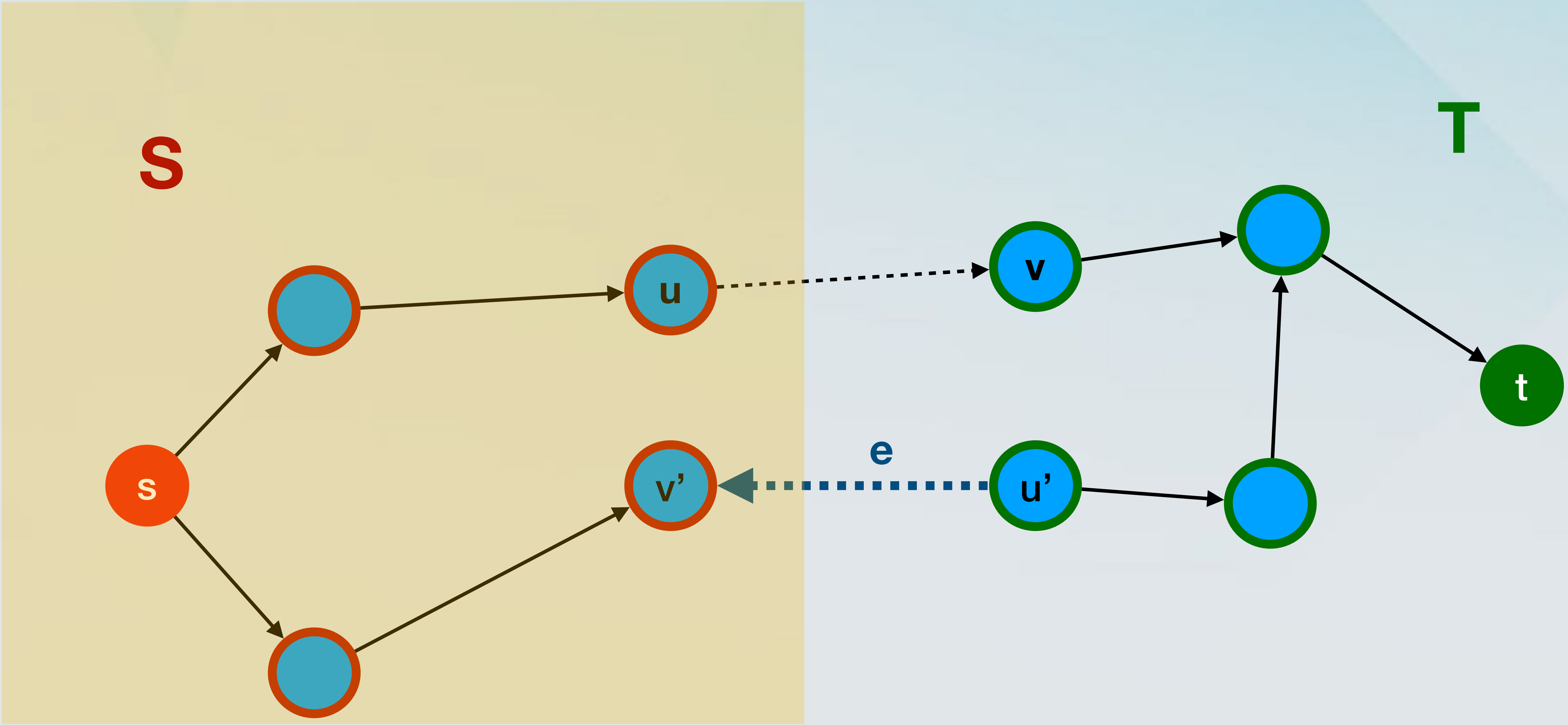

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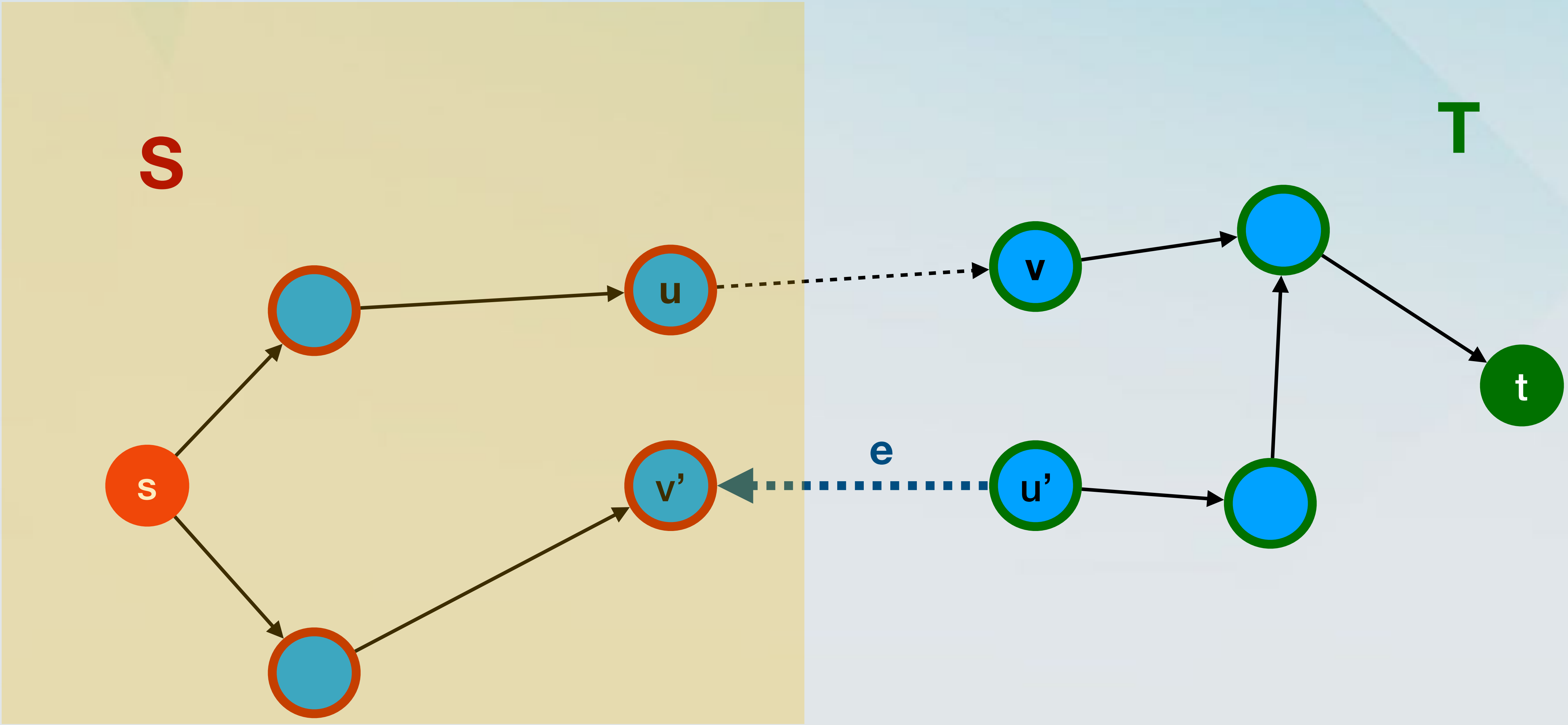

