Advanced Algorithmic Techniques (COMP523)

Network Flows 3

Recap and plan

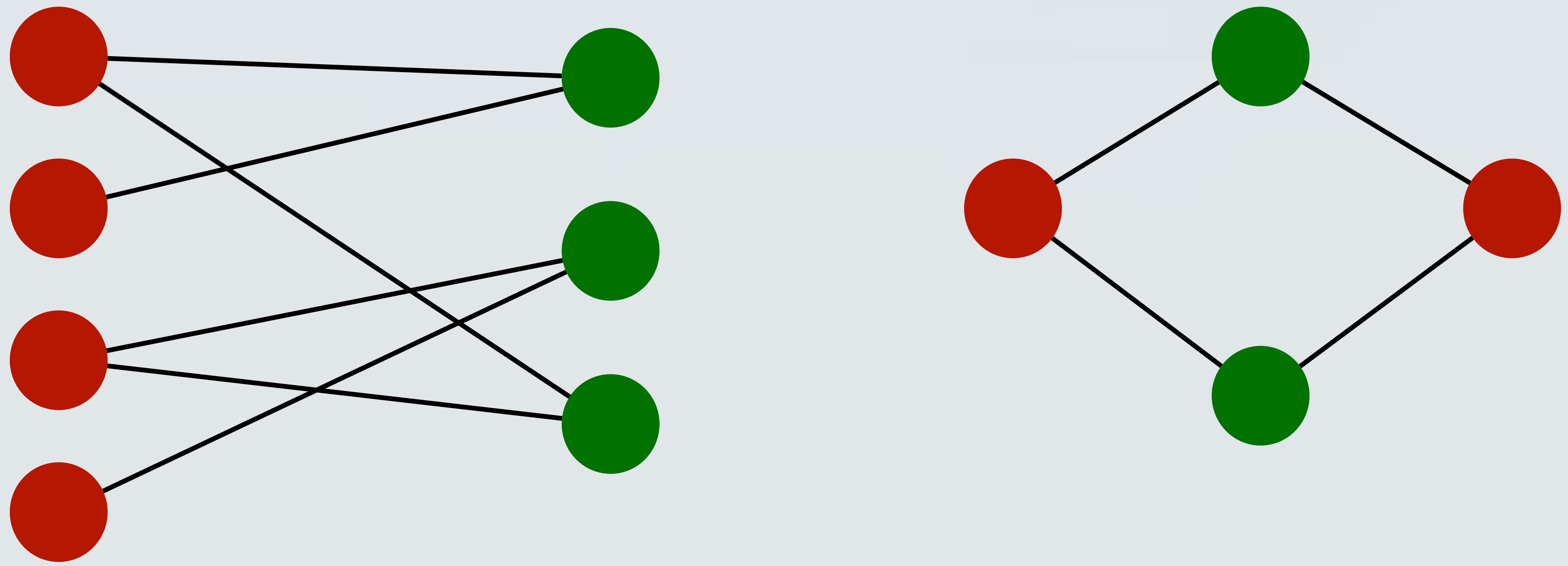
- **Last 2 lectures:**
  - Maximum Flow.
  - The Ford-Fulkerson Algorithm.
  - The Max-Flow - Min - Cut theorem.
  - The Edmonds-Karp algorithm.
- **This lecture:**
  - Modelling with flows.
  - Maximum Bipartite Matching.
  - Baseball Elimination.
  - Open-pit mining.

Bipartite Matching

- *Maximum Bipartite Matching* or Maximum matching on a bipartite graph G.

Bipartite graphs

- A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.
  - Often, we write G=(A U B,E).

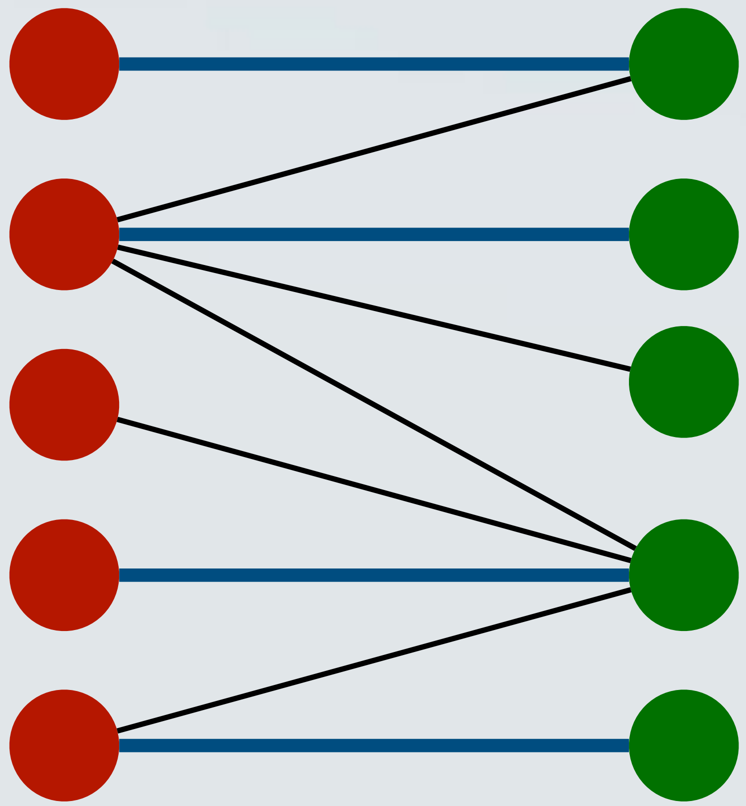


Bipartite Matching

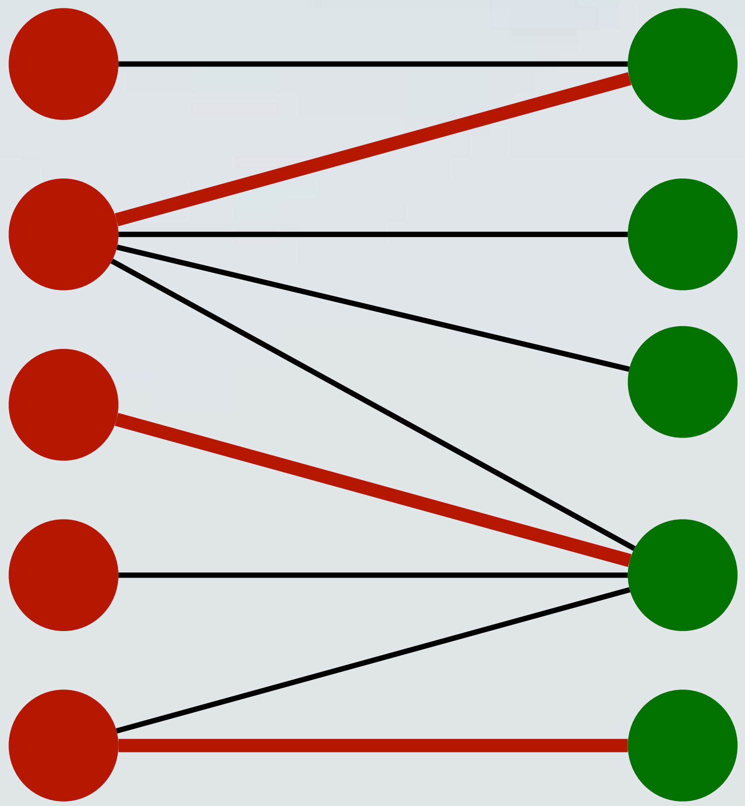
- *Maximum Bipartite Matching* or Maximum matching on a bipartite graph G.
  - Matching: A subset M of the edges E such that each node v of V appears in at most one edge e in E.
  - Maximum matching: A matching with maximum cardinality.(i.e., |M| is maximised).

Example

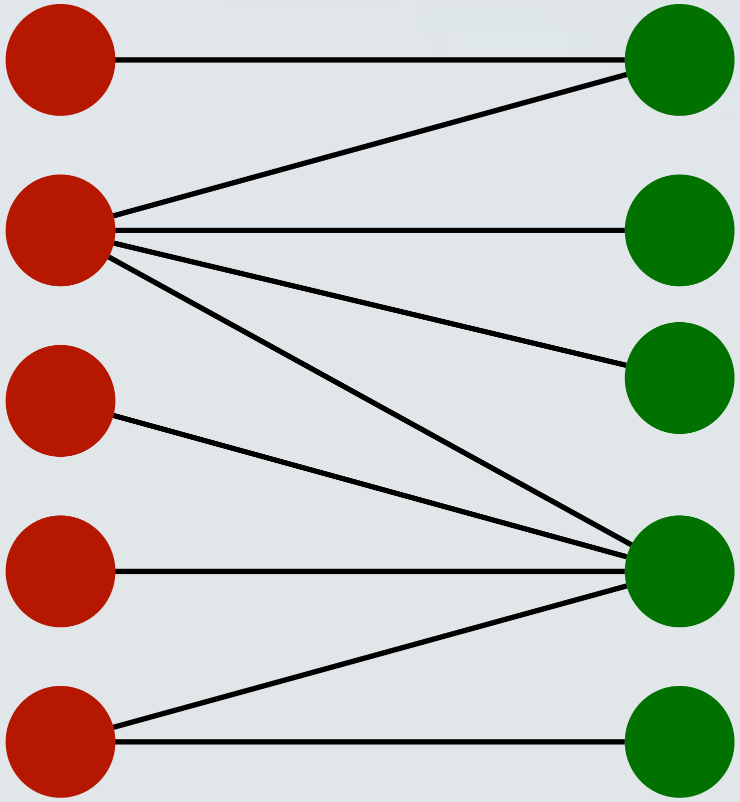
A maximum matching



A maximal matching



From matchings to flows



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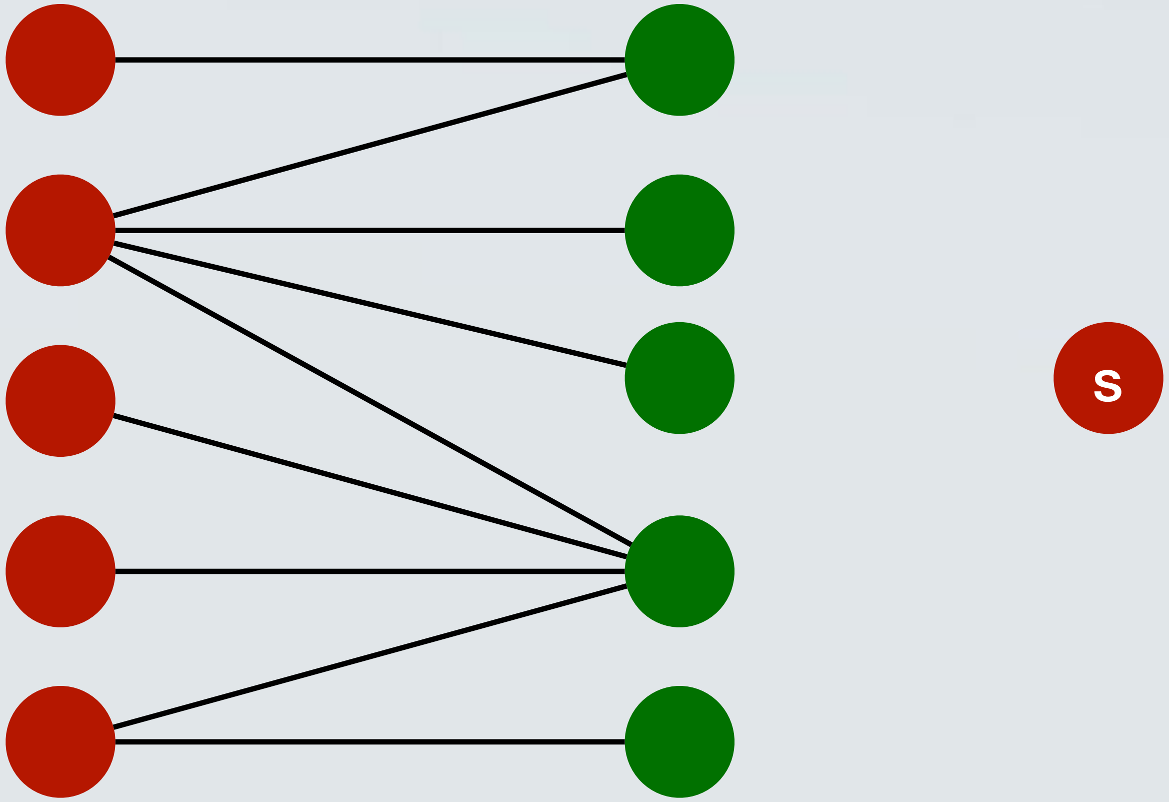

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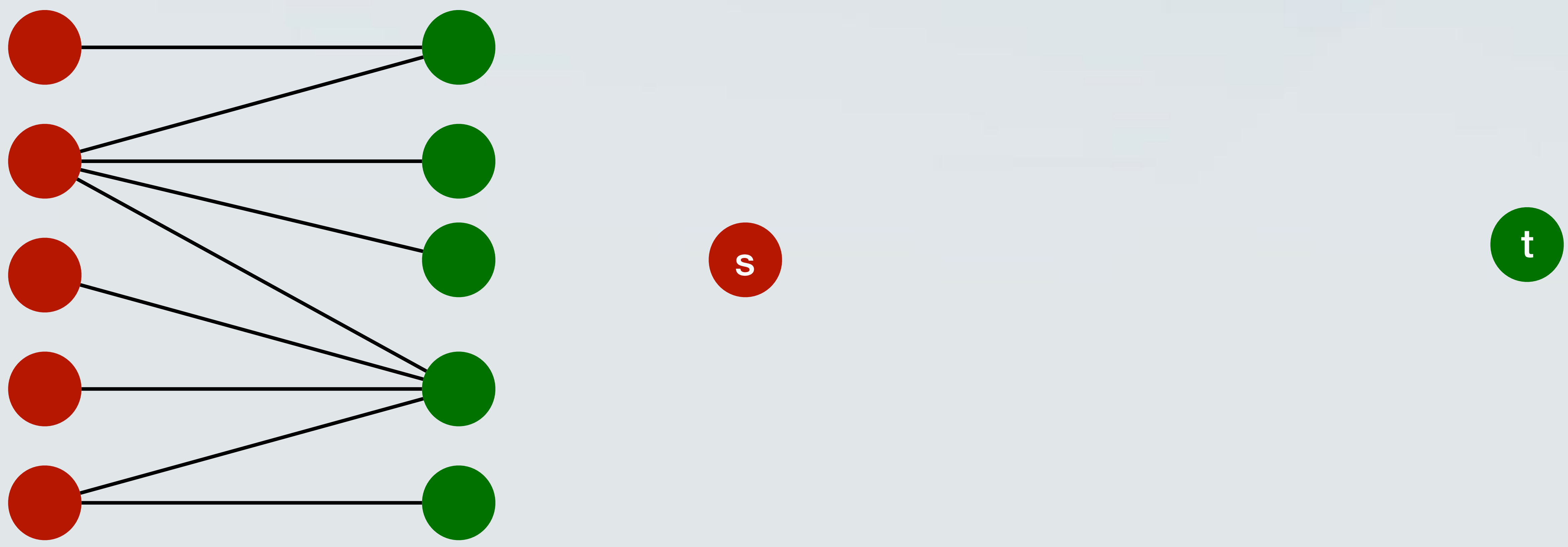

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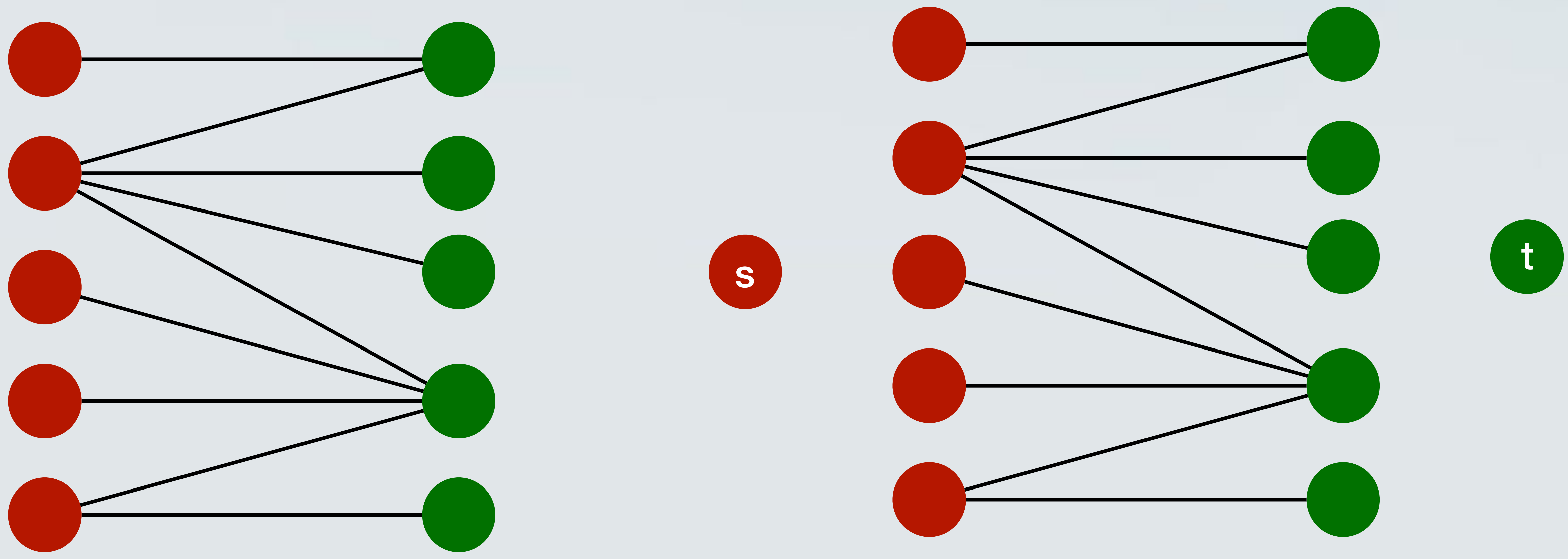

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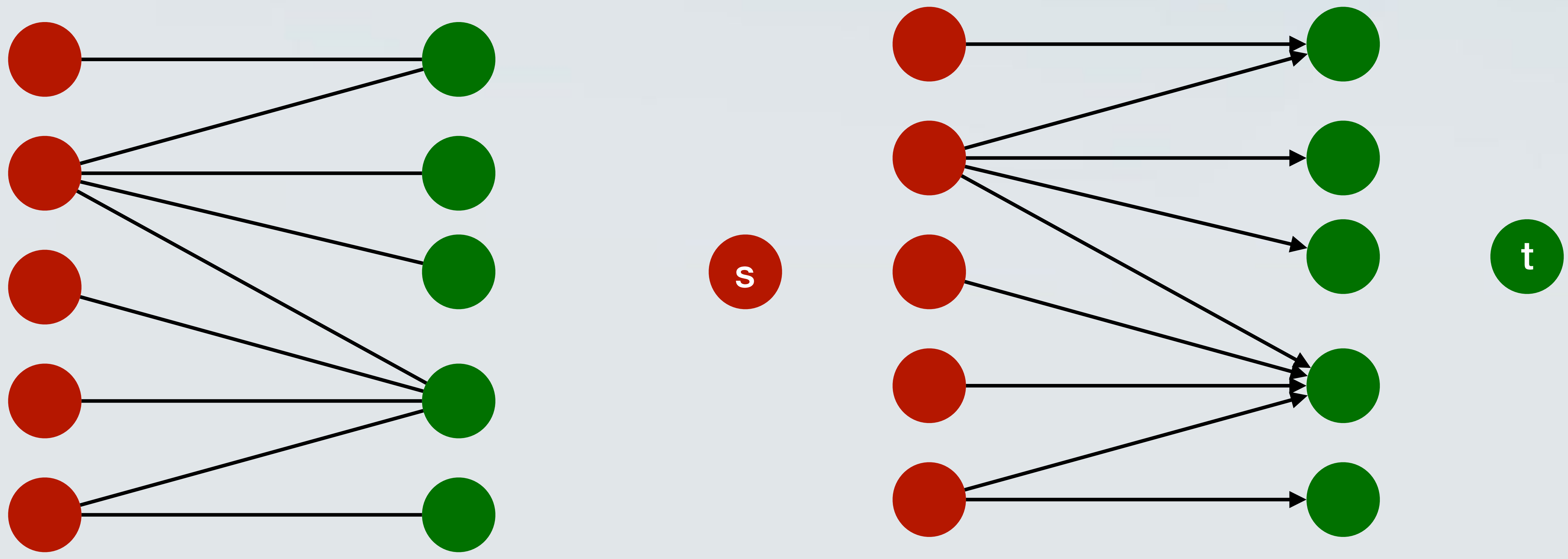