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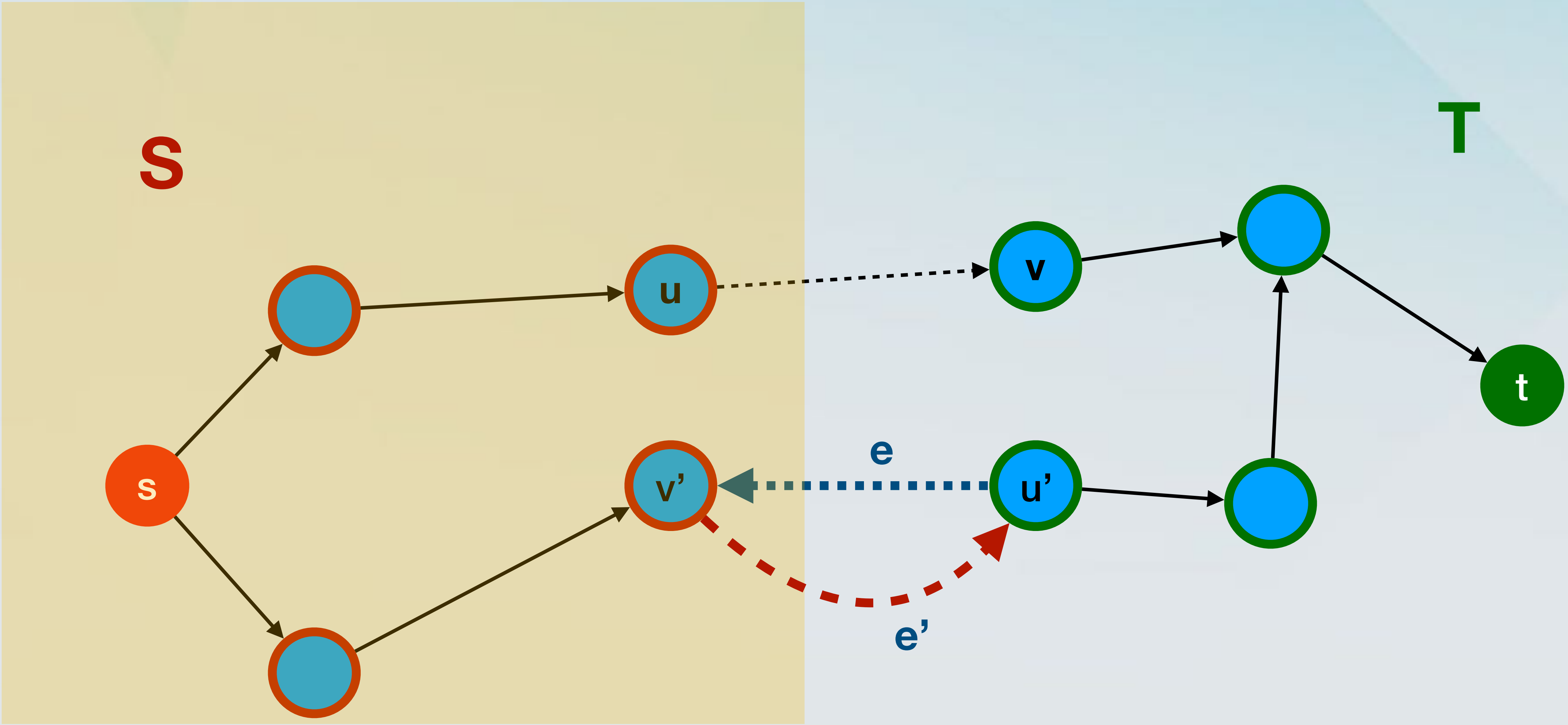

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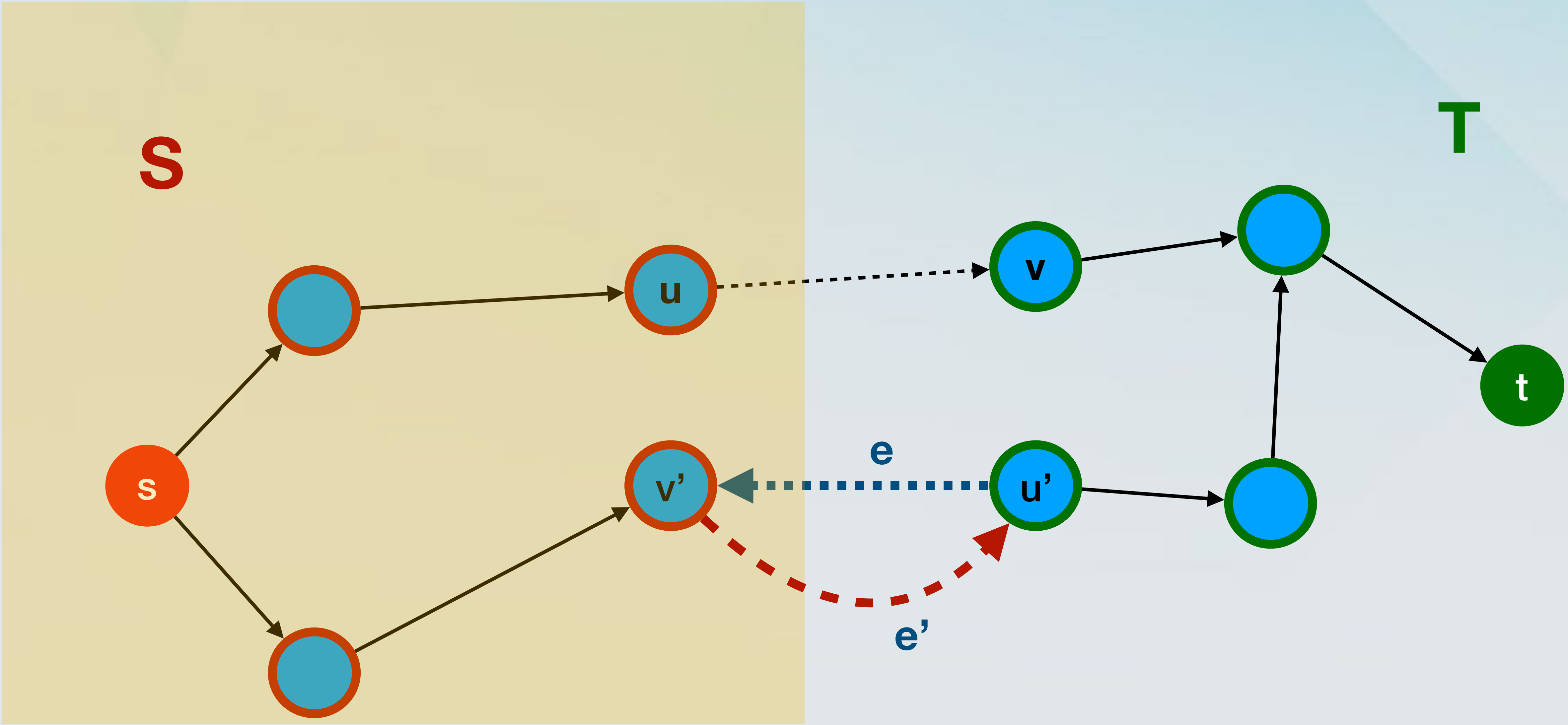

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- What do we get from this?
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Putting everything together

- Fact 4: Let f by any (s-t) flow in G such that the residual graph G_f has no *augmenting paths*. Then there is an (s-t) cut C(S*, T*) in G such that c(S*, T*) = v(f).

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- Ford-Fulkerson stops when there are no augmenting paths in the residual network.

- The value of the flow is equal to the capacity of *some* cut.

- This means that the value of the flow is maximum.

Related question

Related question

- How do we find *the value of the minimum cut* in a flow network?

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- How do we find *a minimum cut* in a flow network?
  - Run Ford-Fulkerson and look at the final residual graph.
  - Put the nodes reachable from s to S and the remaining nodes to T.

The Max-Flow Min-Cut Theorem

- Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.
  - The proof of the theorem follows from the proof of optimality for Ford-Fulkerson!

Ford-Fulkerson analysis

- **Feasibility**
  - Does the algorithm produce a flow if it terminates?
- **Termination**
  - Does the algorithm always terminate?
- **Running Time**
  - What is the running time of the algorithm?
- **Optimality / Correctness**
  - Does the algorithm produce a maximum flow?

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  - This follows from the properties of the Ford-Fulkerson algorithm.
    - It produces a maximum flow.
    - The capacities and flows are integers in every step of the execution.