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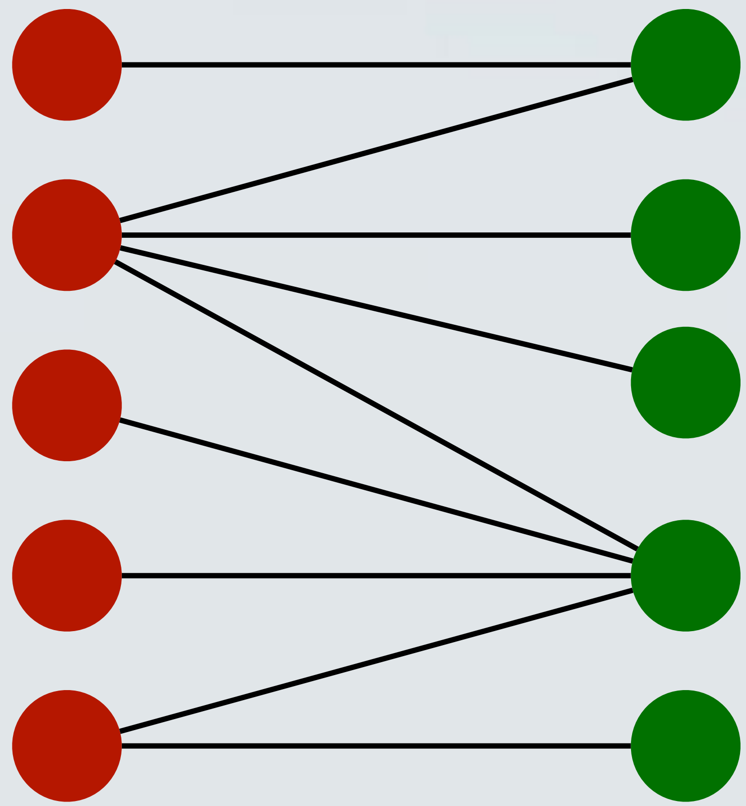

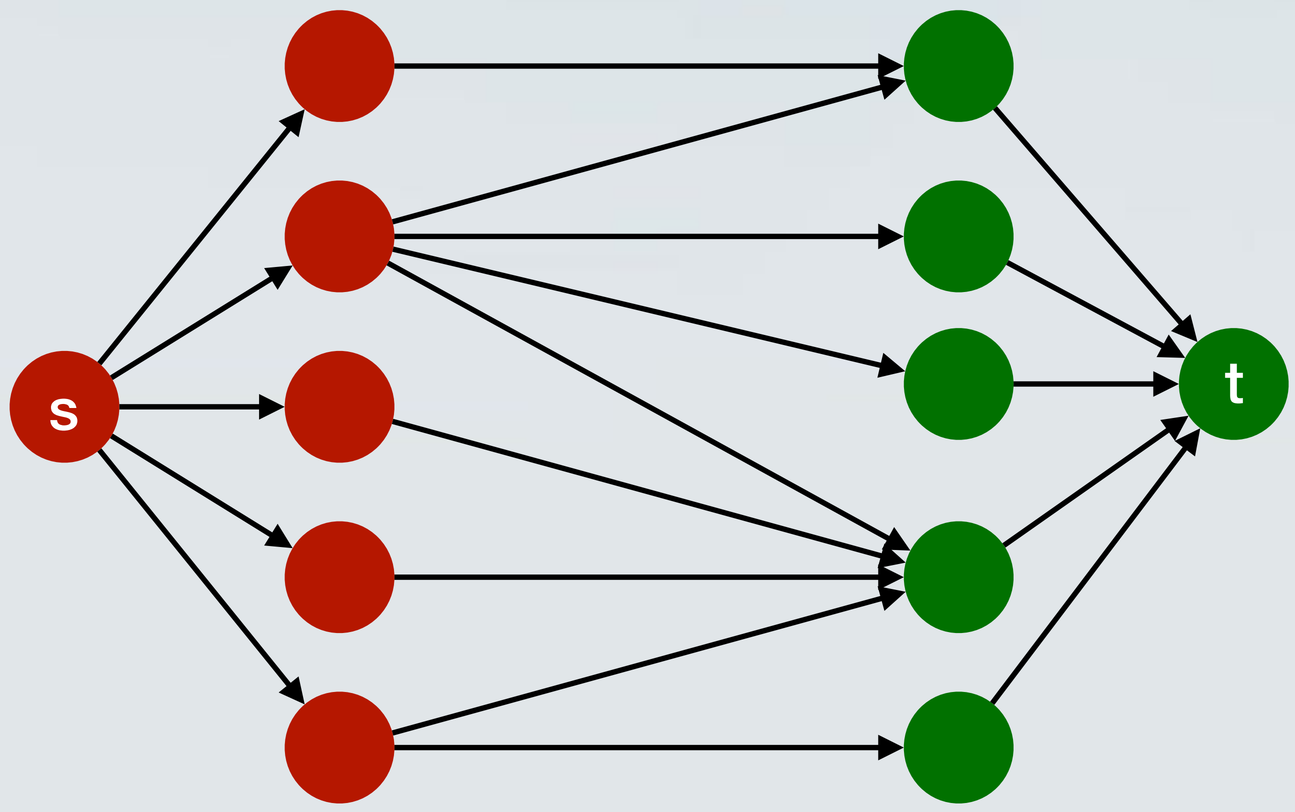

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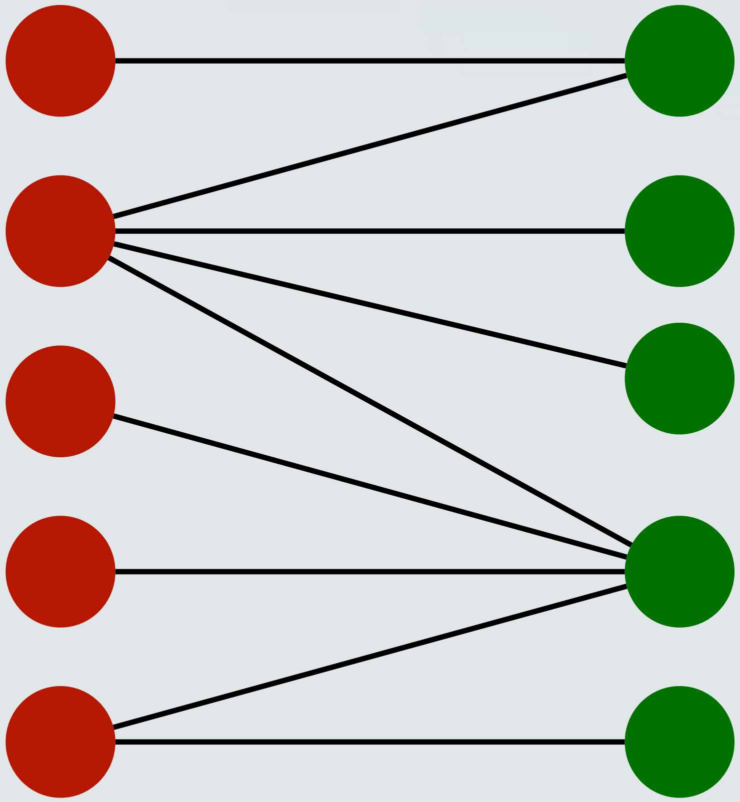

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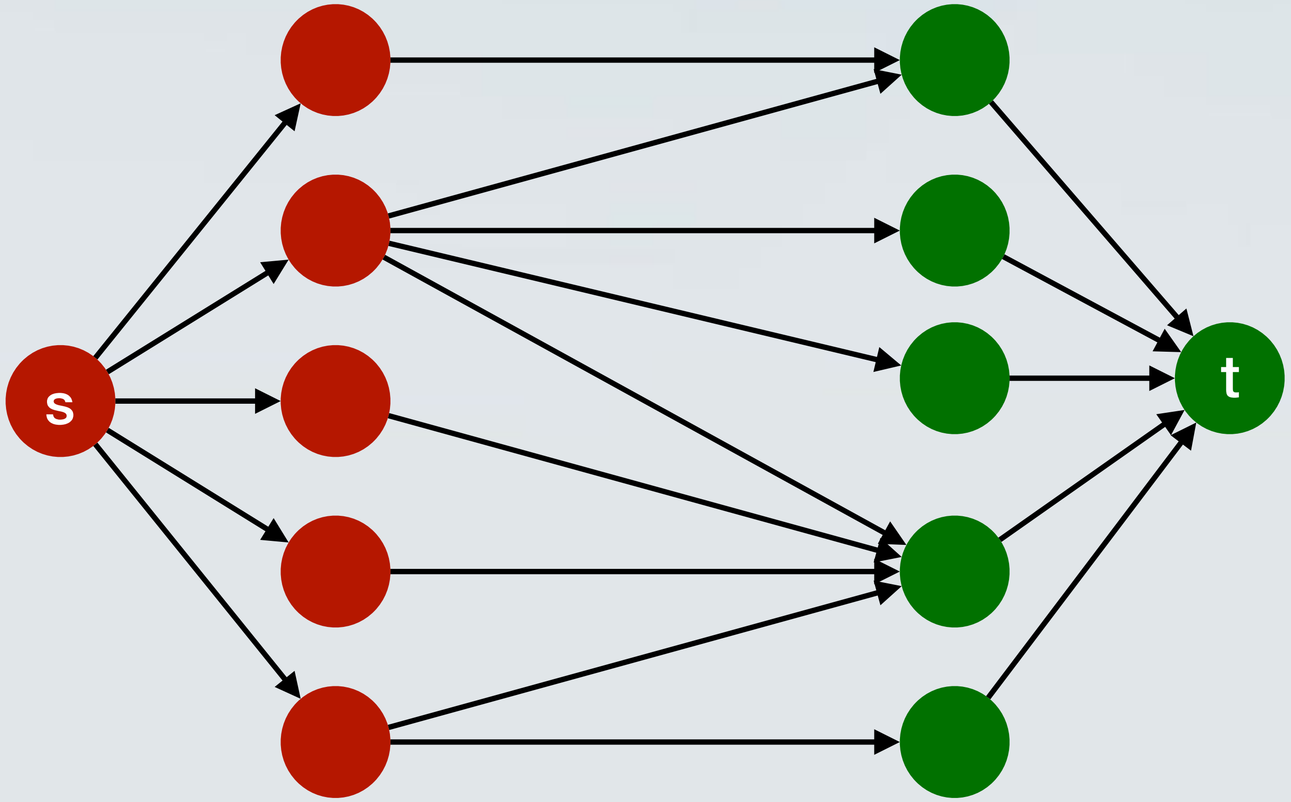

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  - This is a feasible flow and obviously has value k.

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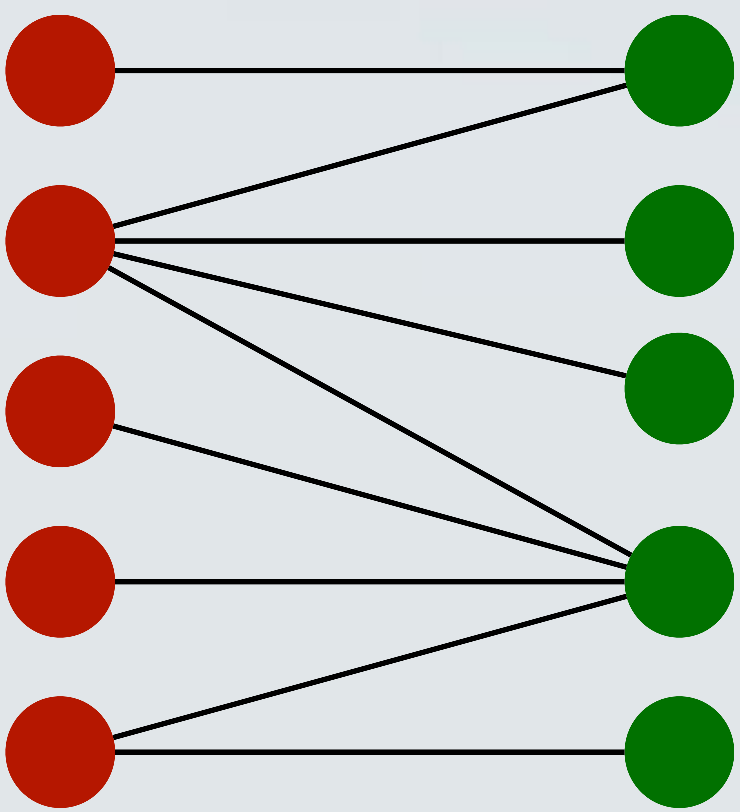

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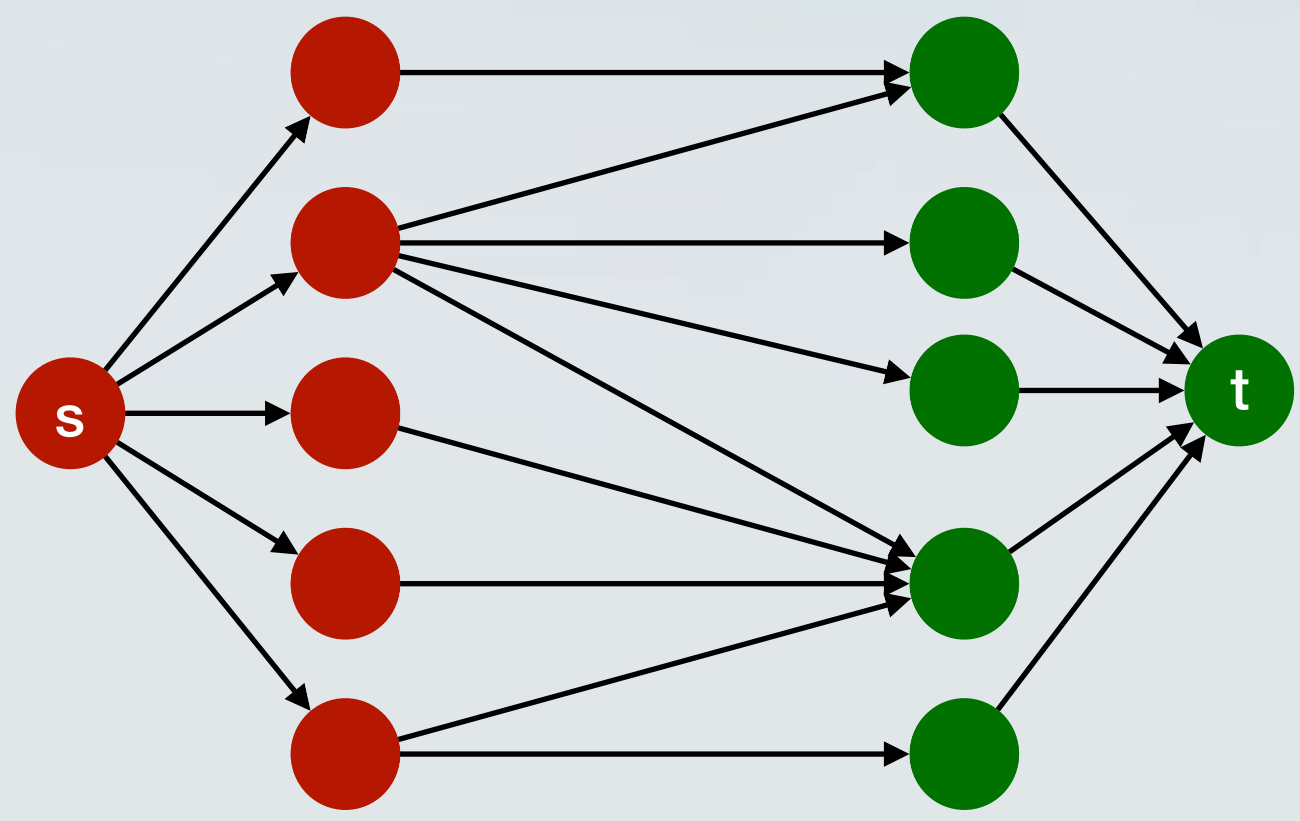

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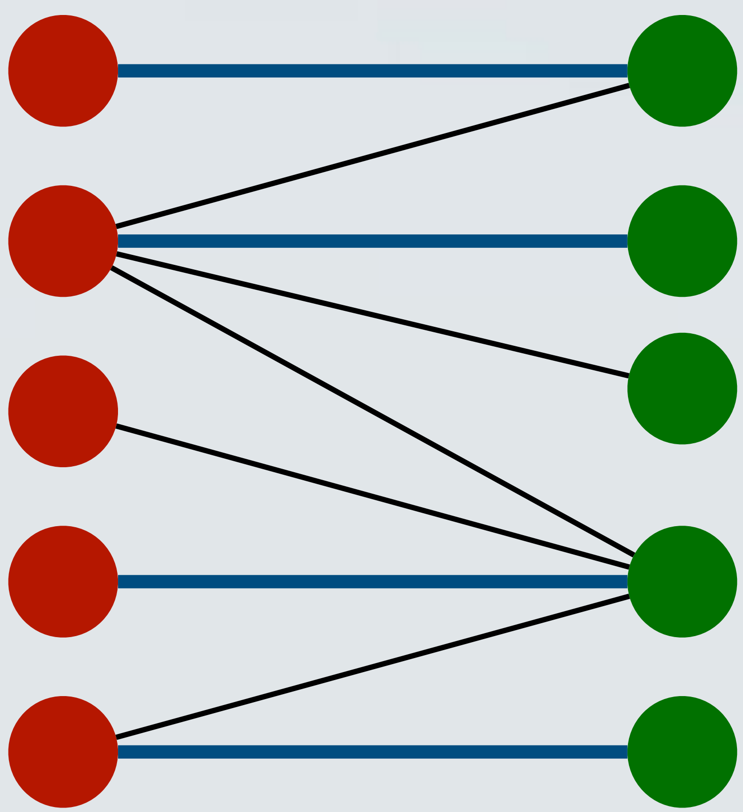