Back to the running time

- The running time of *FF* is **O(***m*F**)**, where F is the value of the maximum flow.

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    - Should we be happy about this?

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- The running time of *FF* is **O(*m*F)**, where F is the value of the maximum flow.
- Is this a polynomial time algorithm?
  - It runs in *pseudo-polynomial* time.
  - Should we be happy about this?
  - Is this problem NP-hard?

The Ford-Fulkerson Algorithm

Max-Flow

Initially set f(e) = 0 for all e in E.

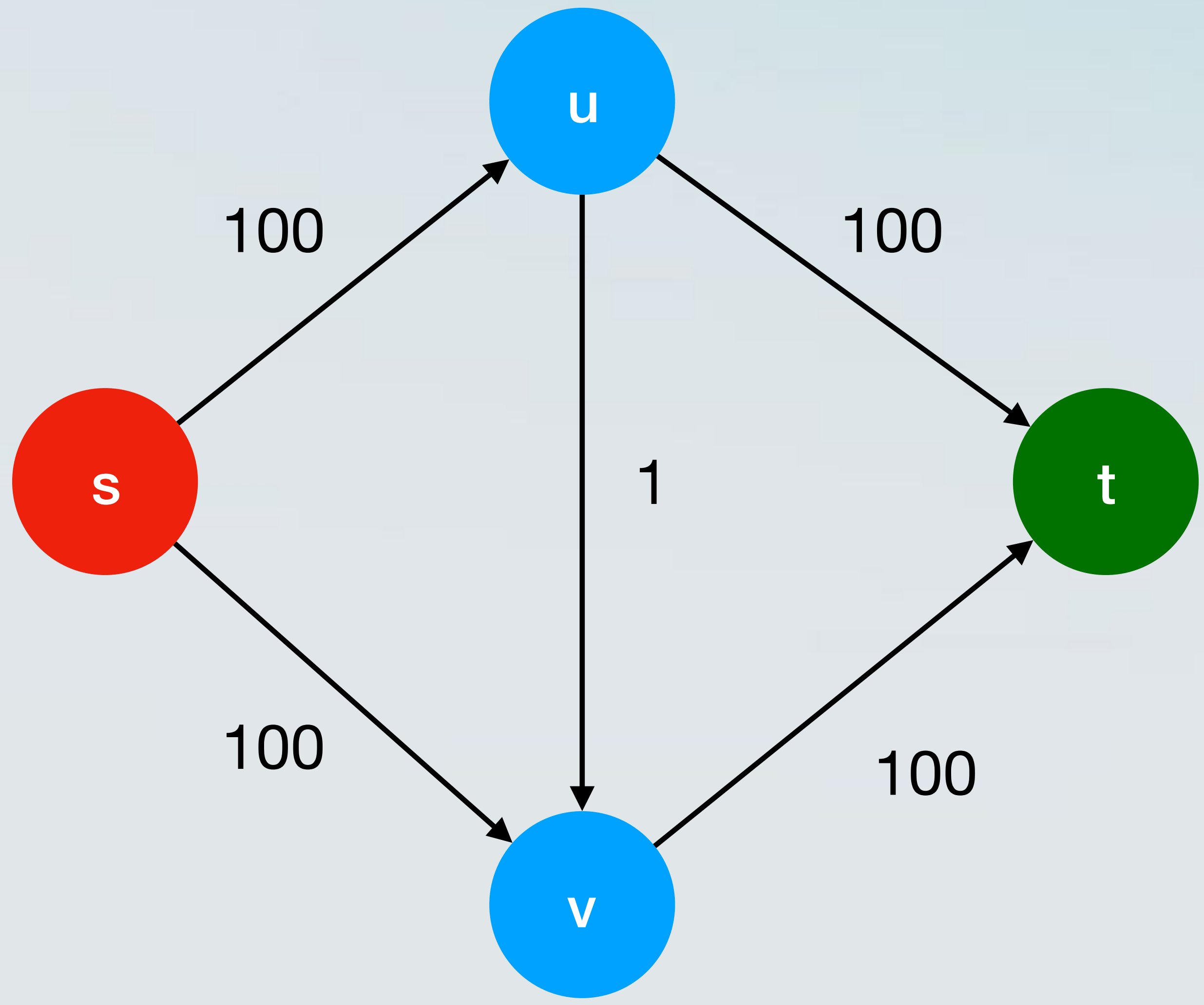
While there exists an s-t path in the residual graph Gf

Choose such a path P
f' = **augment**(f, P)
Update f to be f'
Update the residual graph to be Gf'

Endwhile

Return (f)

Example



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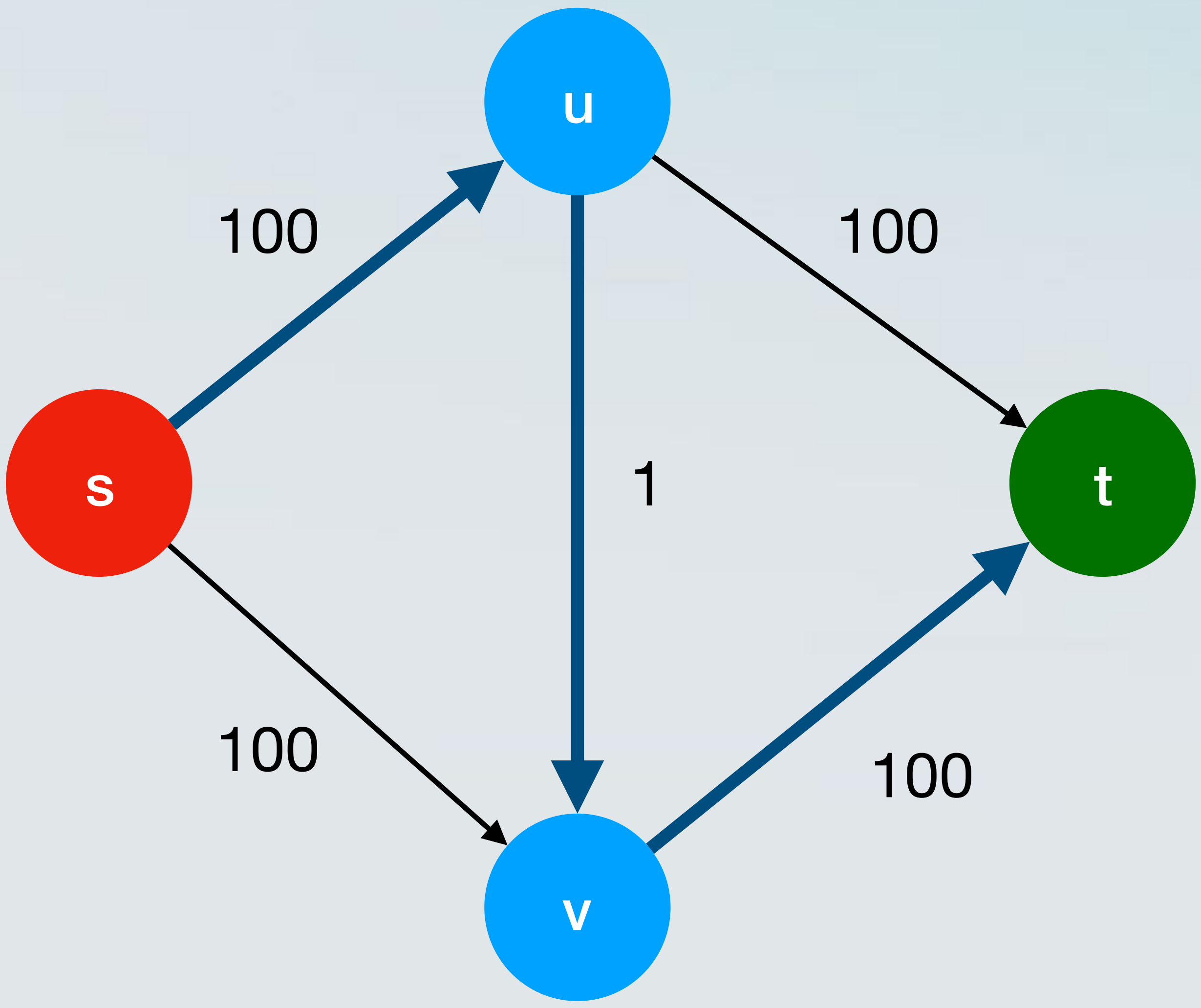