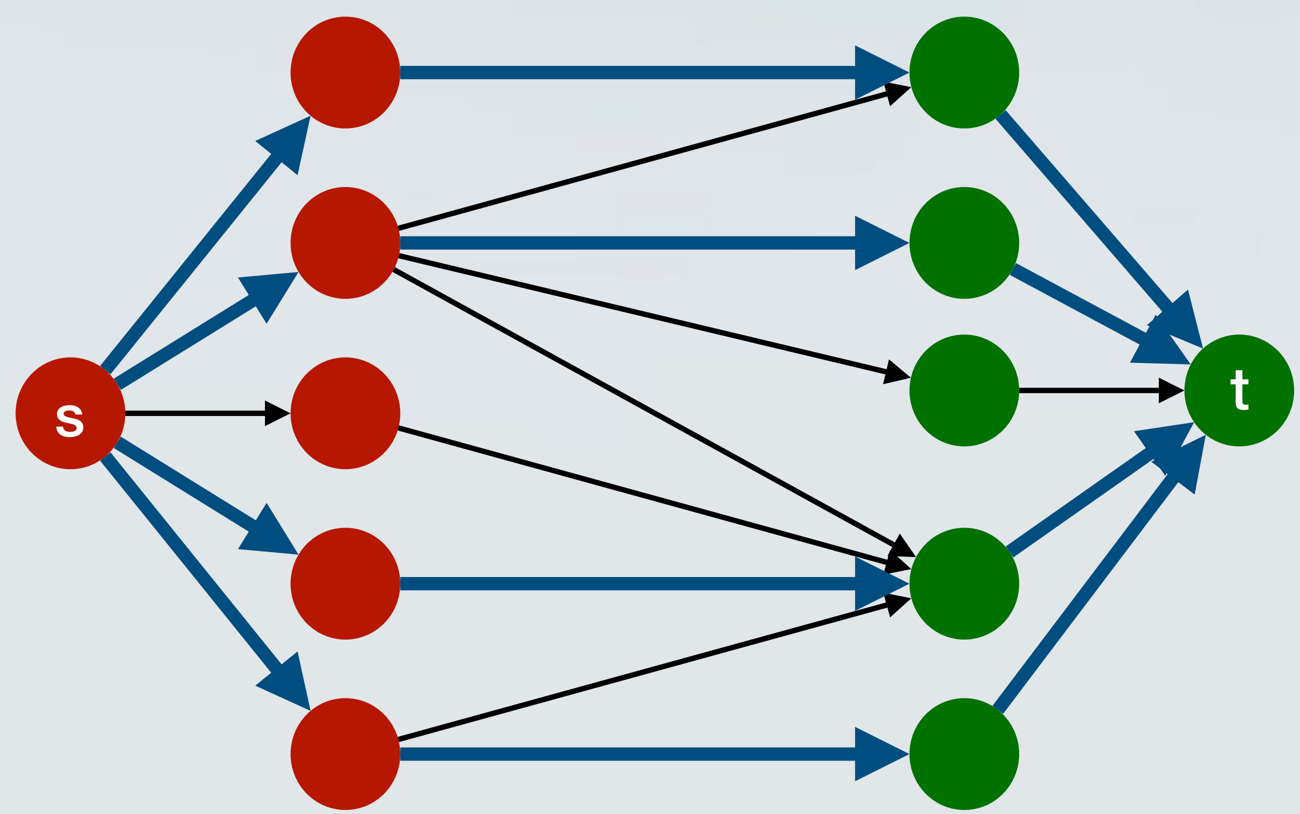

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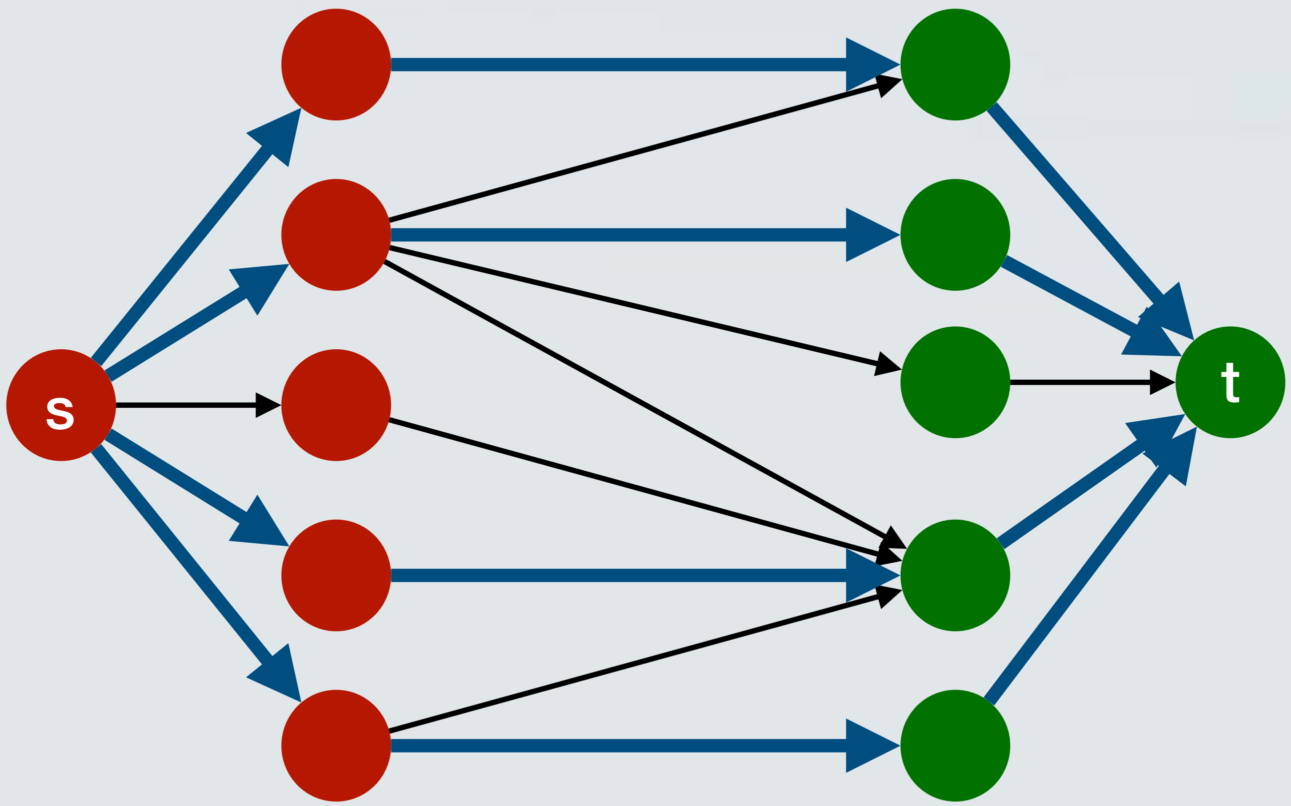

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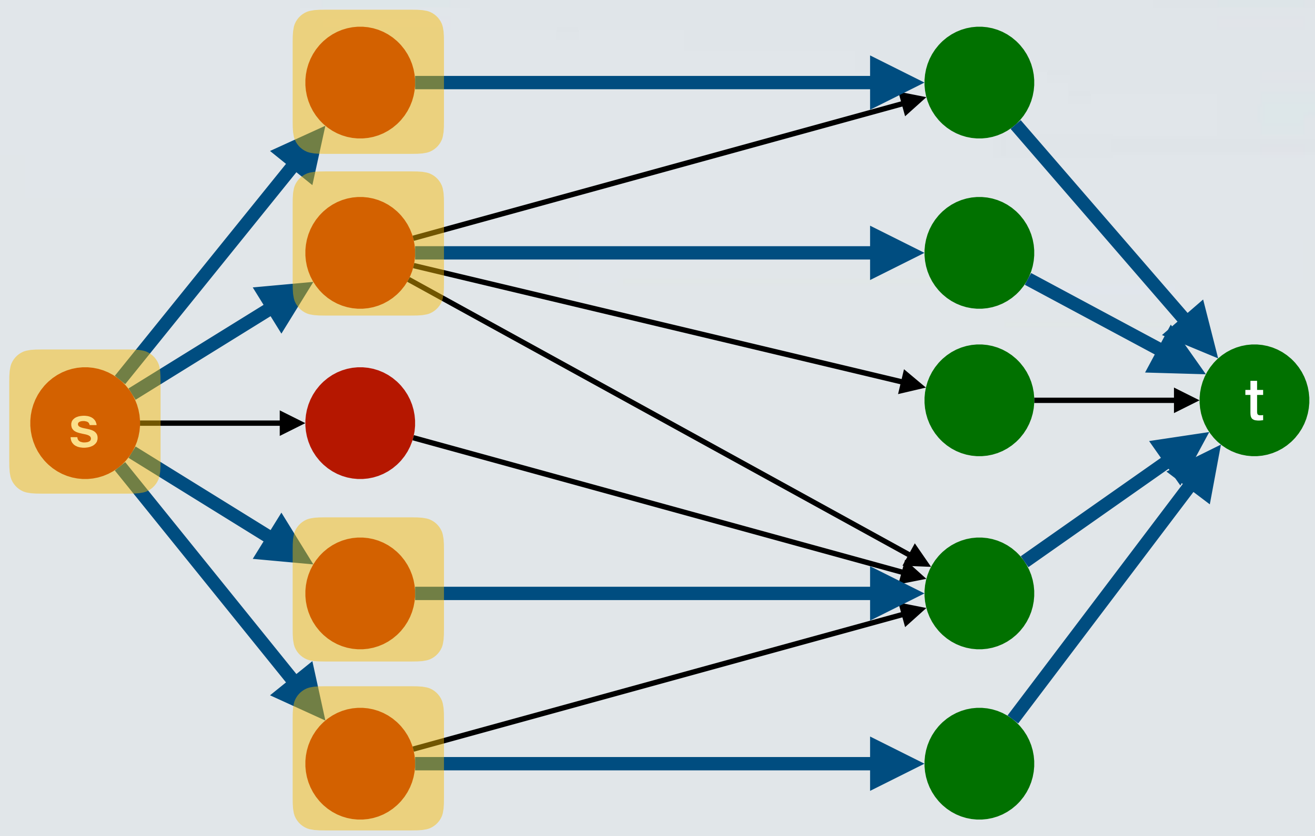

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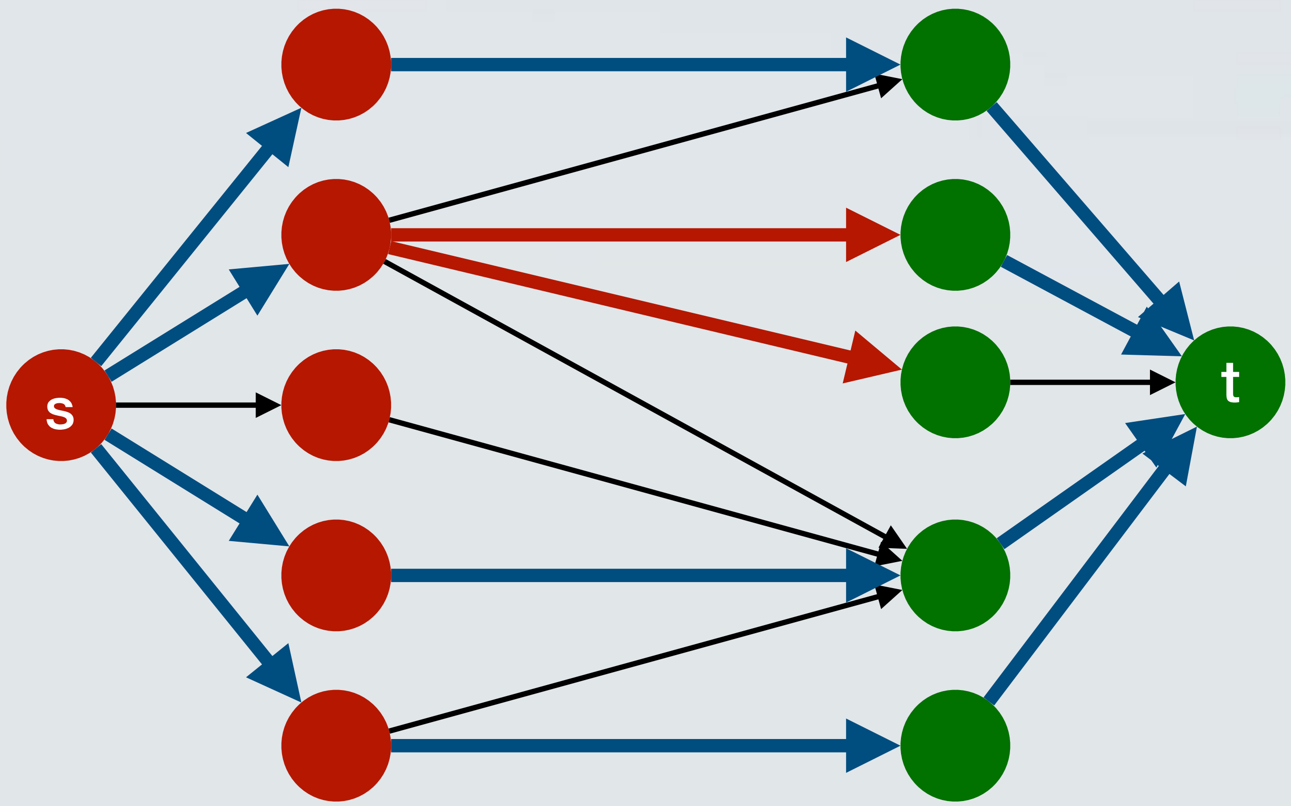

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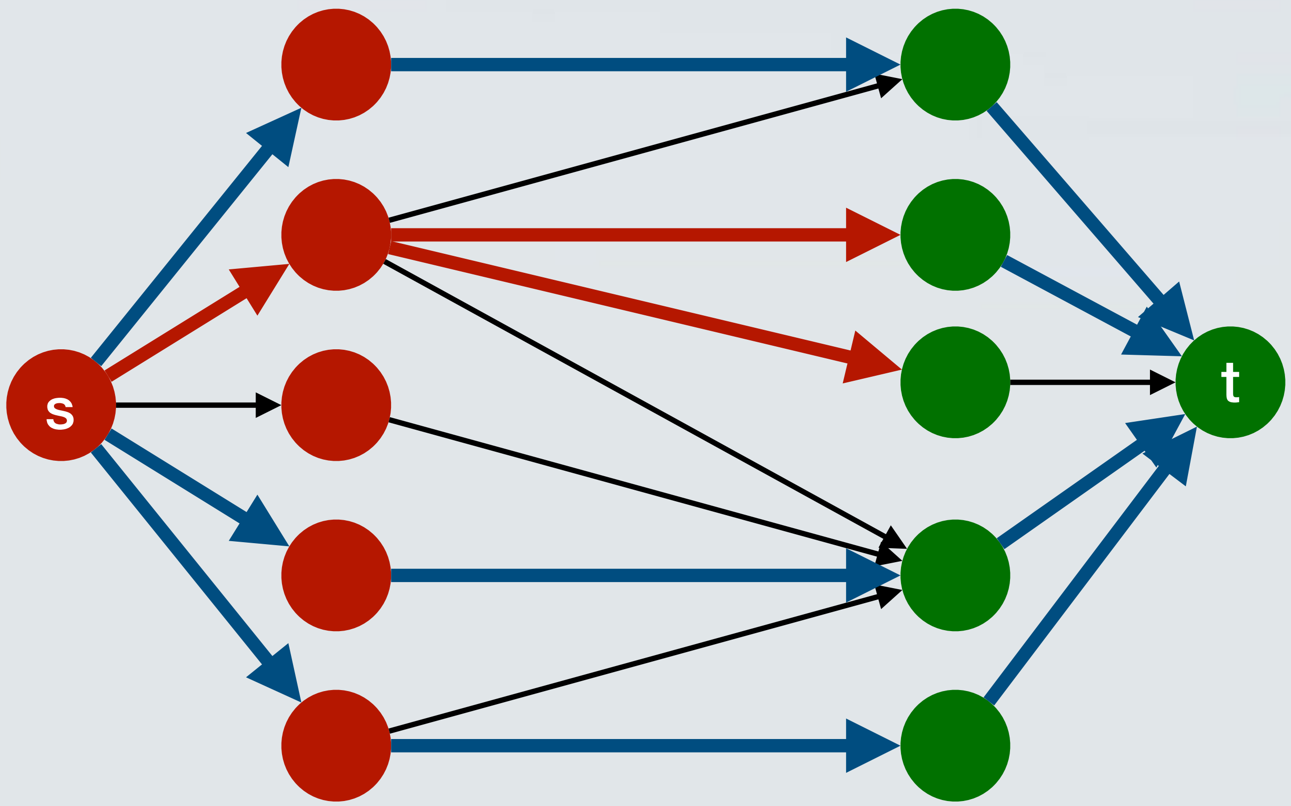

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- The edges of M are the edges that carry flow from A to B in G^f.
- What was the crucial part, that allows us to establish this?
  - The integrality theorem.

Running time

- What is the running time of the algorithm?
- By Edmonds - Karp, we get **O(**nm^2**).**

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    - Running time **O(**nm**).**

Baseball Elimination

- In the baseball league, there are 4 teams with the following number of wins:`

New York	92
Baltimore	91
Toronto	91
Boston	90

- There are five games left in the season.
  - NY vs BLT, NY vs TOR, BLT vs TOR, BLT vs BOS, TOR vs BOS
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The answer is no.

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Baseball elimination

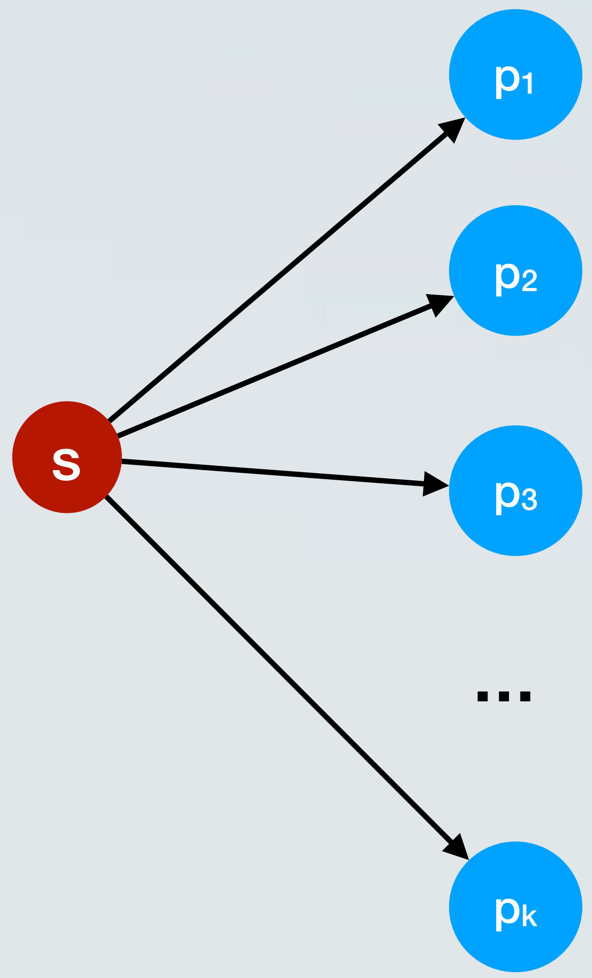
- Generally:
  - We have a set S of teams.
  - For each team x in S, the current number of wins is w_x.
  - For teams x and y in S, they still have to play g_{xy} games against each other.
  - We are given a designated team z.
  - Can z win the tournament (possibly in a tie?)

From baseball to flows

From baseball to flows

- Observation: If there is a way for z to be first, there is a way for z to be first *when winning all remaining games*.
  - Suppose that in the end, team z has m wins.
  - What are we looking for?
    - Is there an allocation of all the remaining g^* games (between the other teams) such that no team ends up with more than m wins?

From baseball to flows



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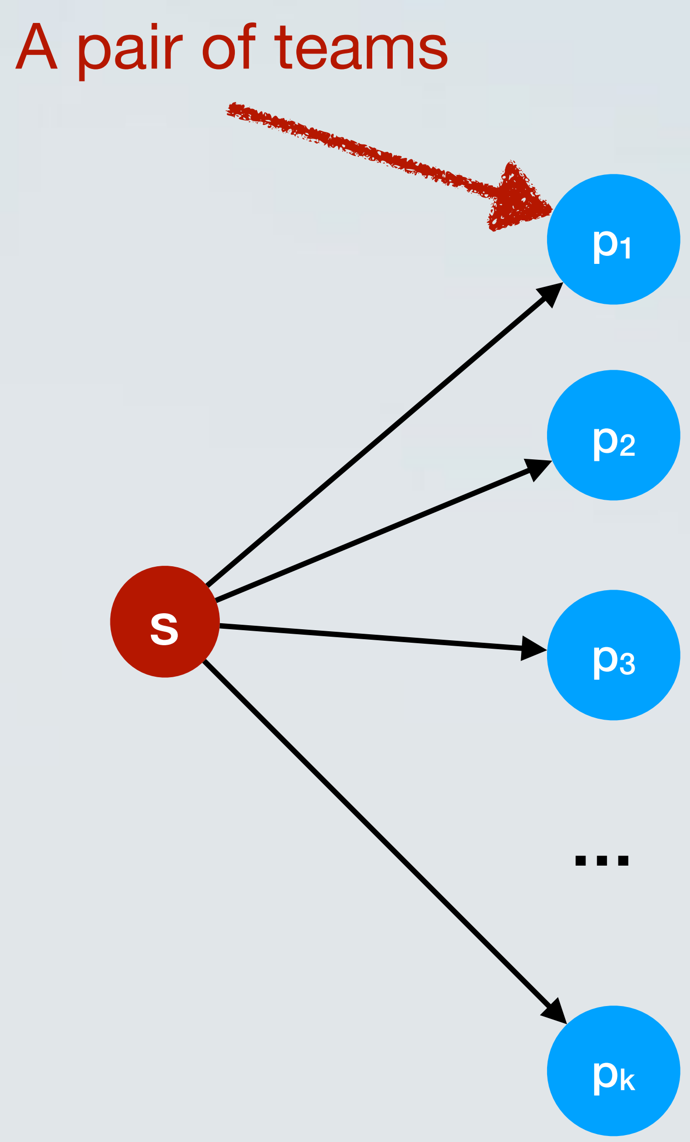

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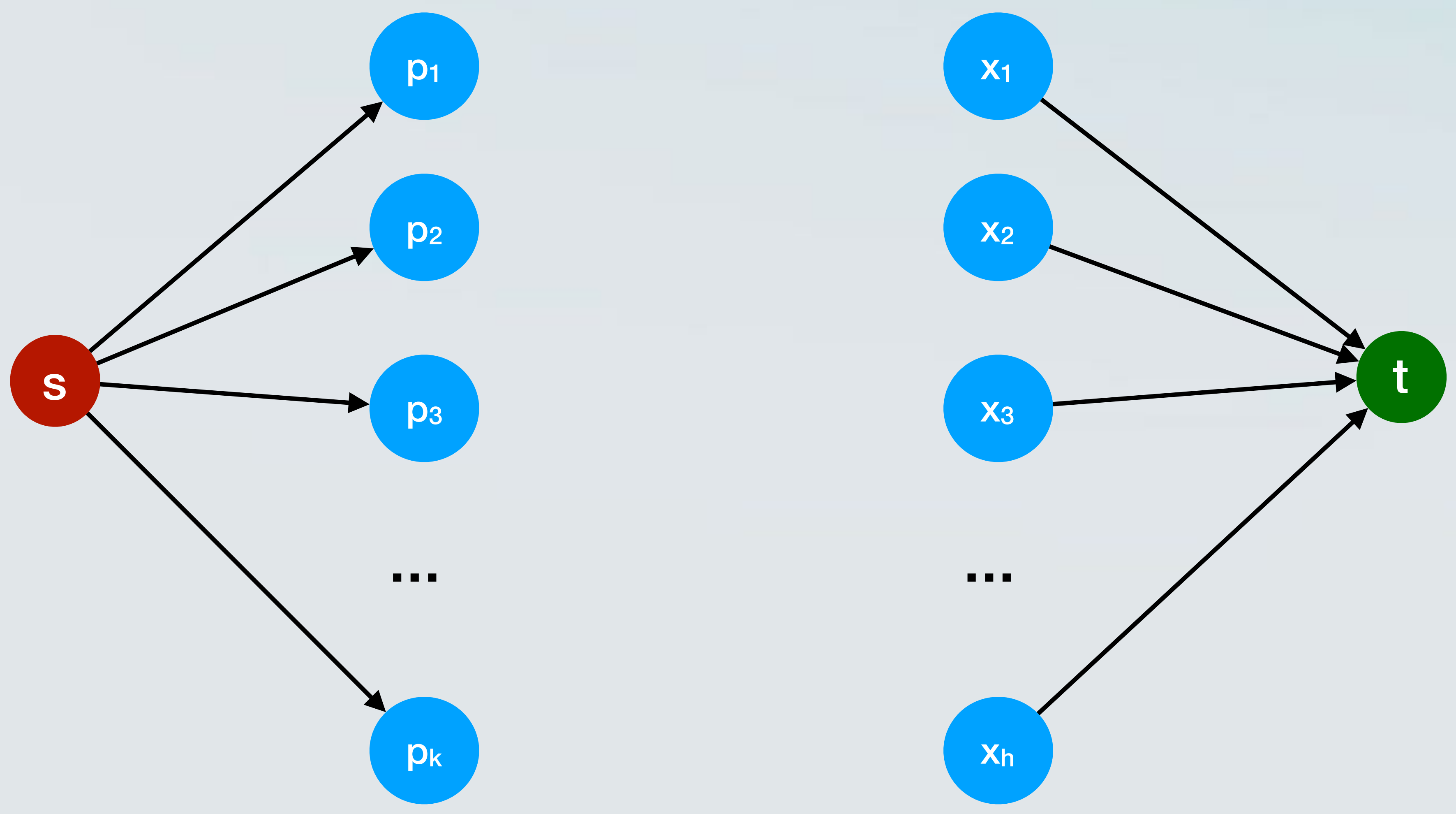

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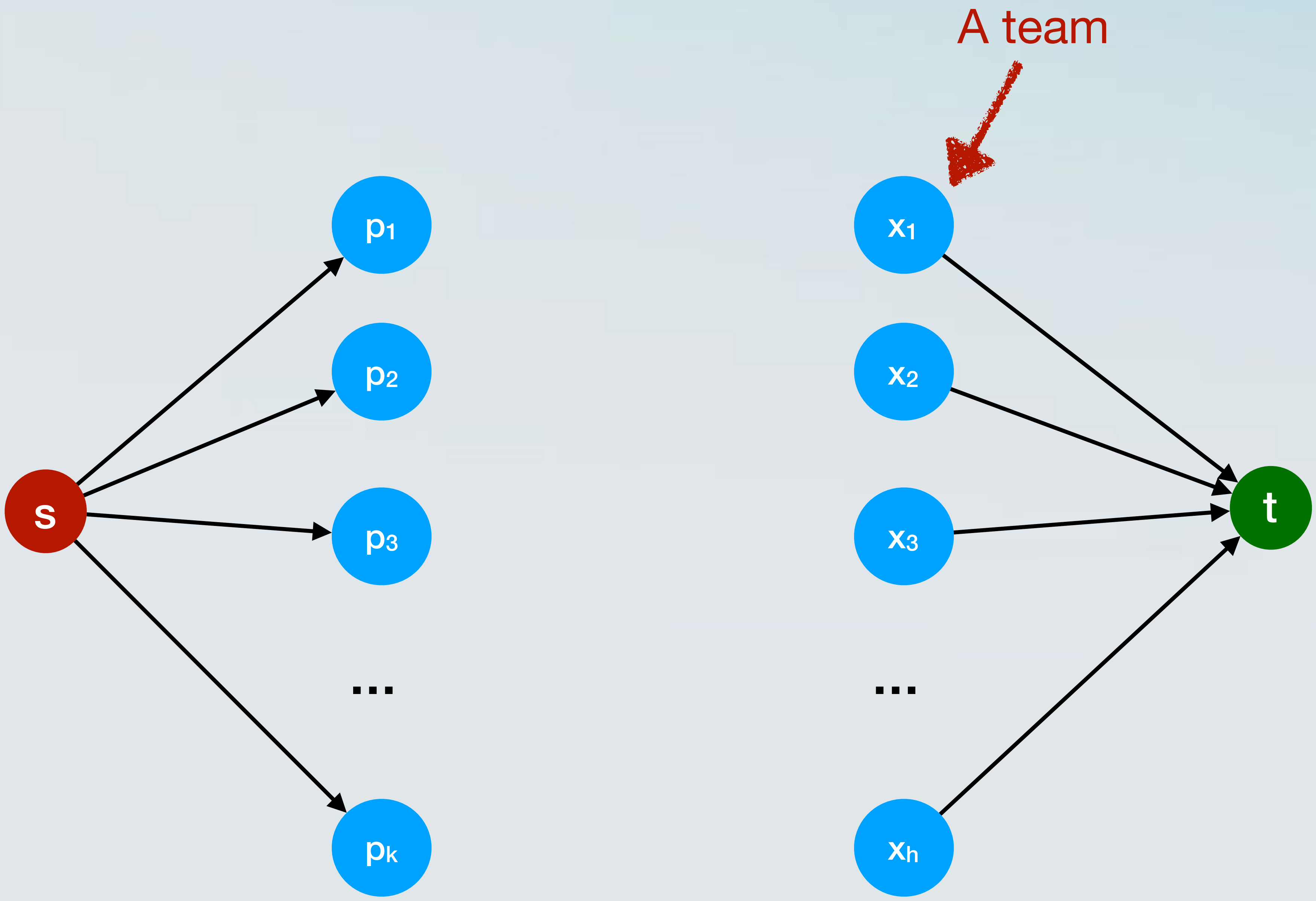

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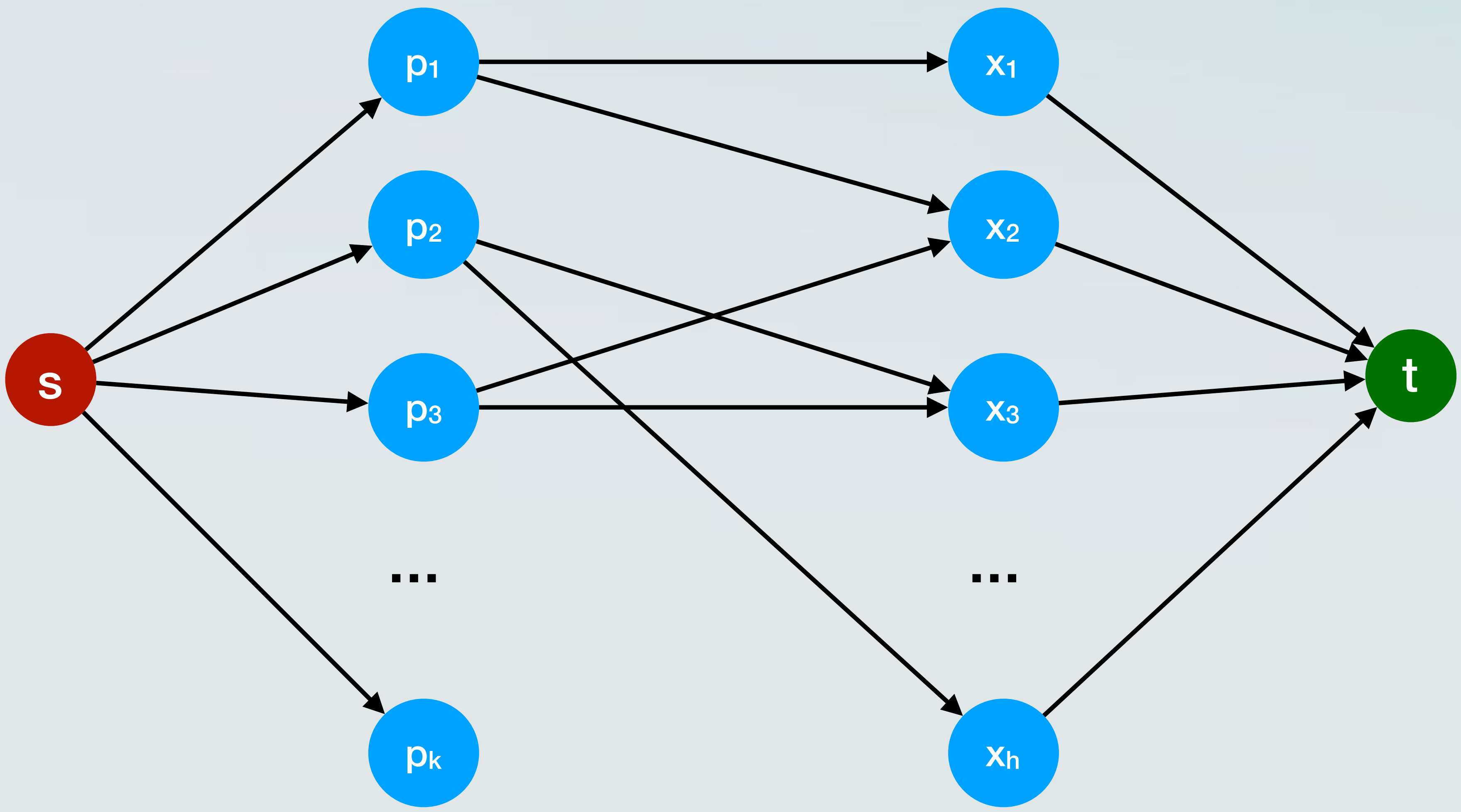

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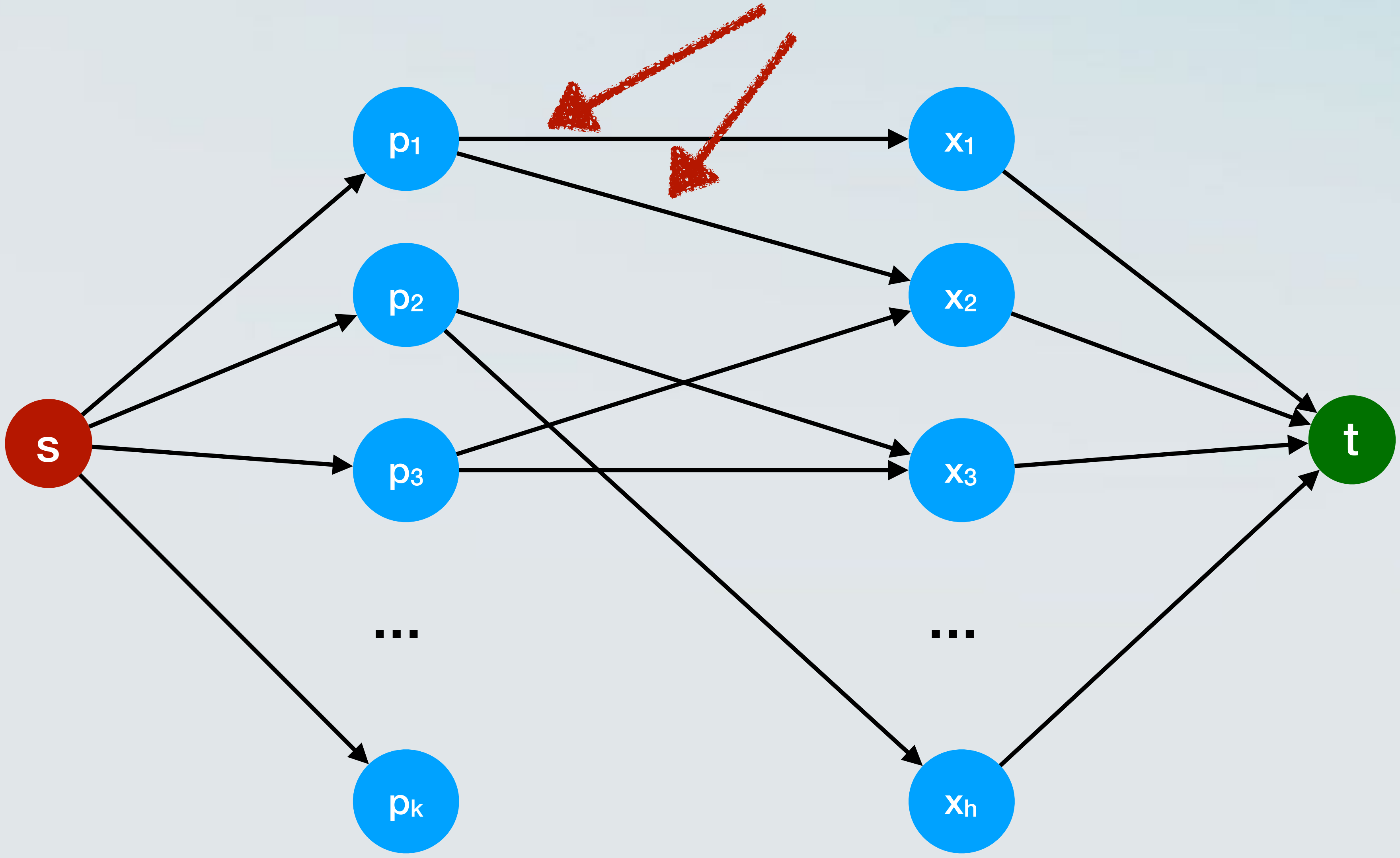
Two edges if teams in p_j still have games to play between them.



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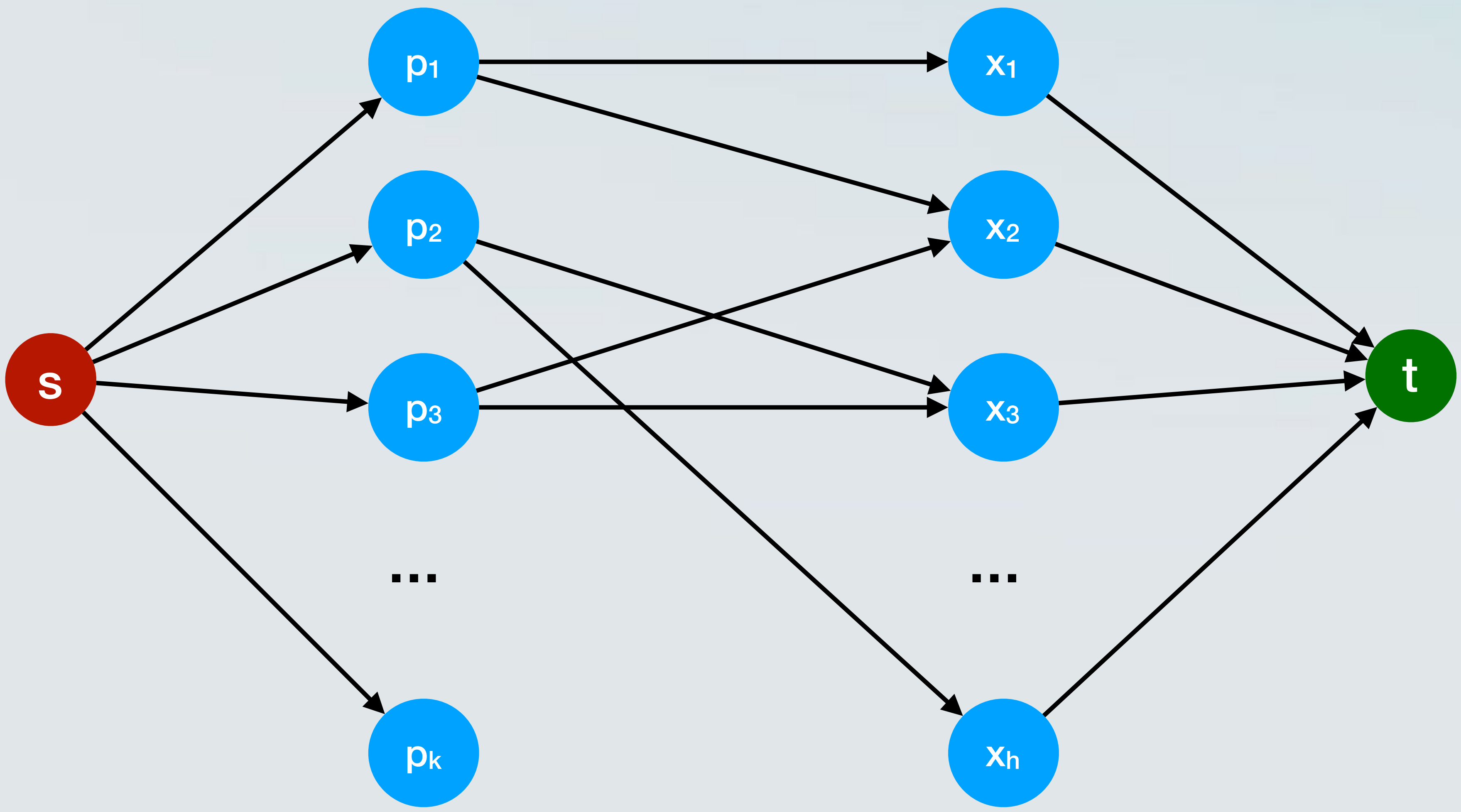