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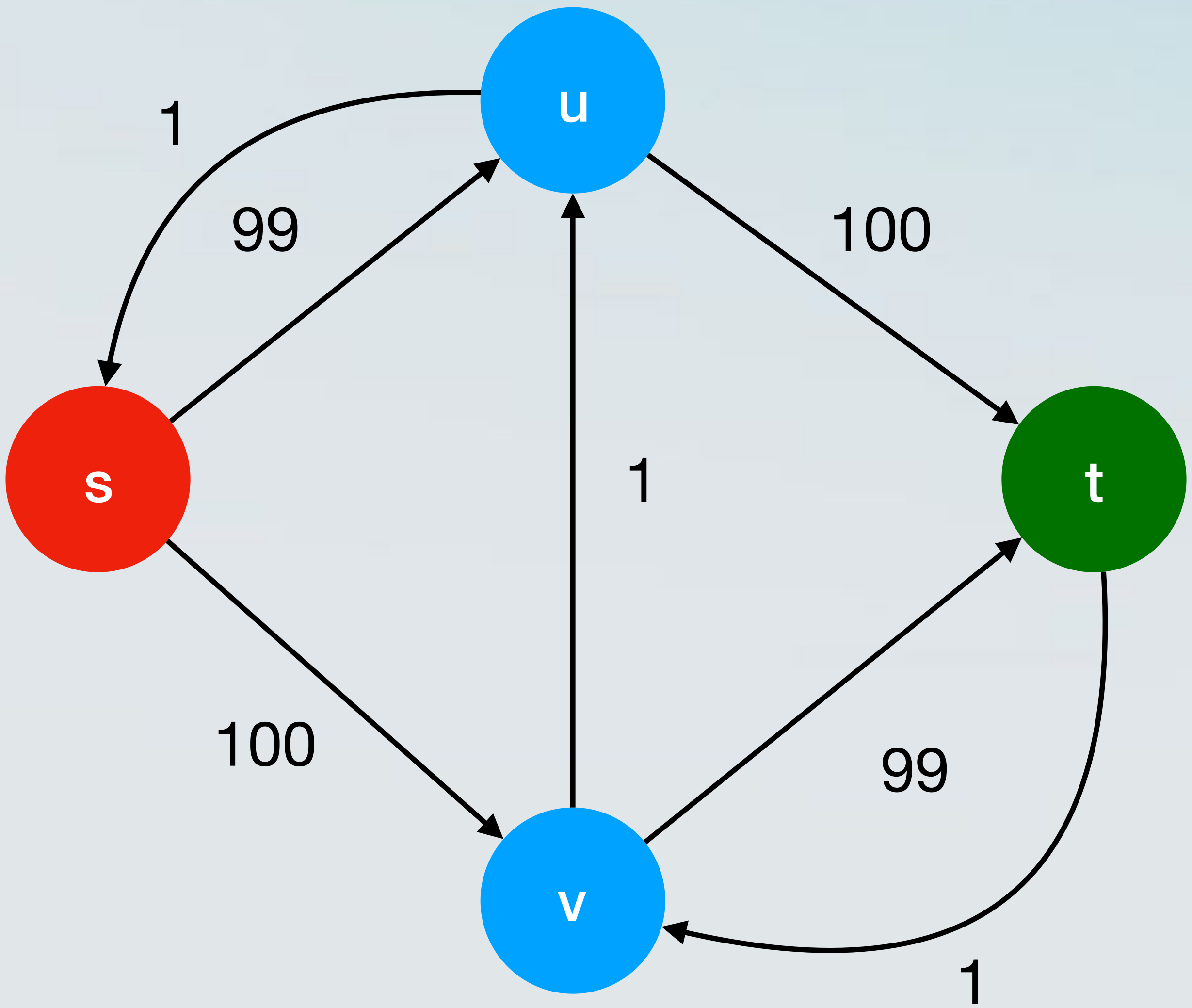

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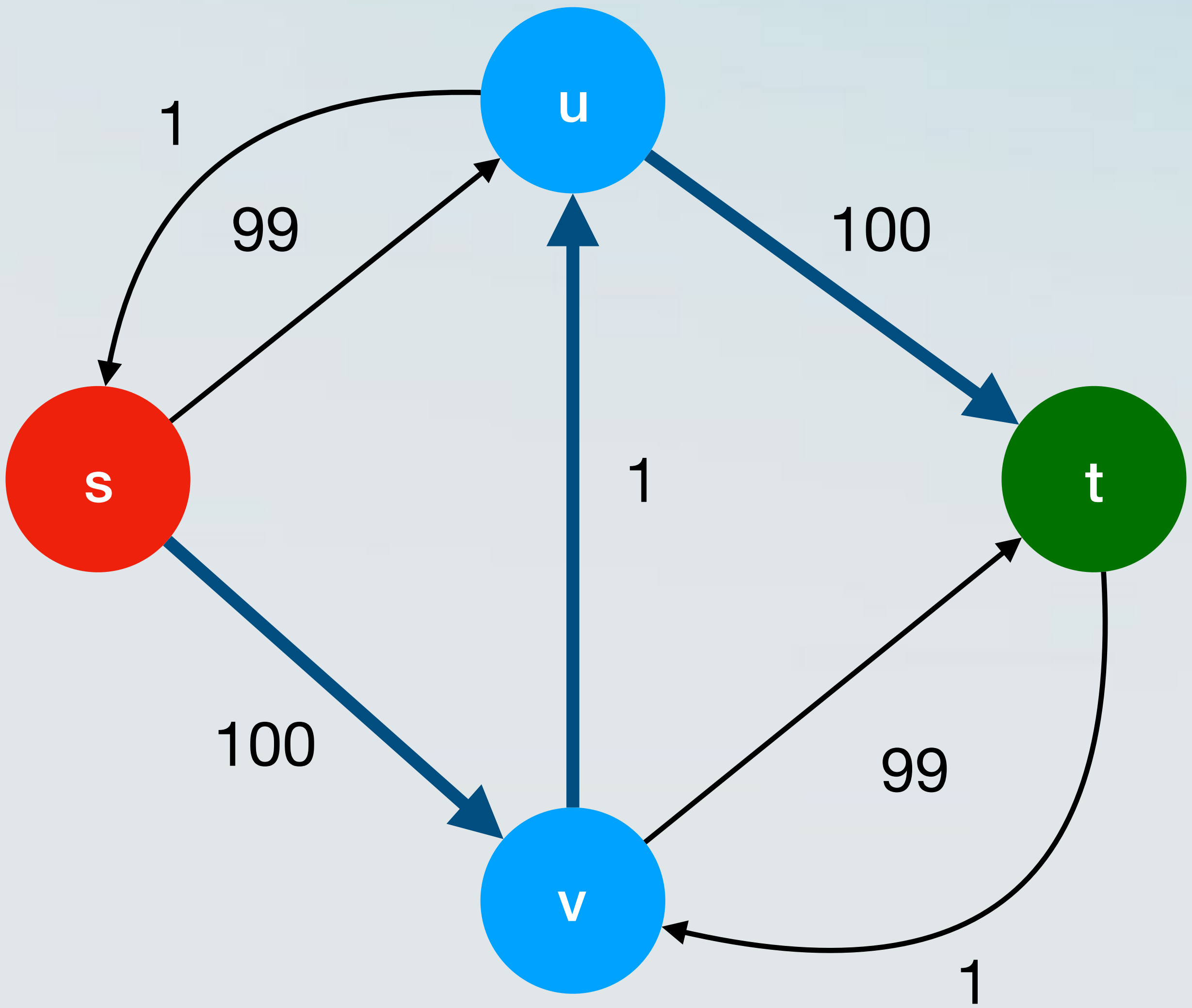