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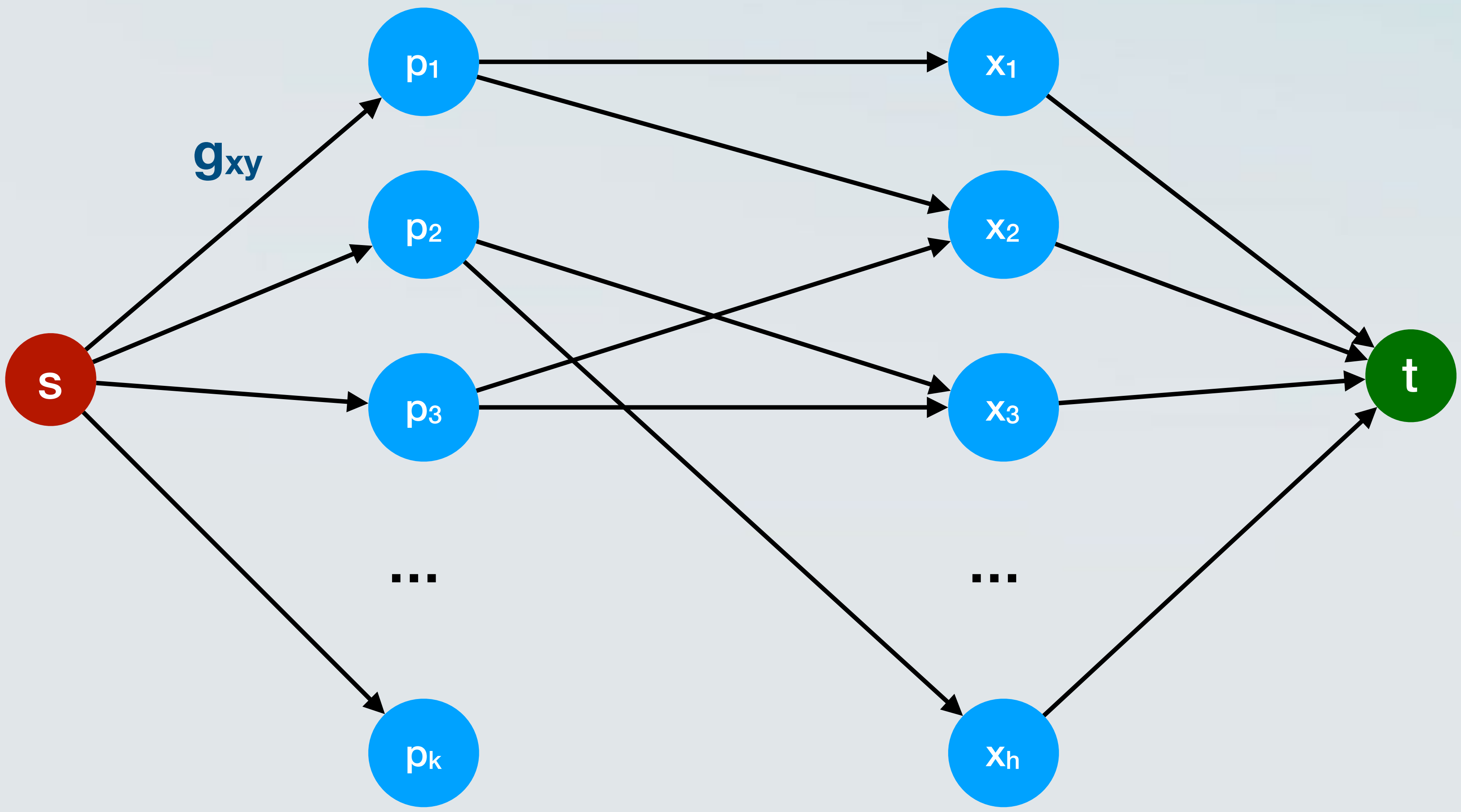

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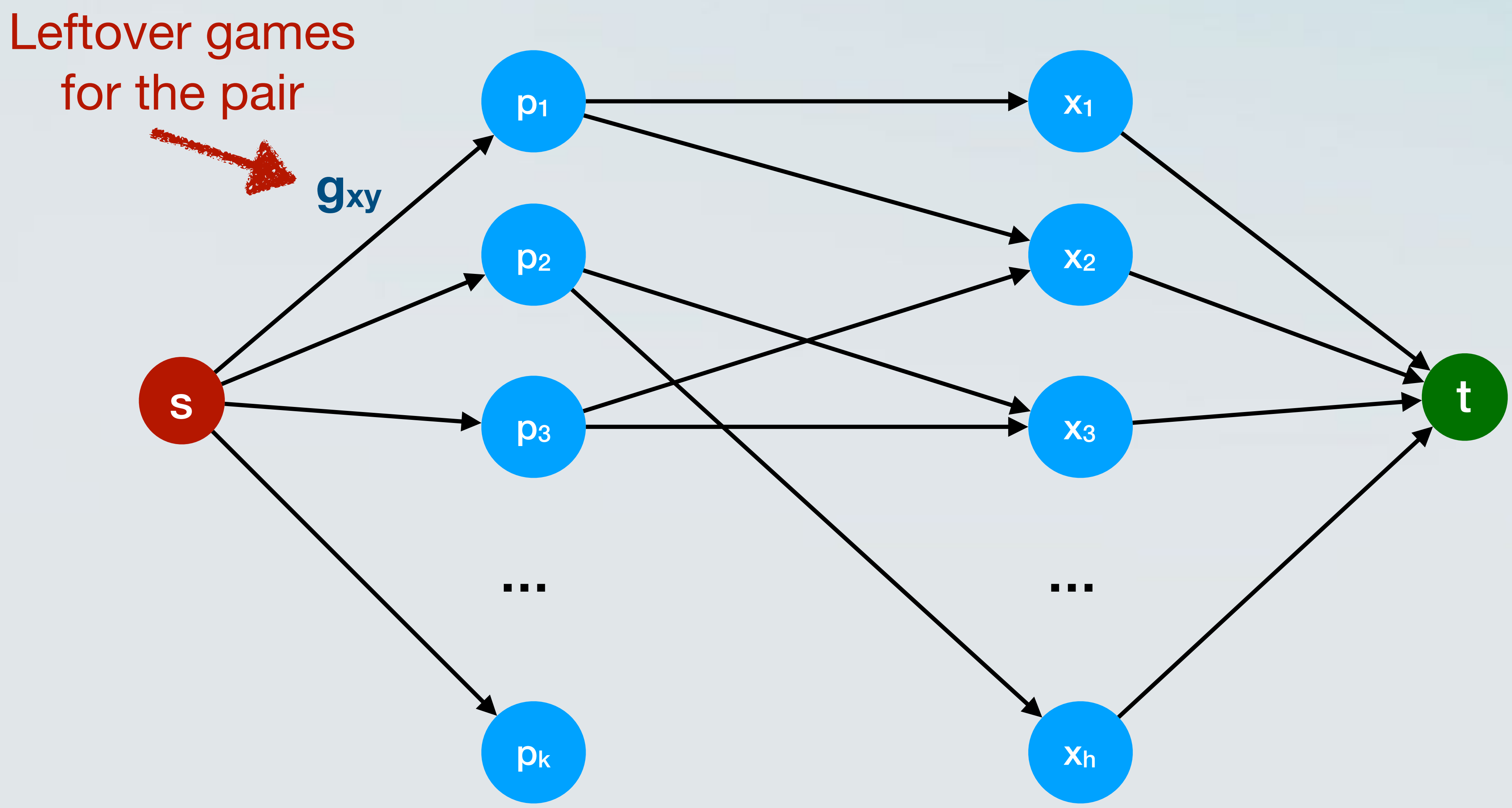

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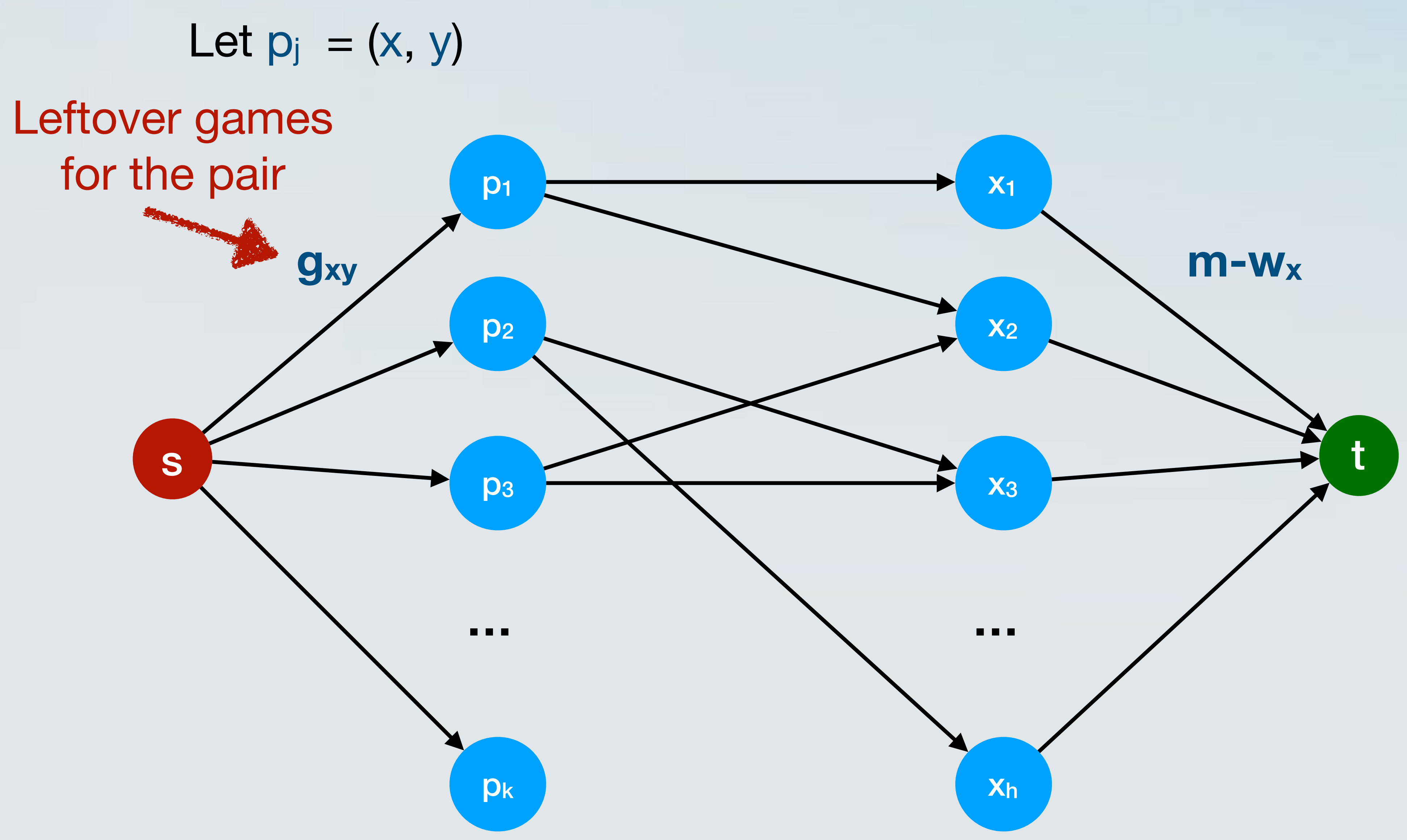

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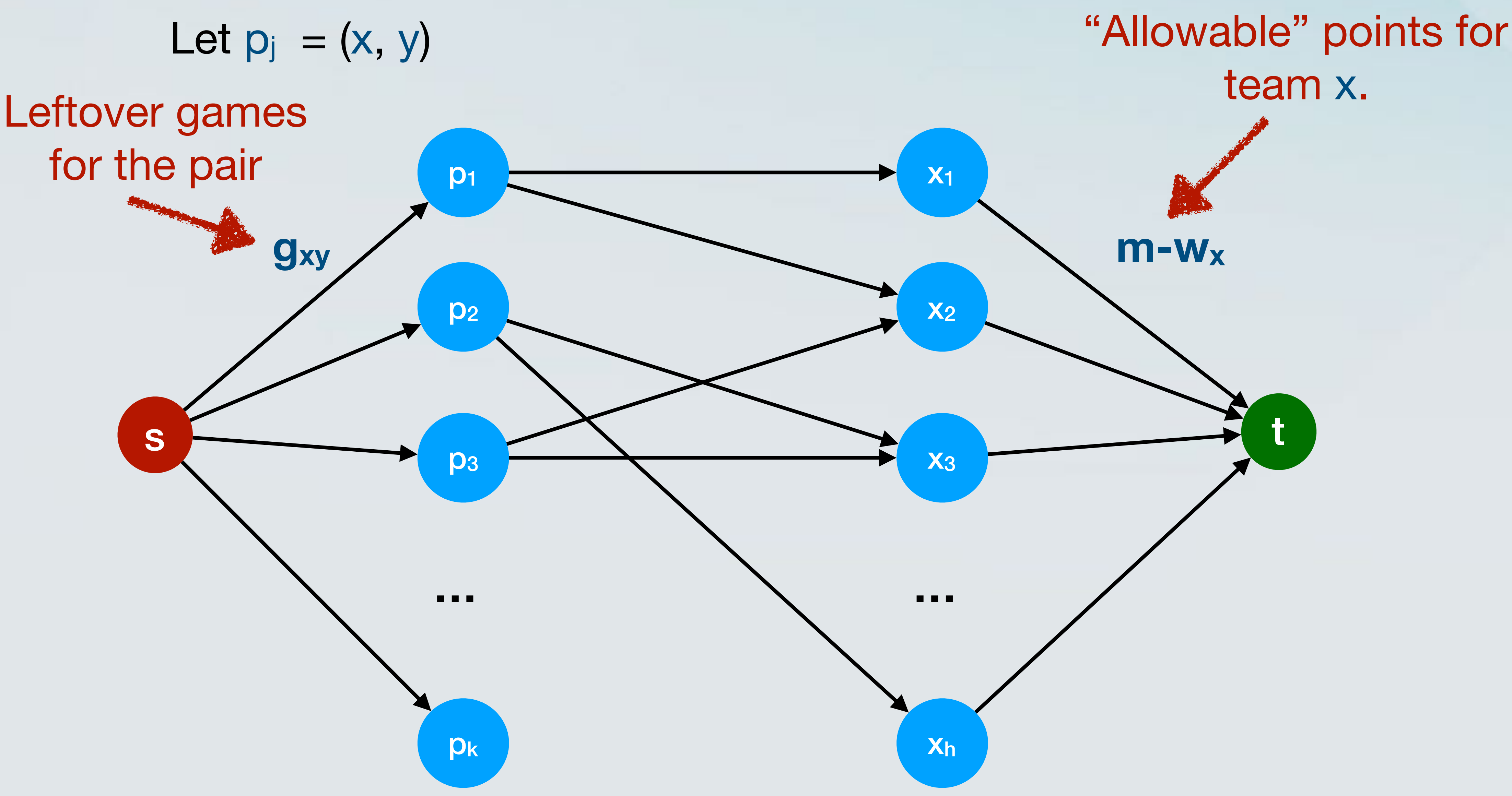

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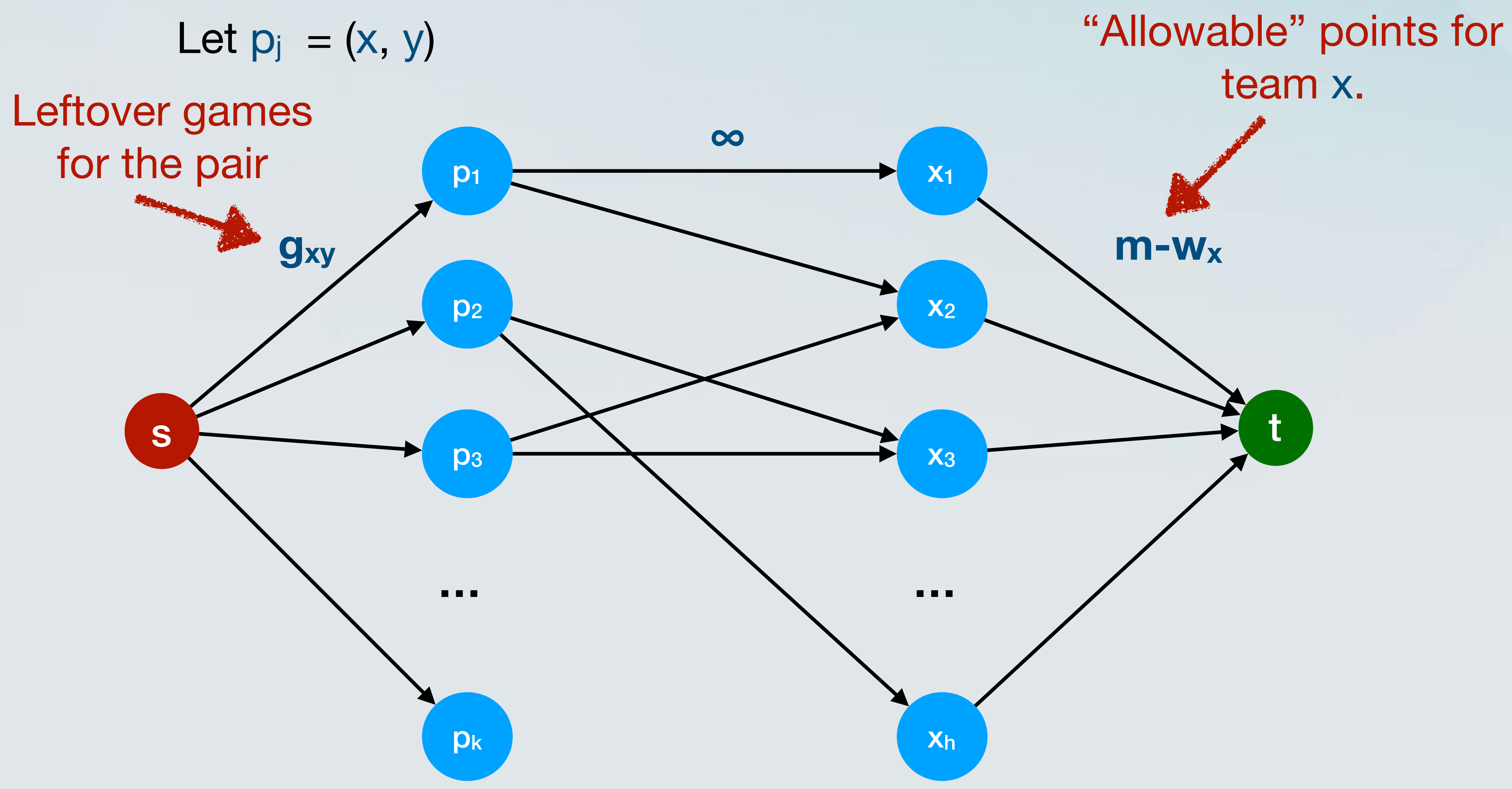

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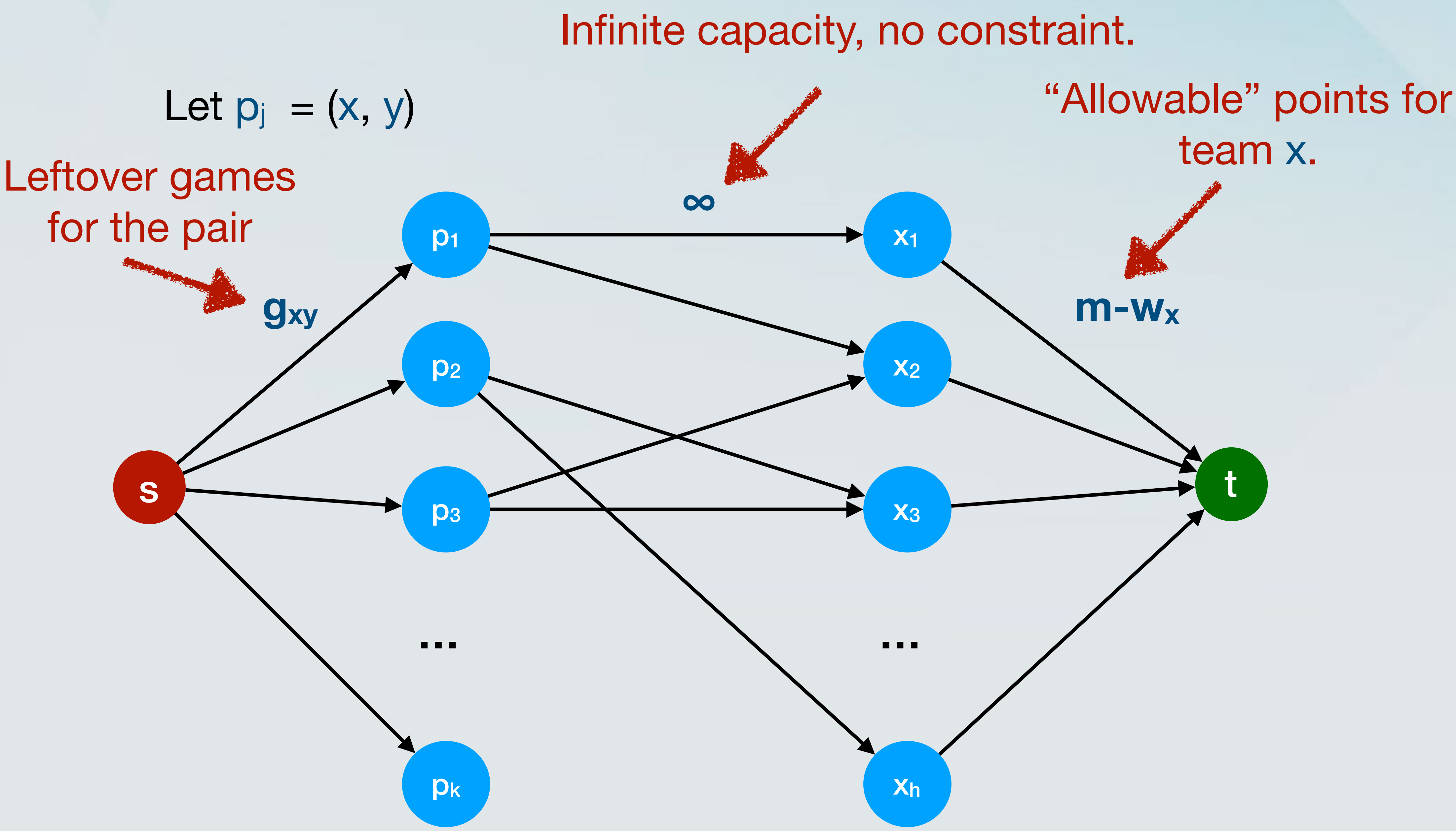

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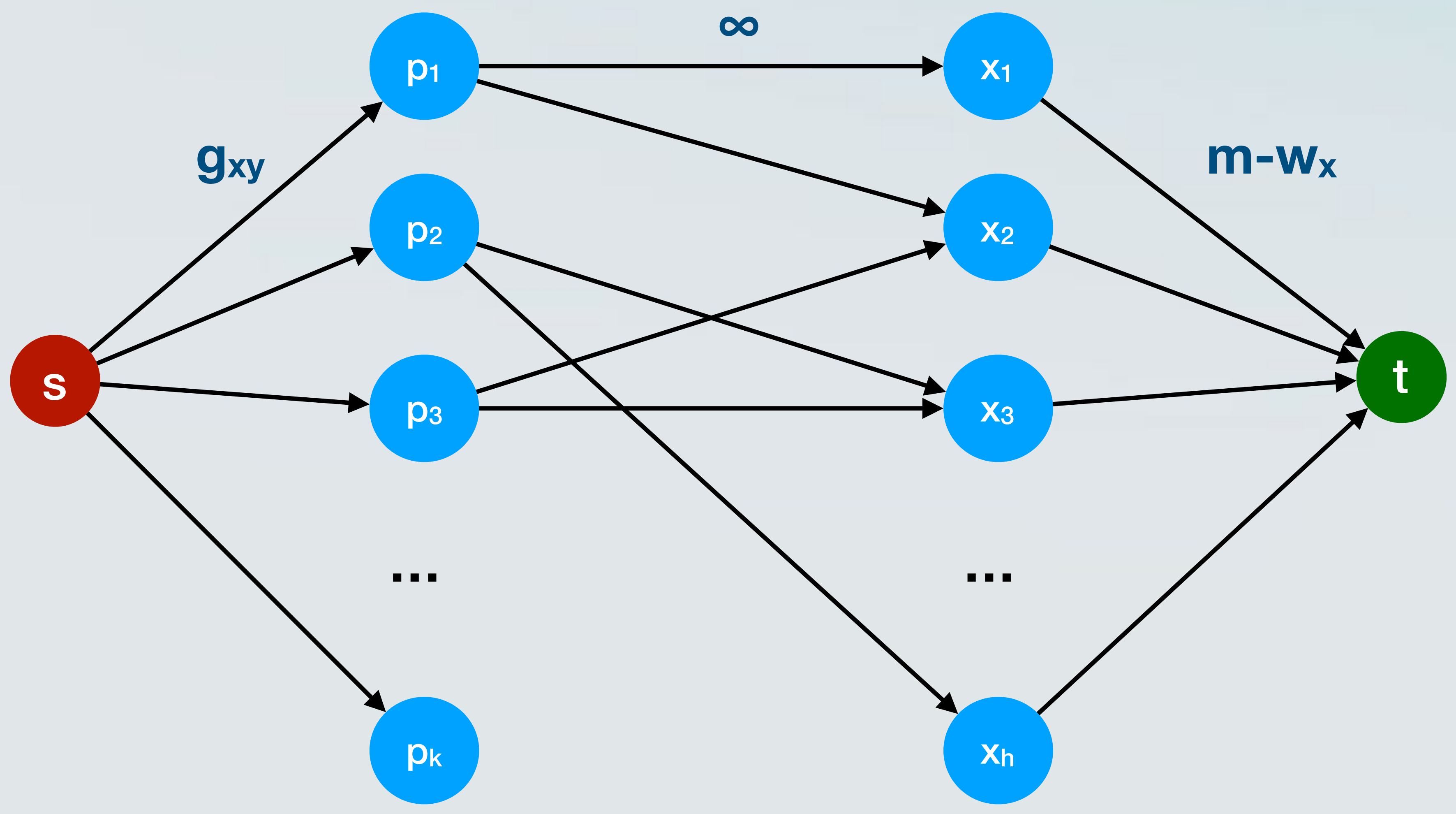

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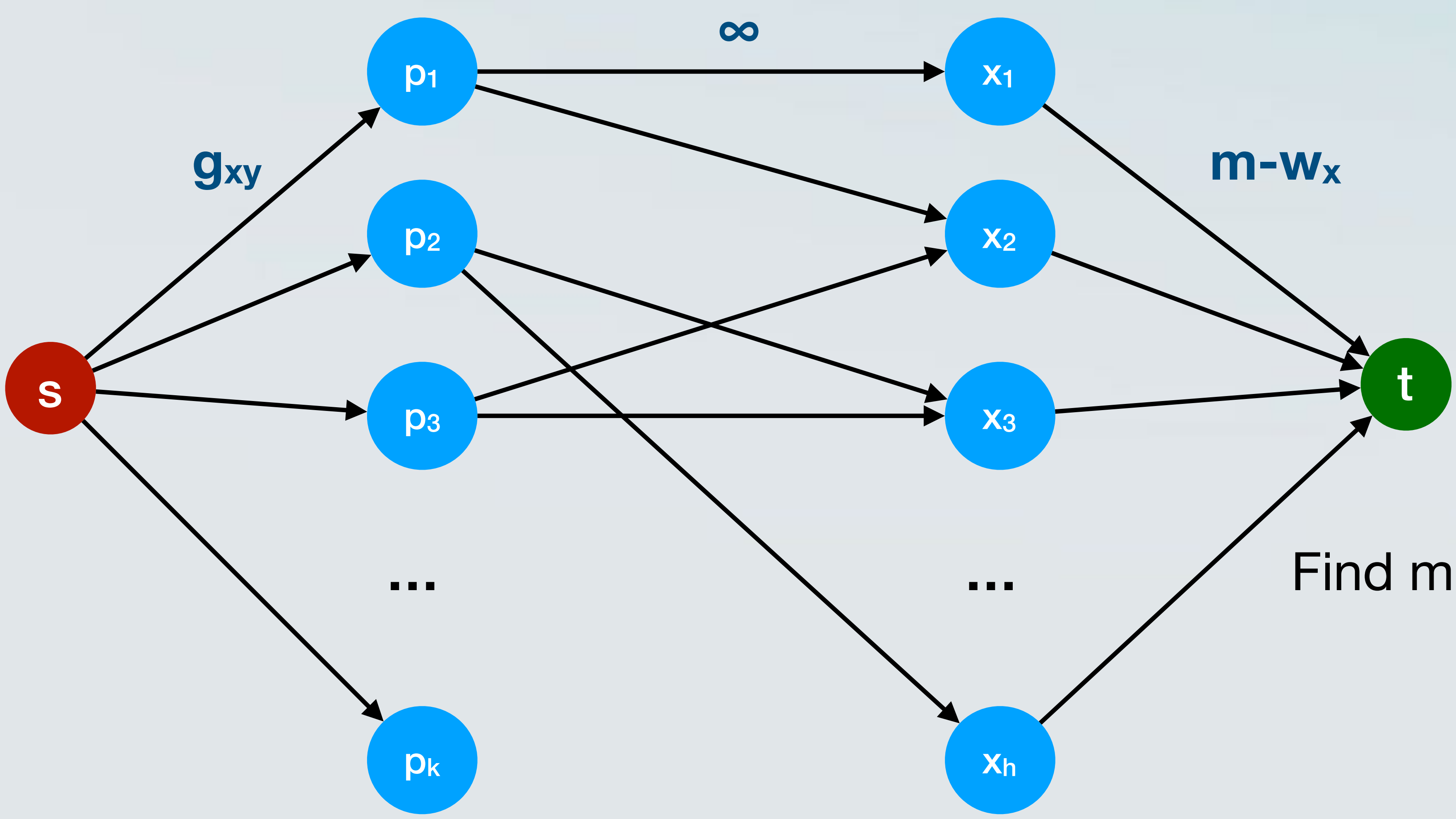

Find max flow in the network

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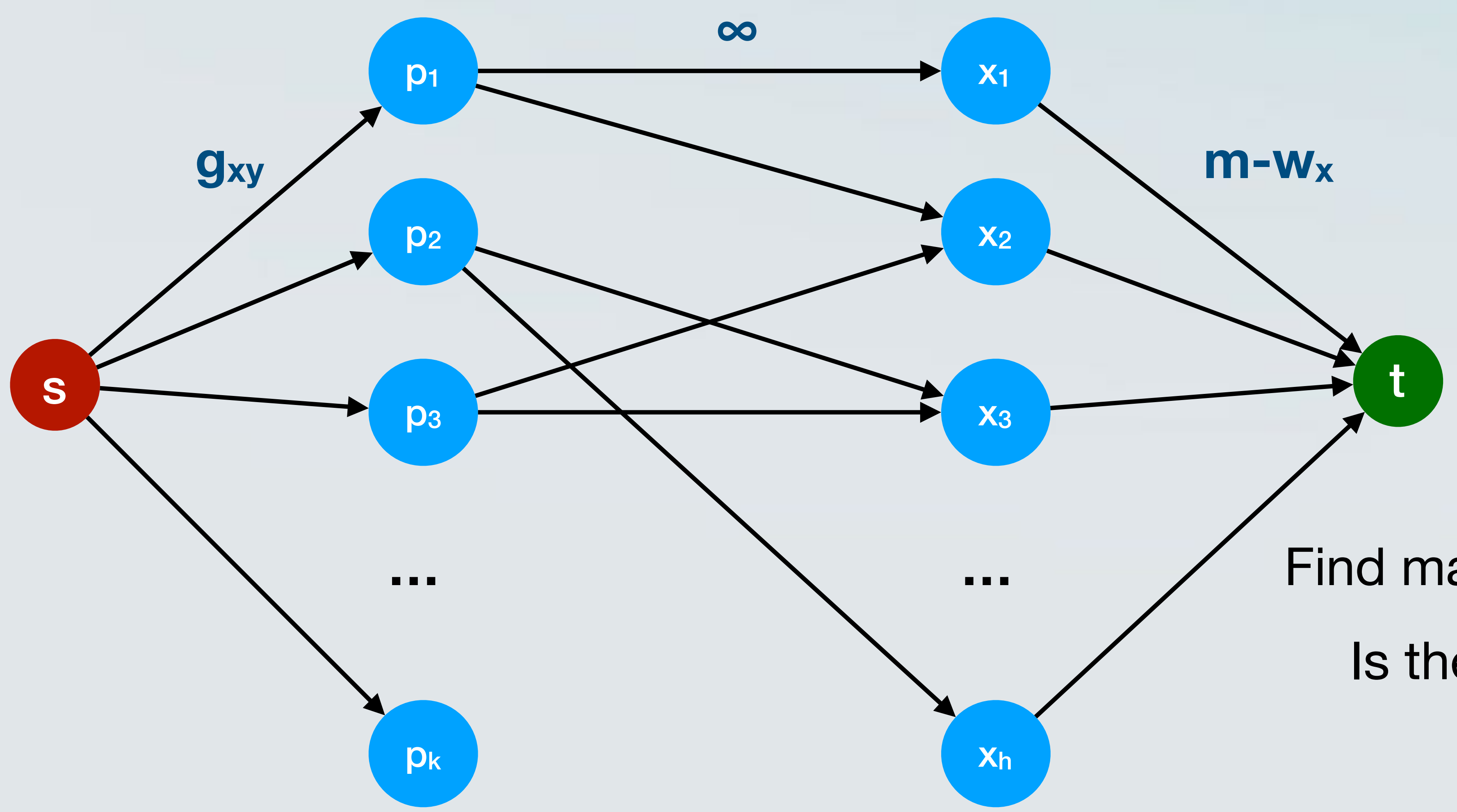

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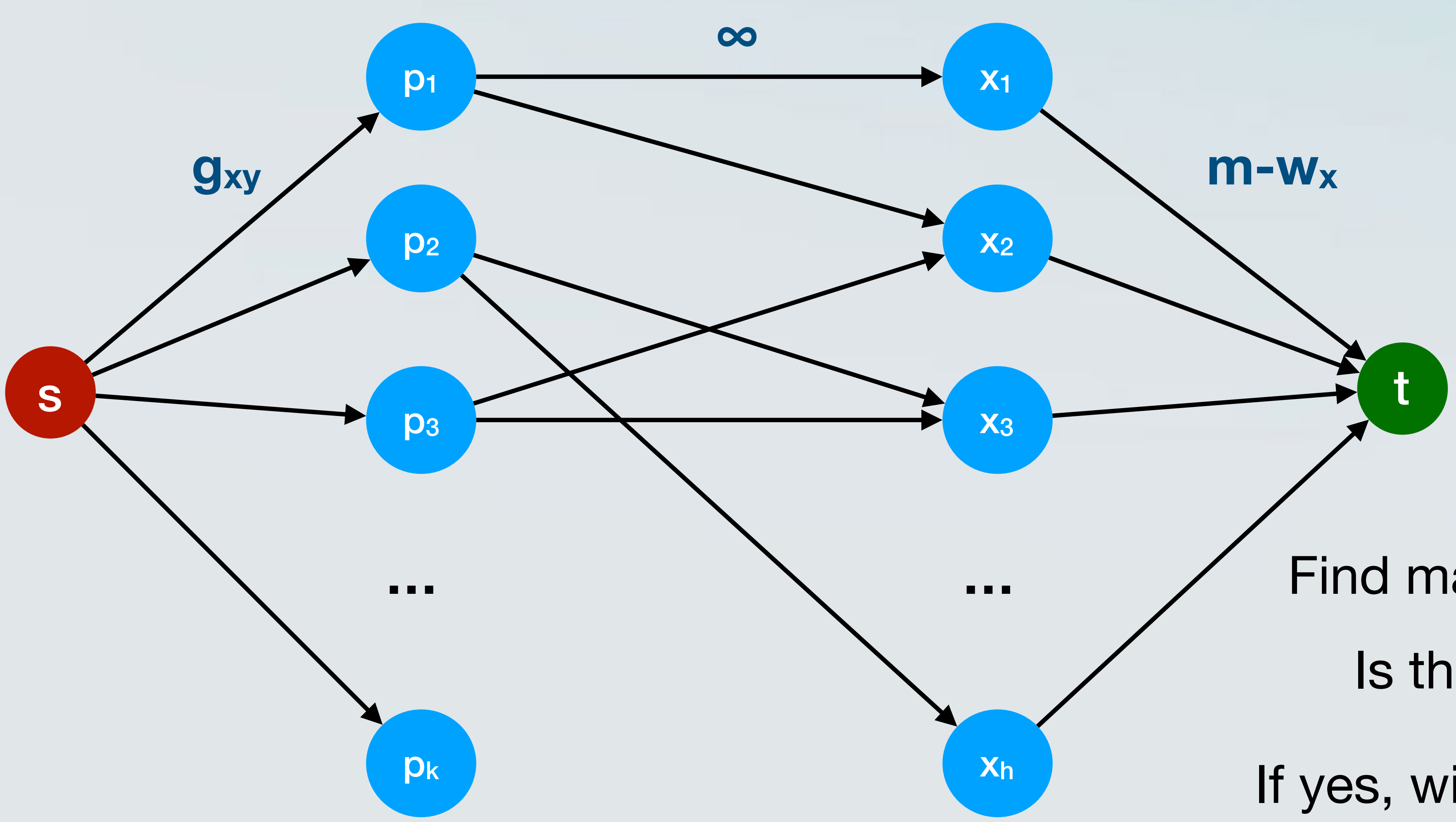

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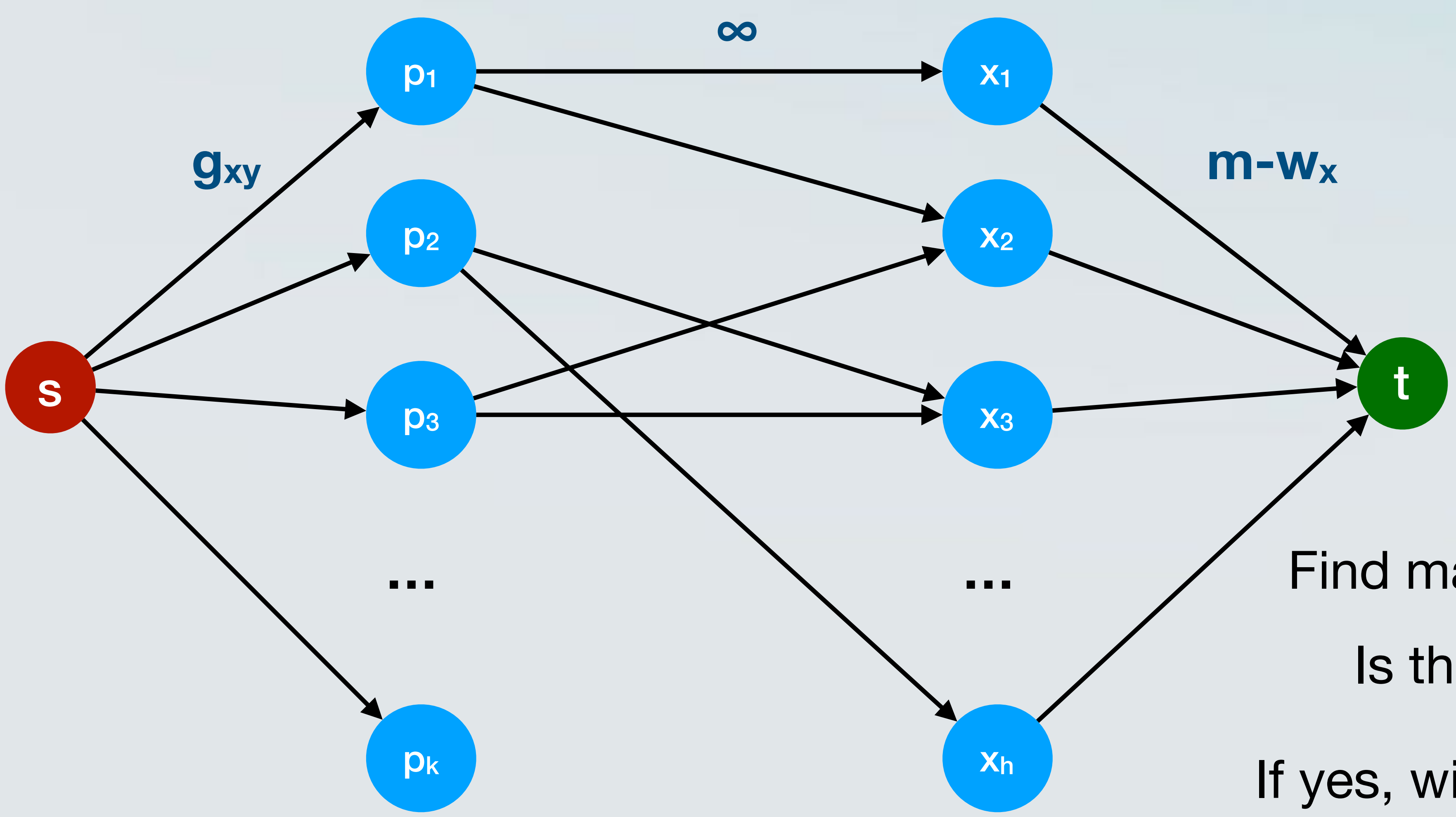

Find max flow in the network

Is the value at least g^*?