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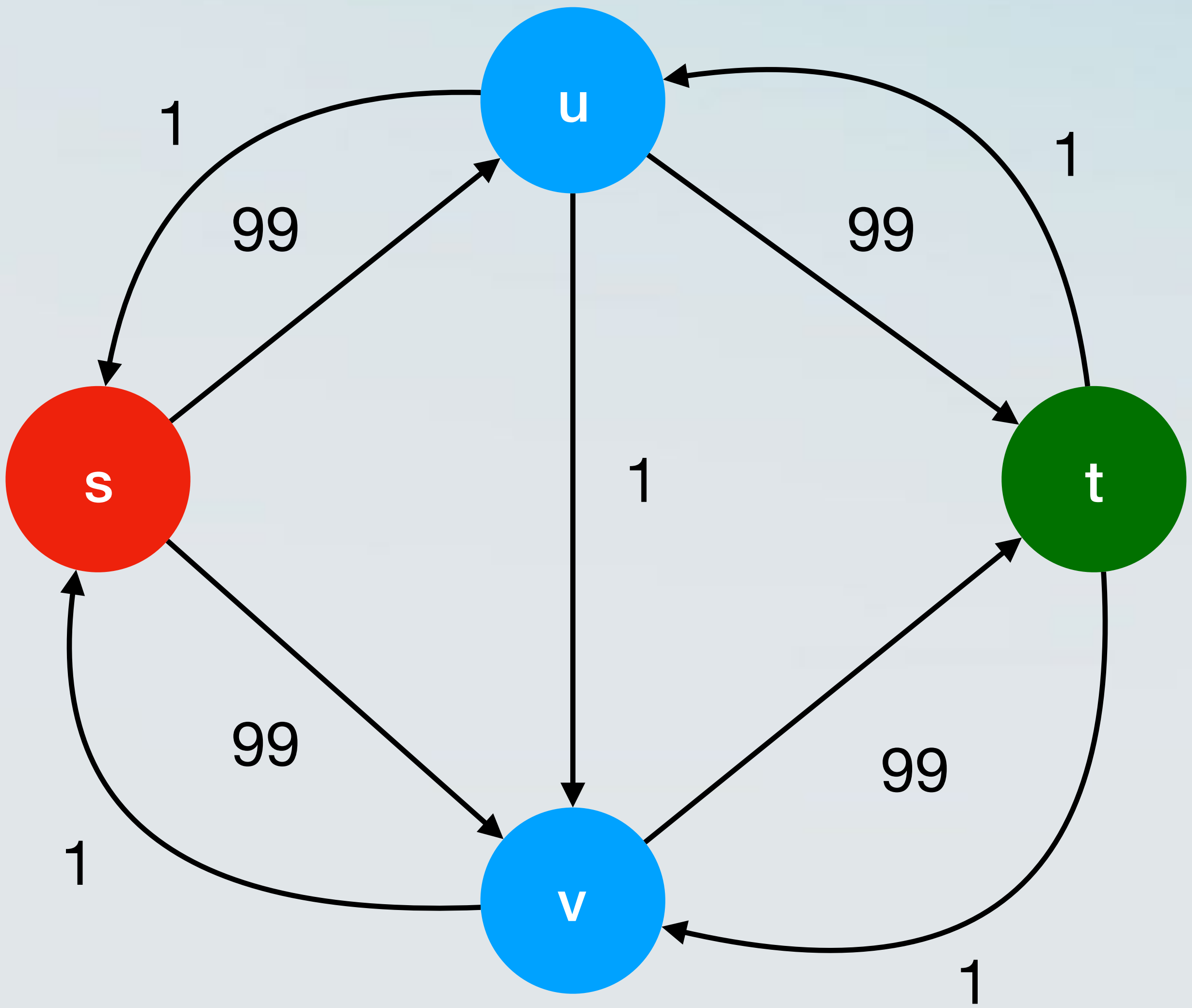

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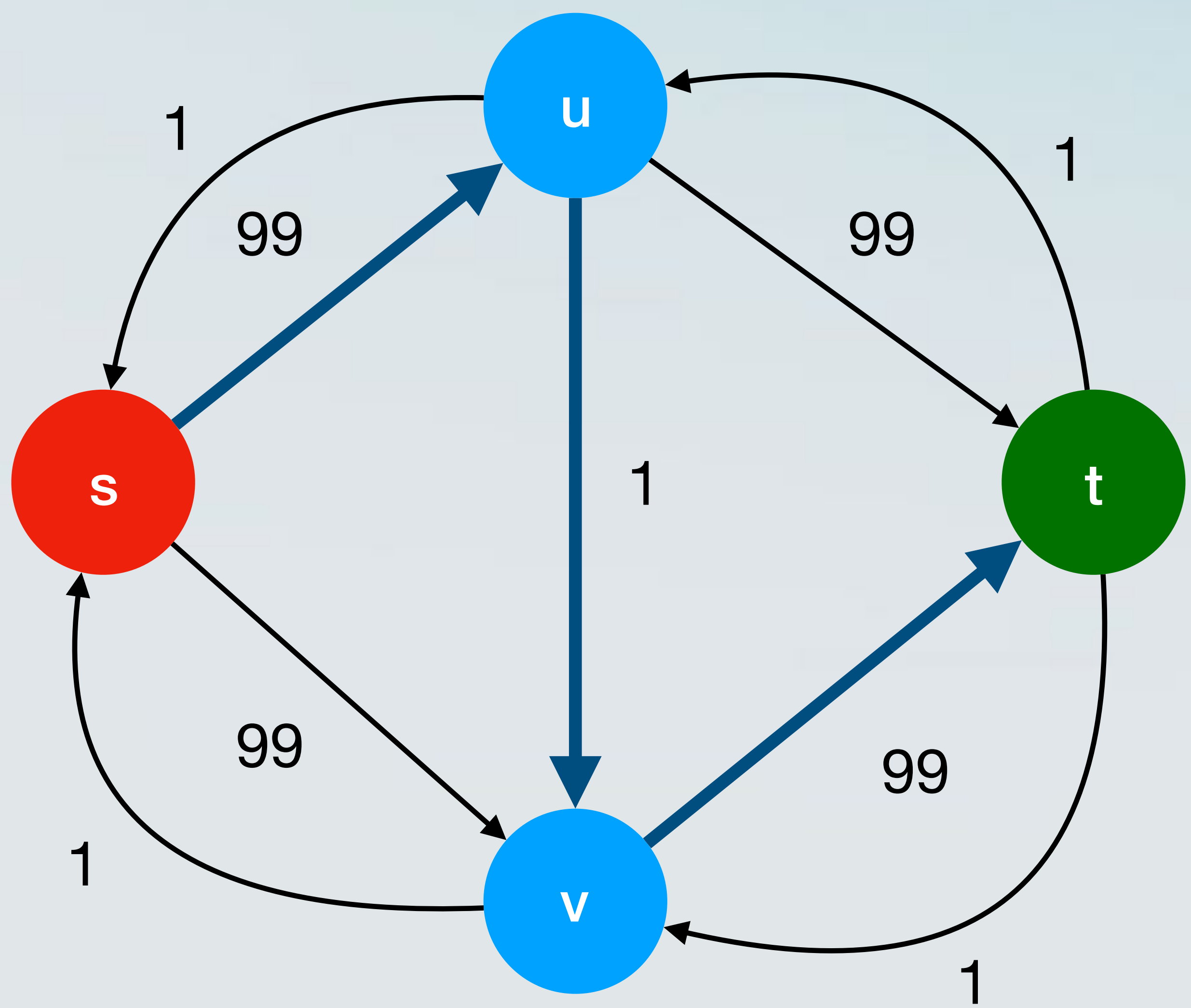