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If no, winning is not possible

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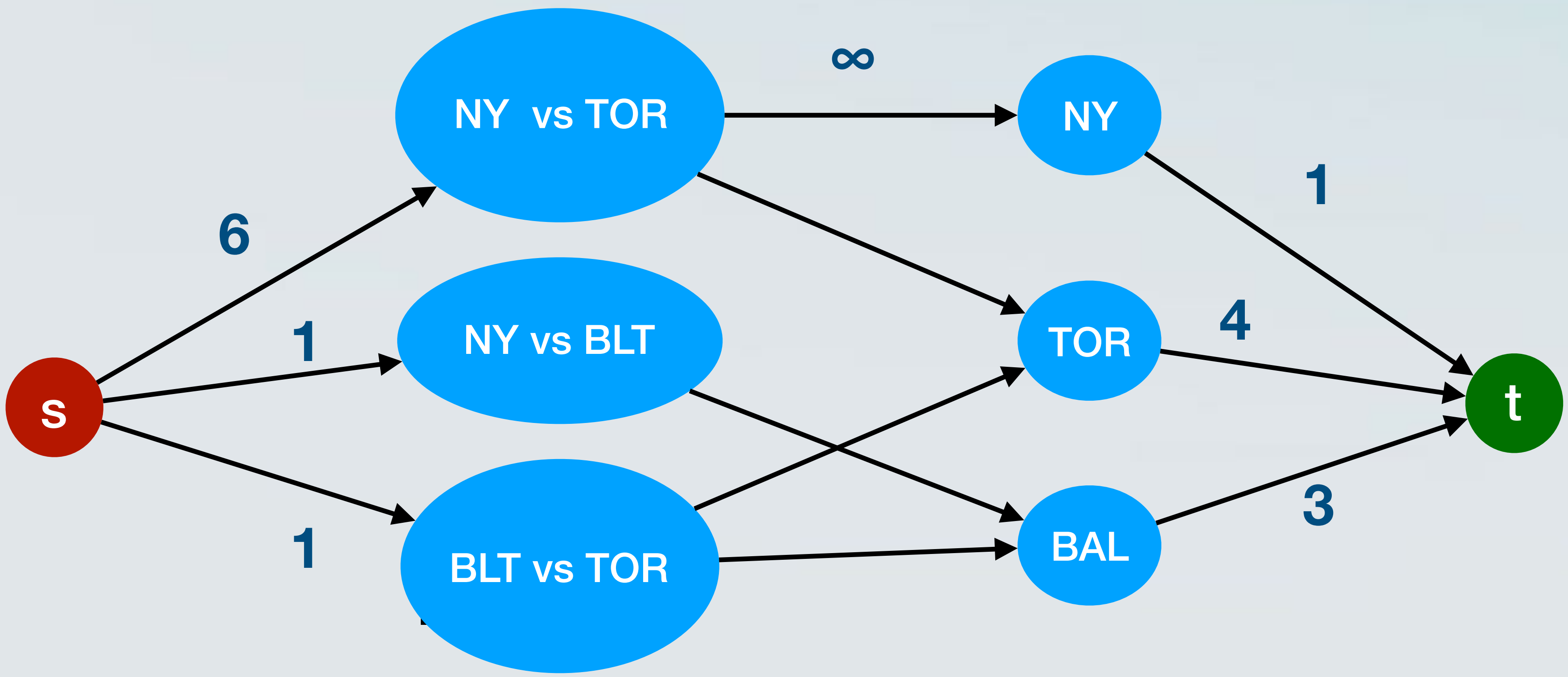
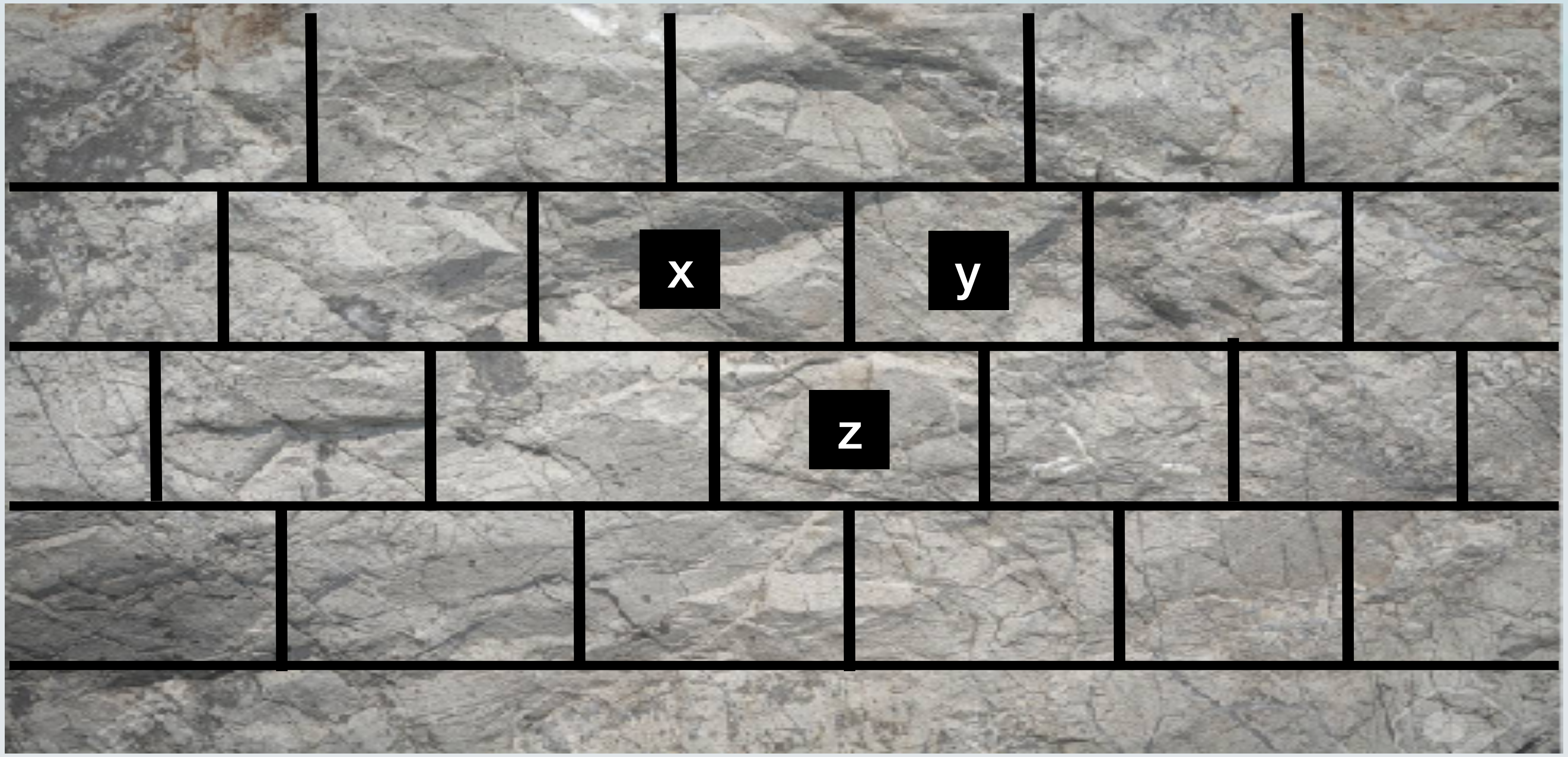


m = 91

Open pit mining

- We extract blocks of earth from the surface, trying to find gold.
- Each block z that we mine has
  - a value p_z
  - a mining cost c_z
- Constraint: We can not mine a block z unless we mine the two blocks x and y on top of it.
- We want to earn as much money as possible.

Open pit mining



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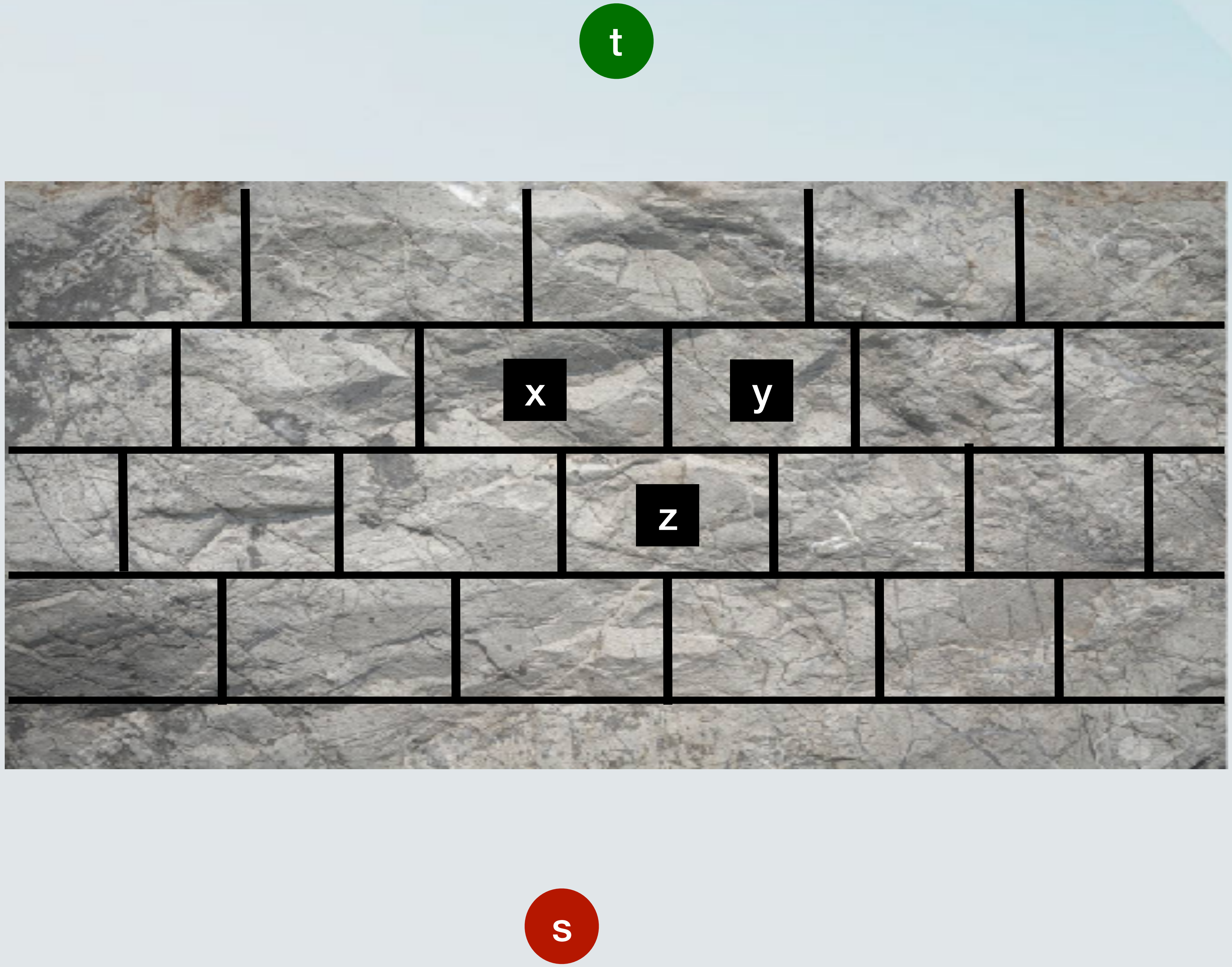
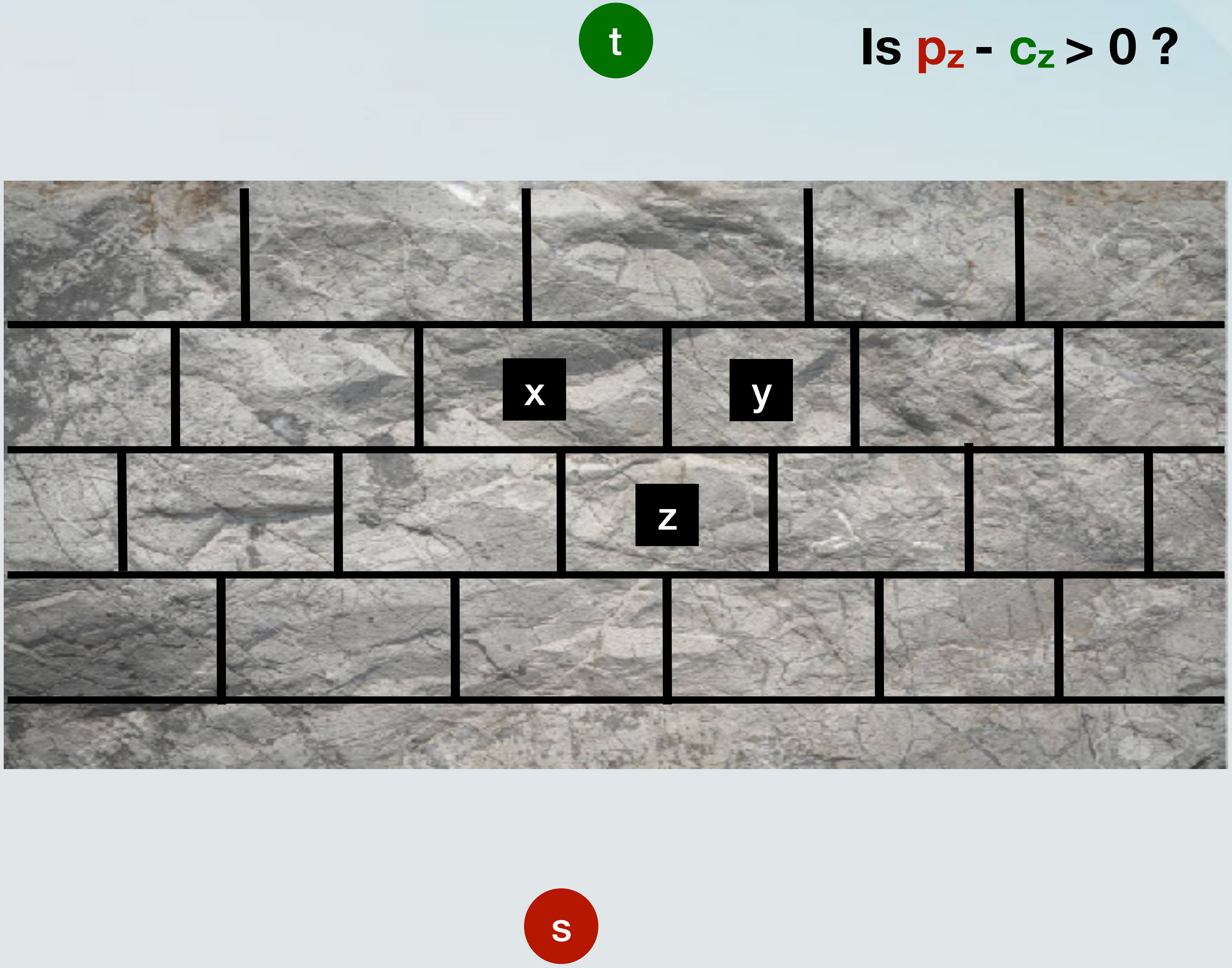
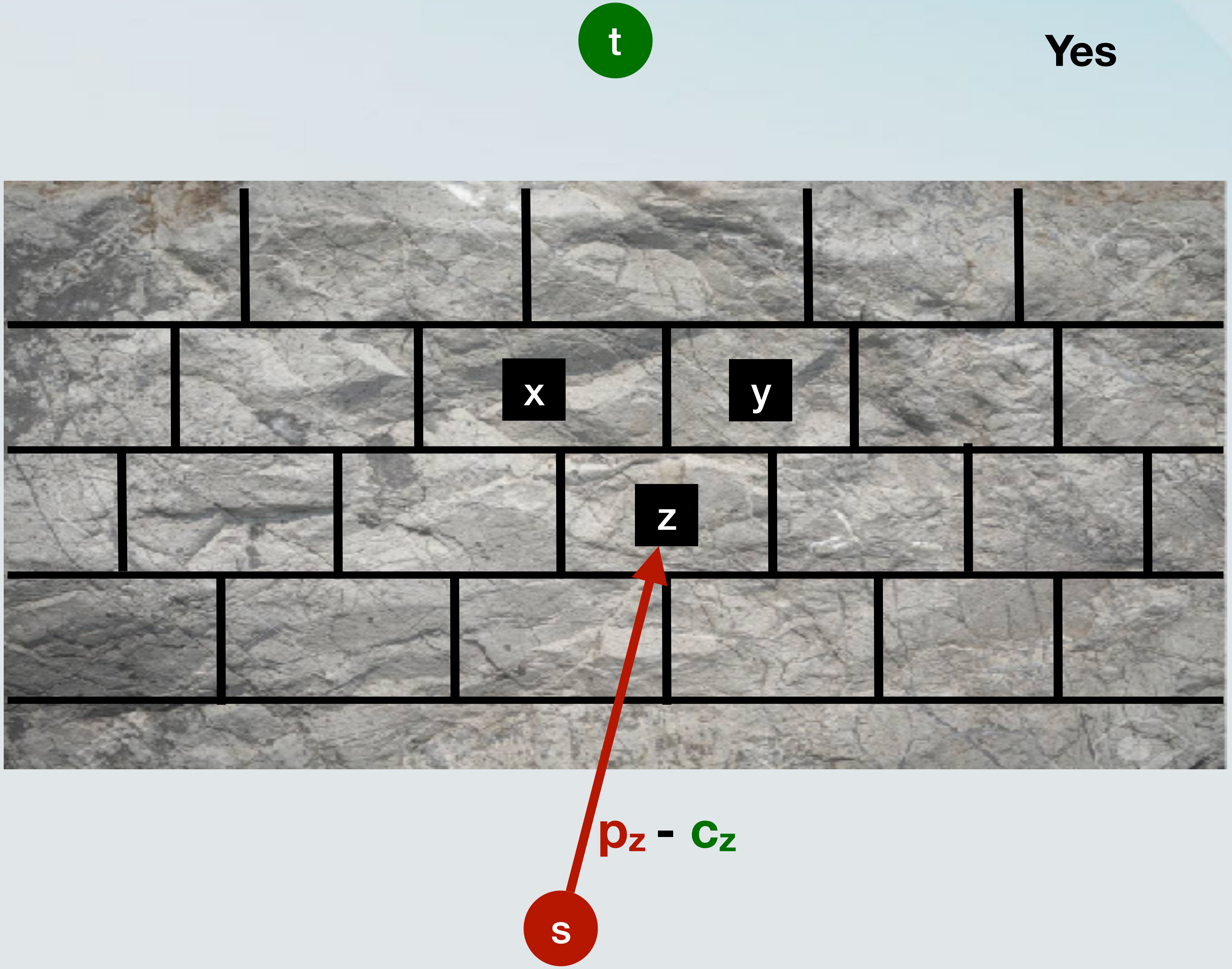


From pits to flows

Is $p_z - c_z > 0$?