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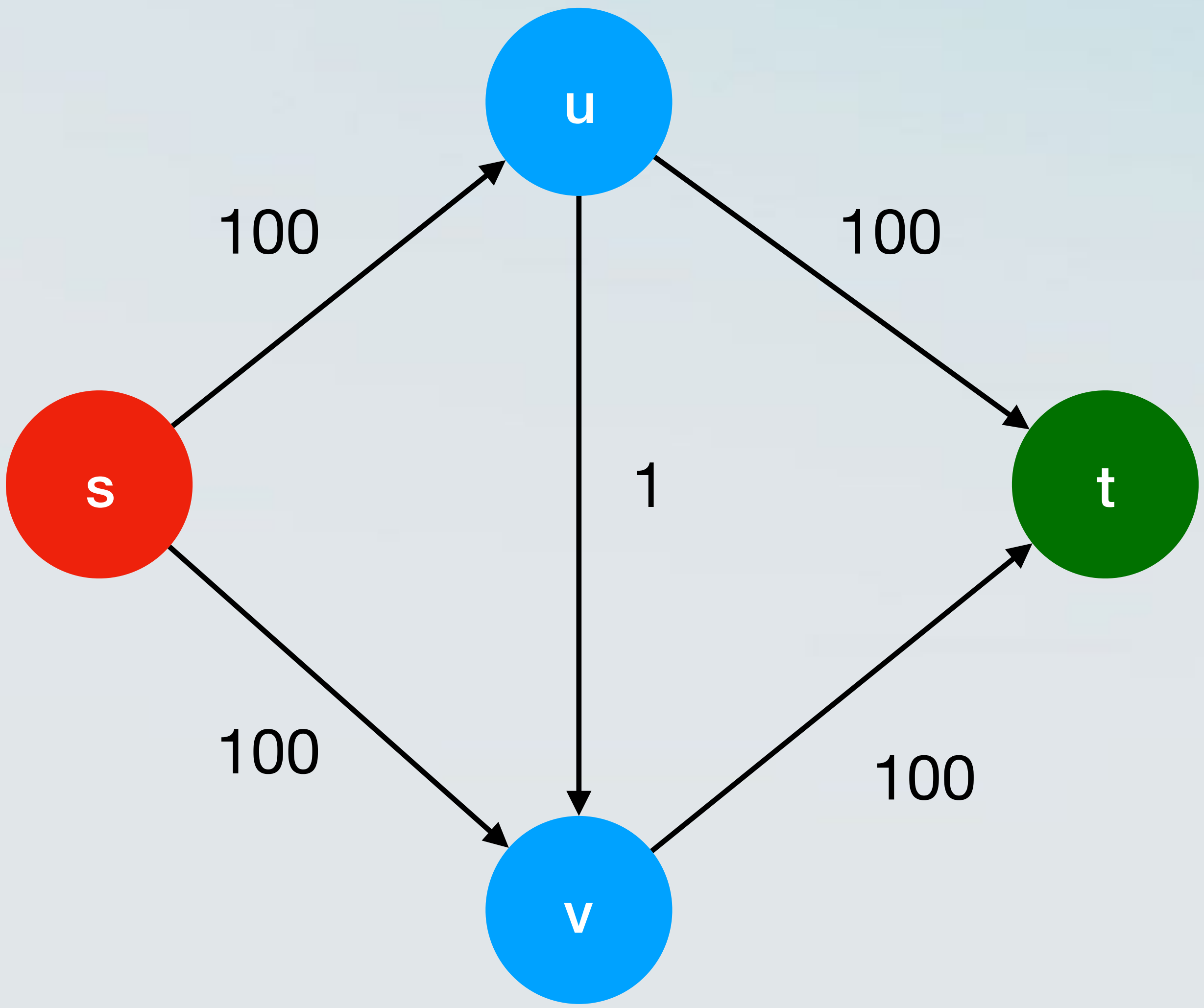

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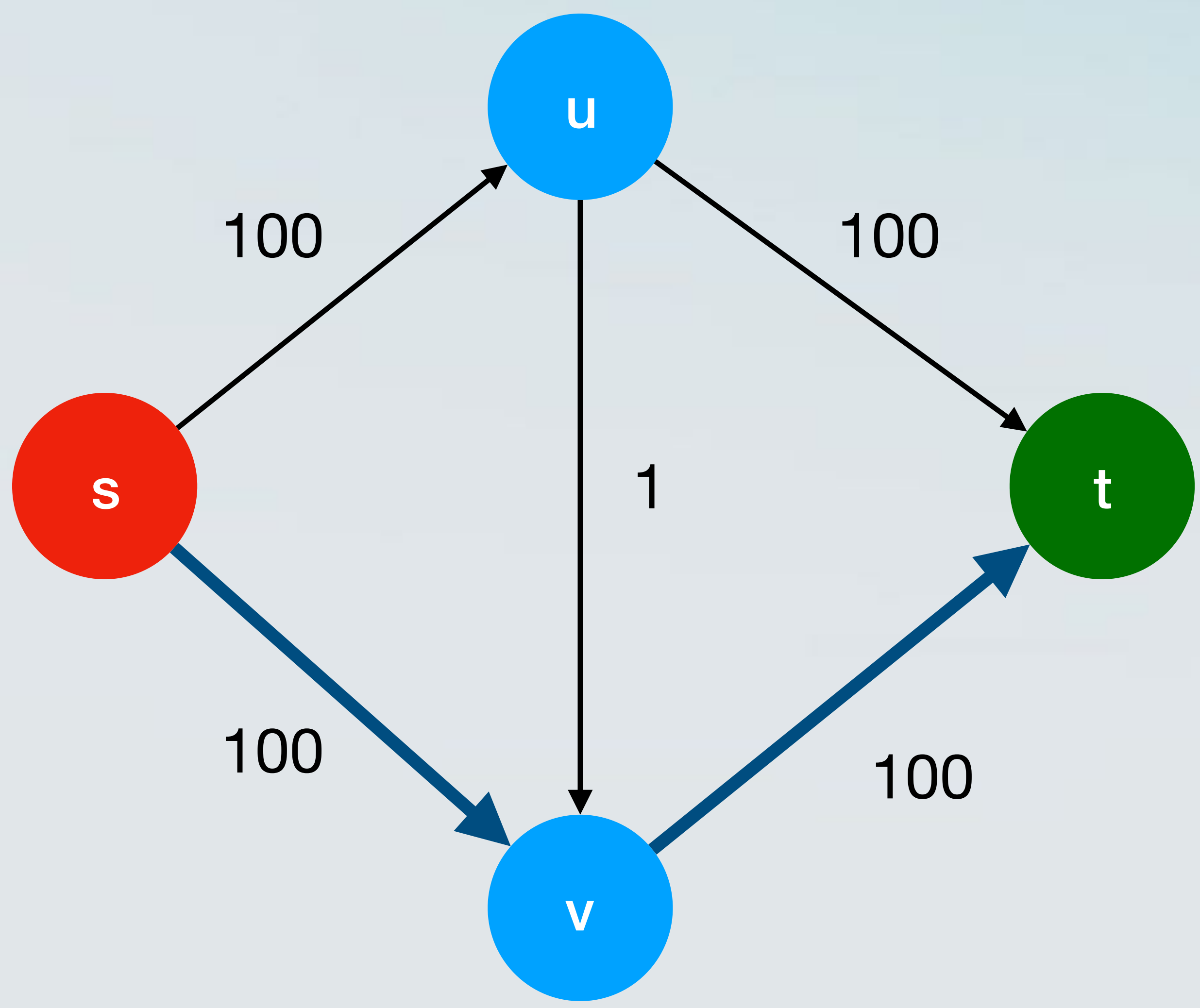

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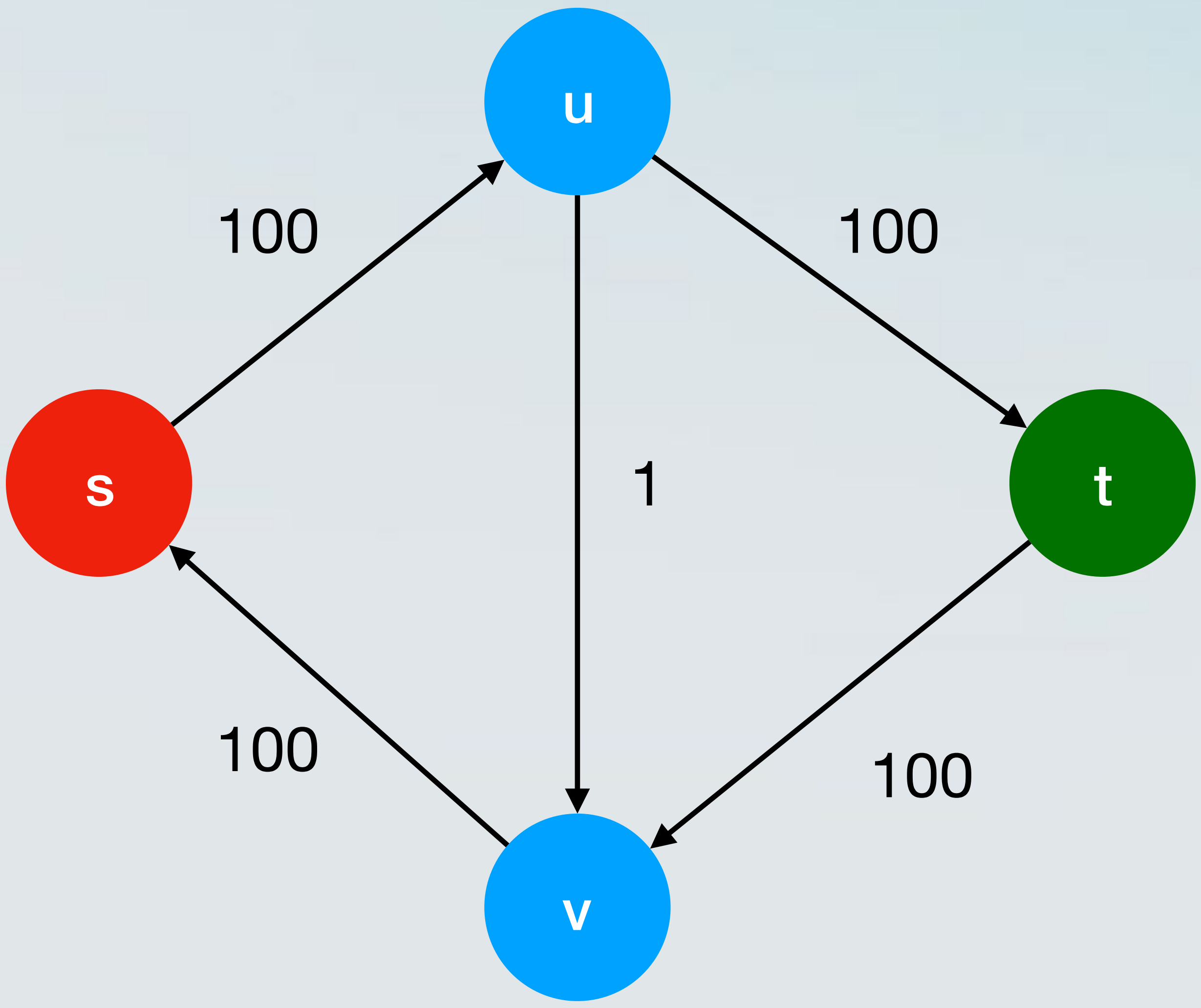

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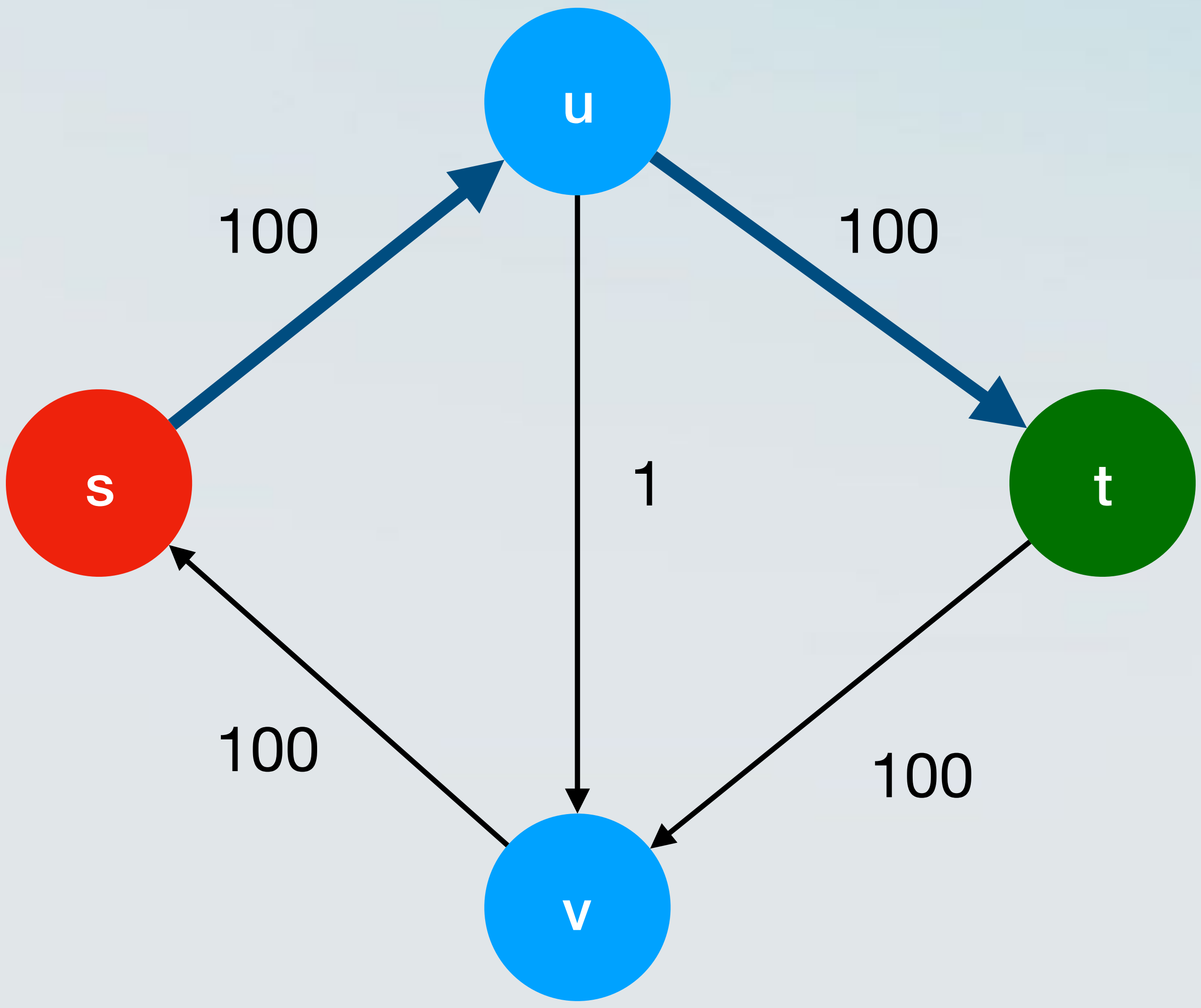

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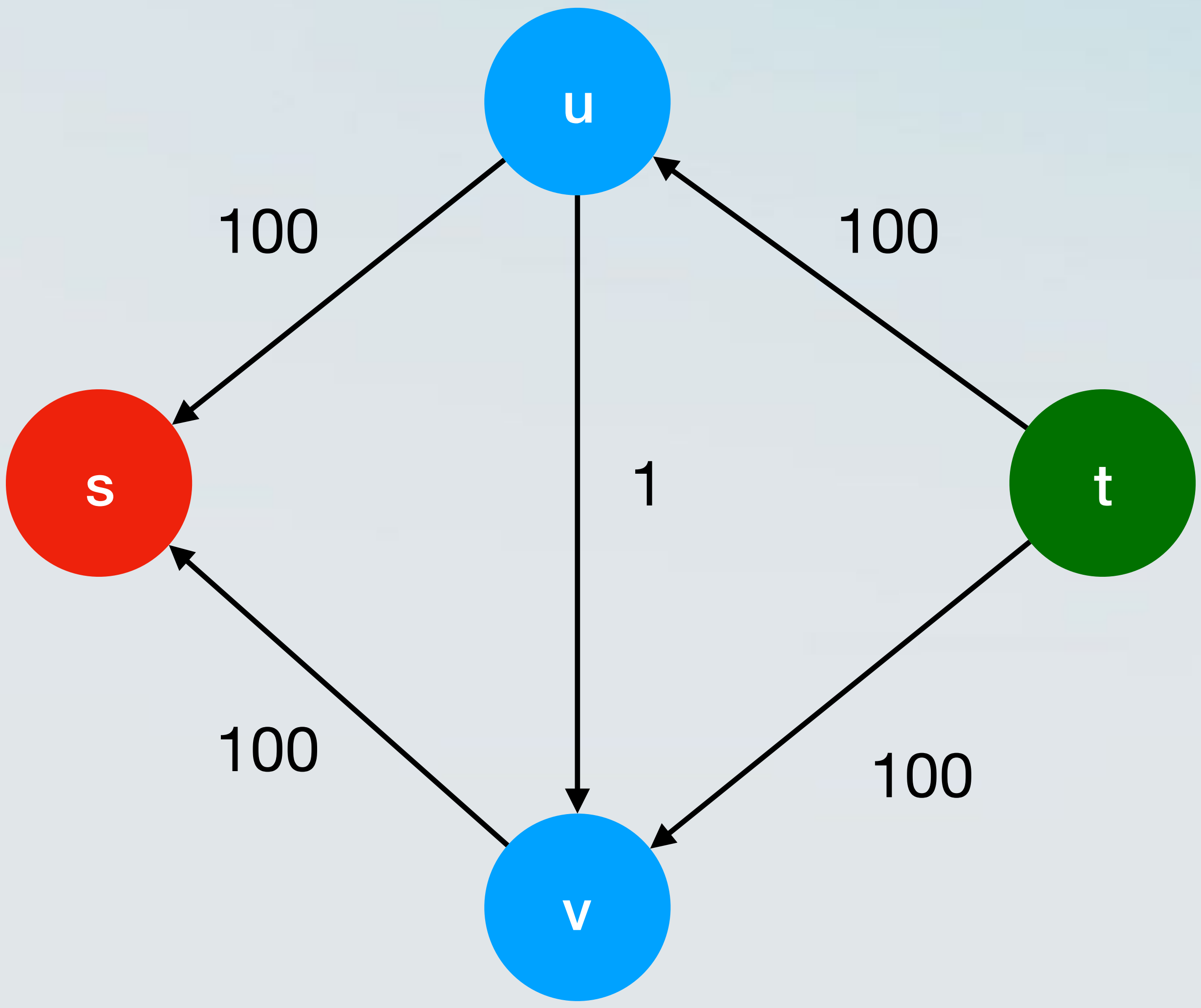

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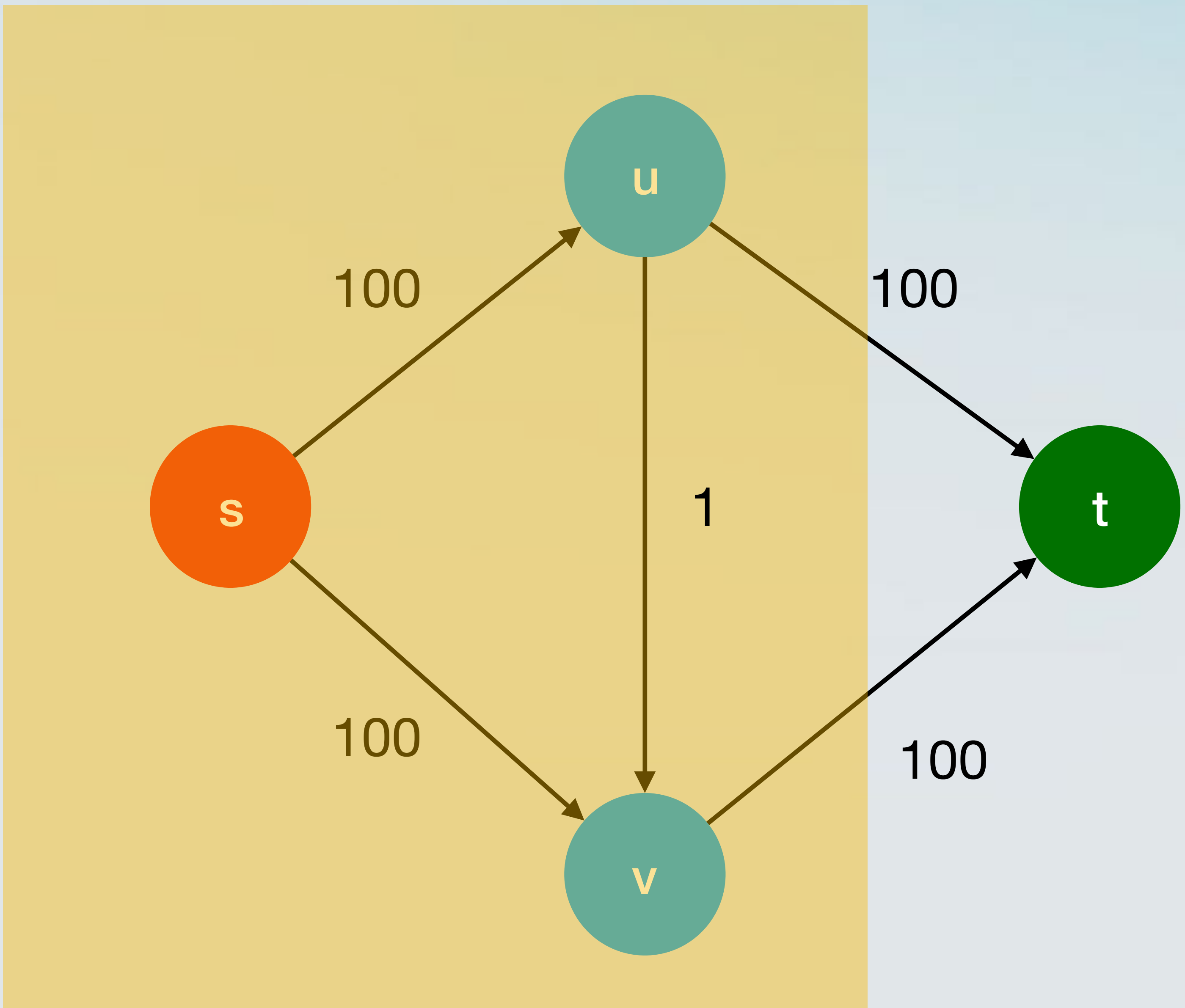

Max-Flow in polynomial time

- We made the algorithm must faster by simply selecting the shortest path with available capacity.
- Can we always hope to do that?

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The Edmonds-Karp Algorithm

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Endwhile

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The Edmonds-Karp Algorithm

- The Edmonds-Karp version of the Ford-Fulkerson algorithm runs in time **O(**nm^2**).**
- The shortest path can be found using a BFS search.