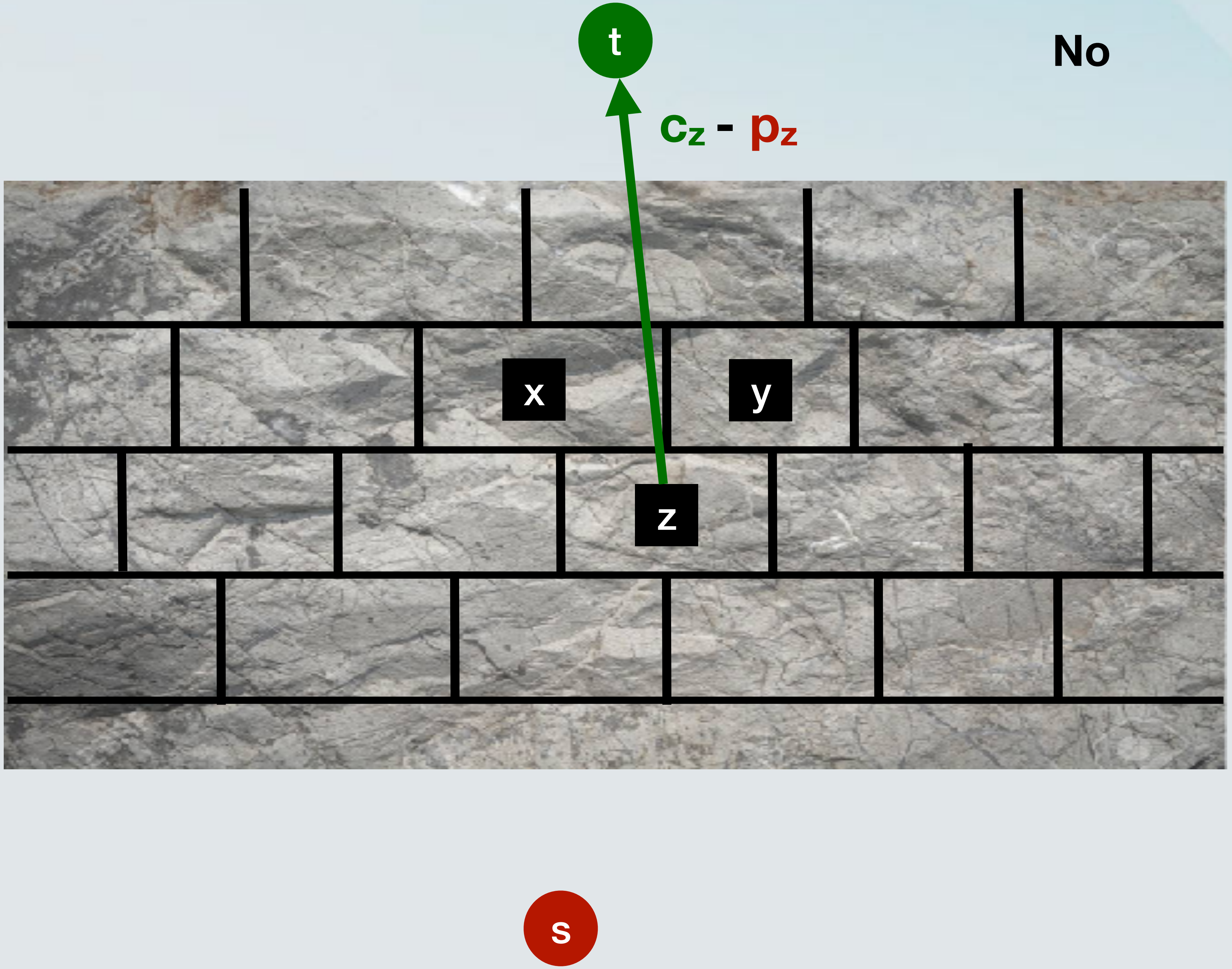


From pits to flows

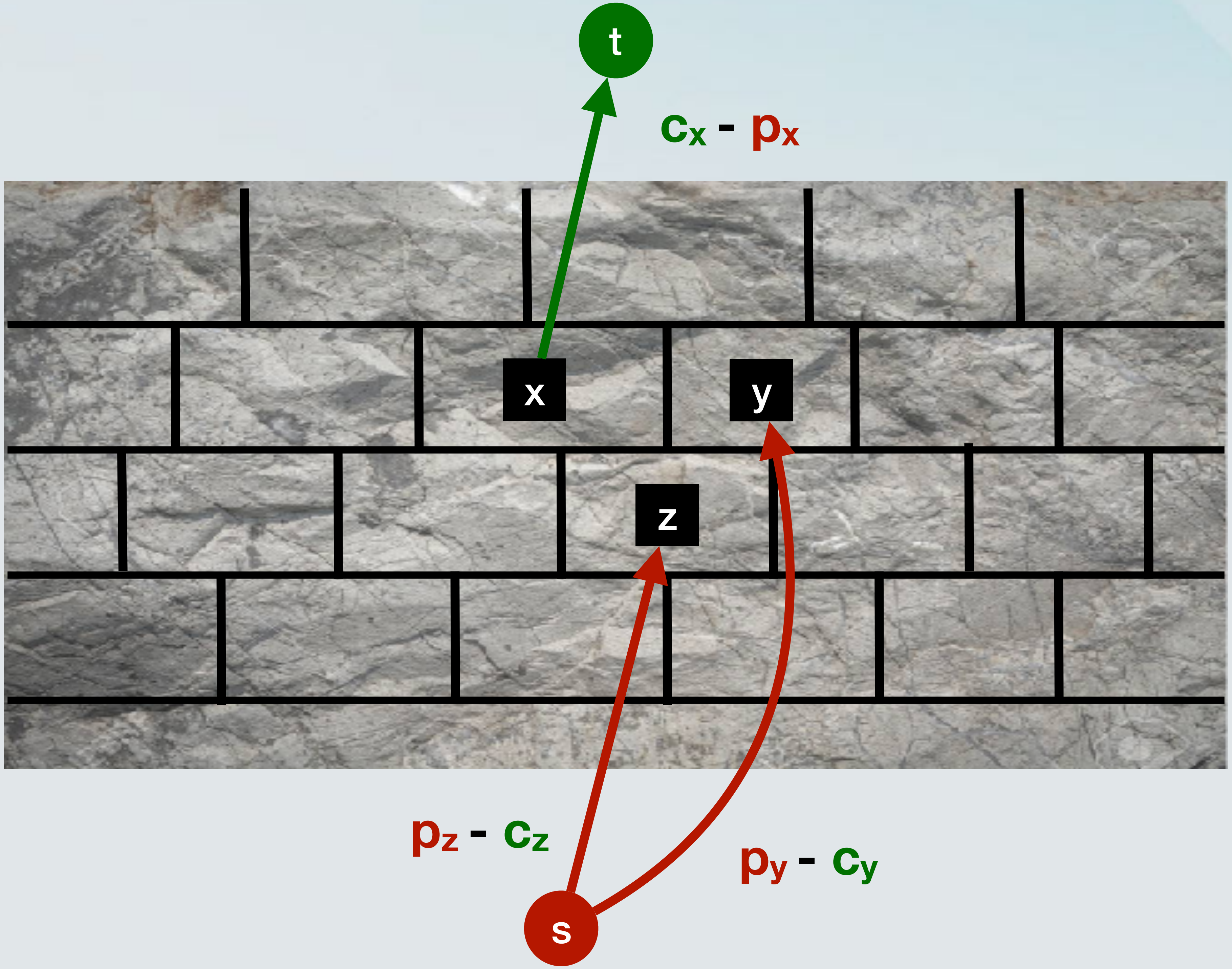


From pits to flows

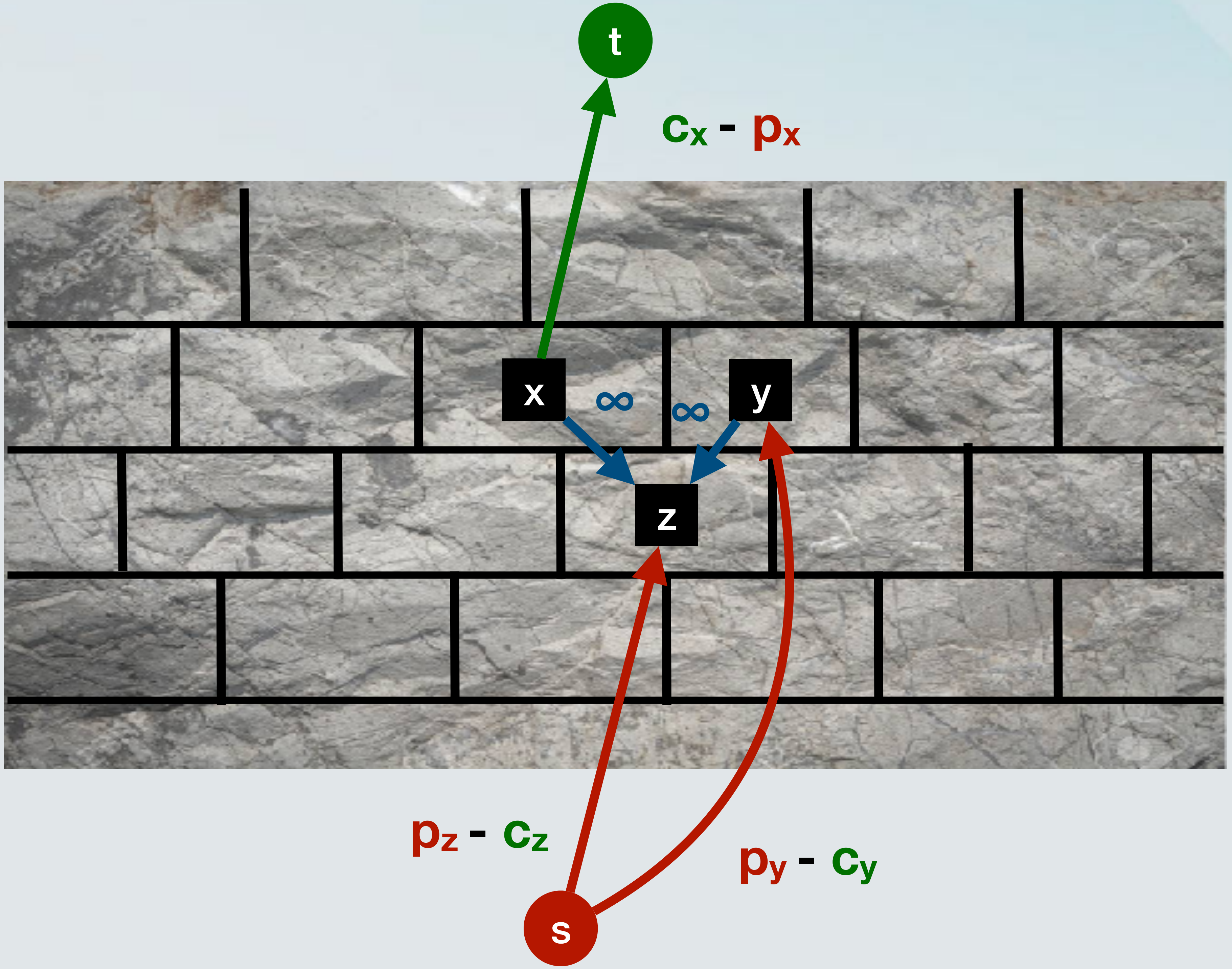
No



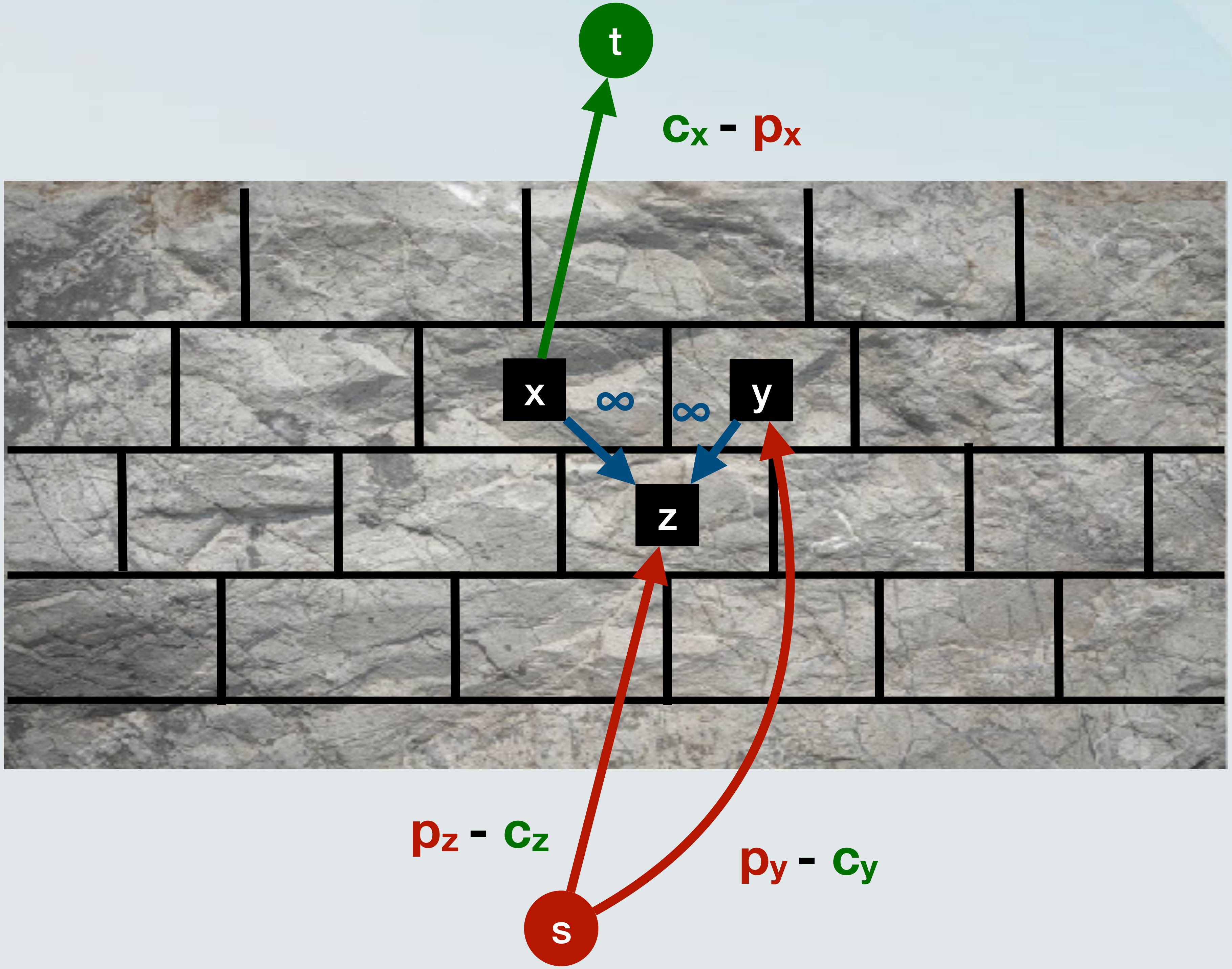
From pits to flows



From pits to flows



From pits to cuts



From pits to cuts

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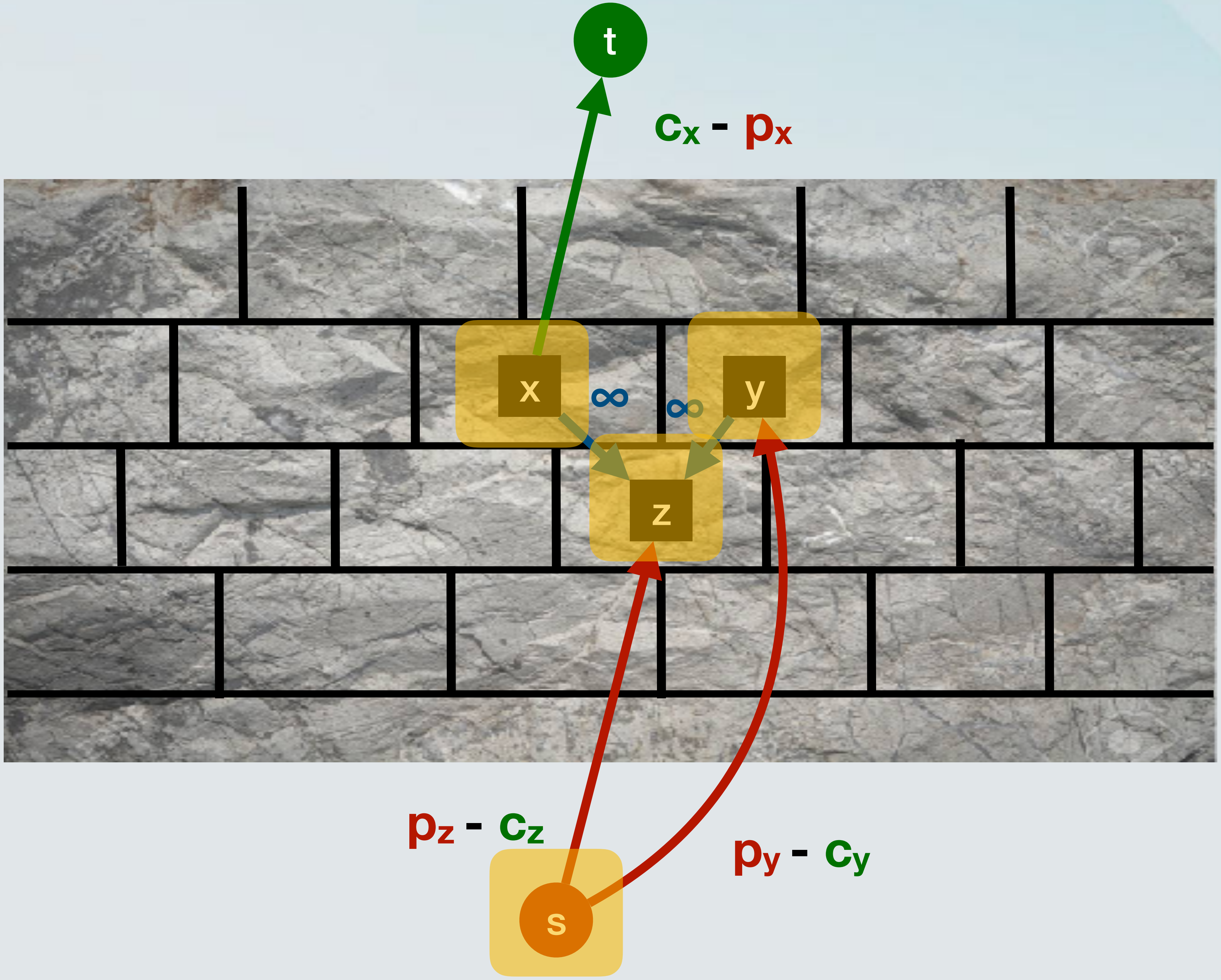
    - If it contains z, it must contain x and y.
- We will mine S - {s}.
    - Feasibility guaranteed by the above fact.
    - Optimality?

Optimality of our mining set.

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

From pits to cuts



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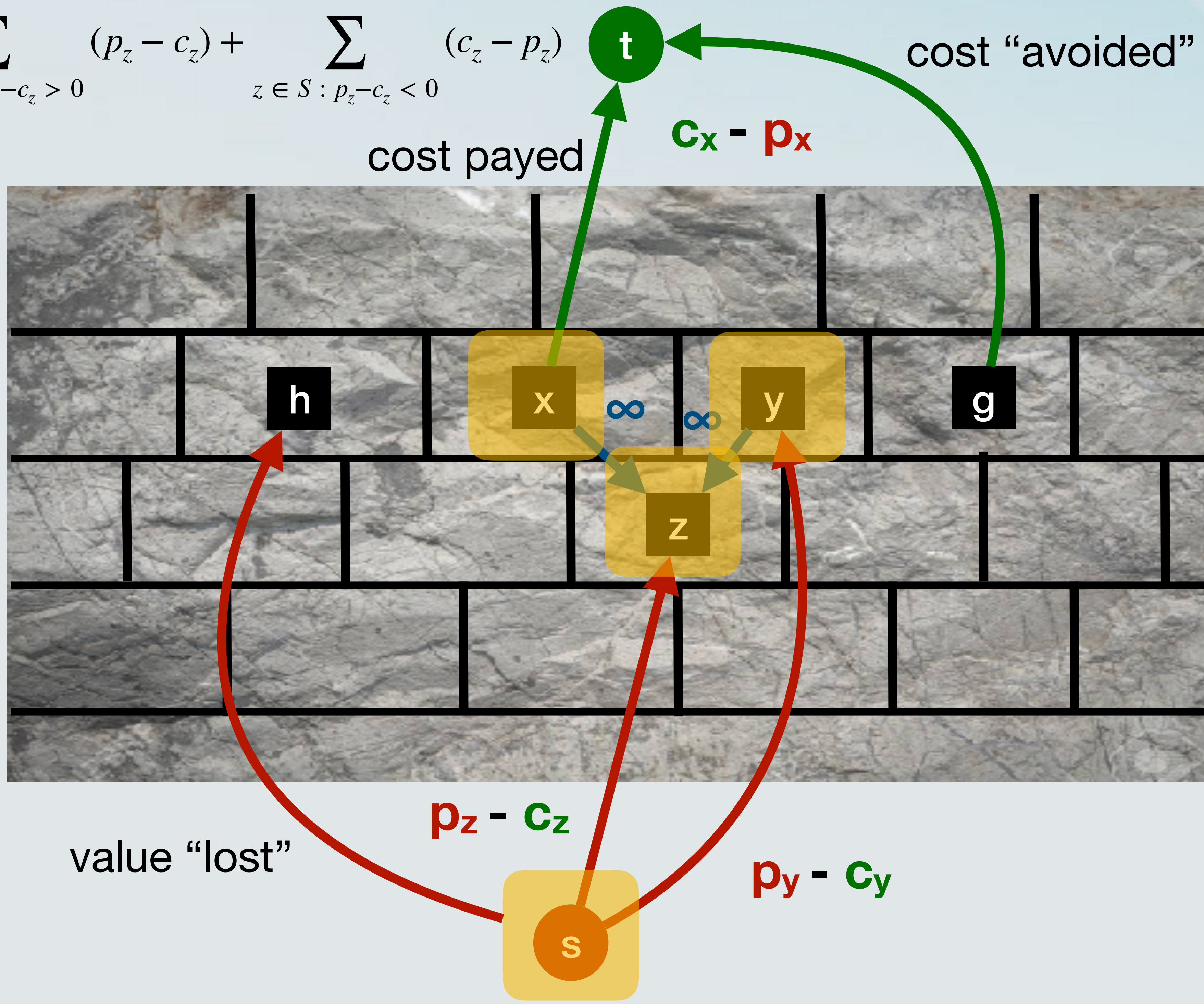

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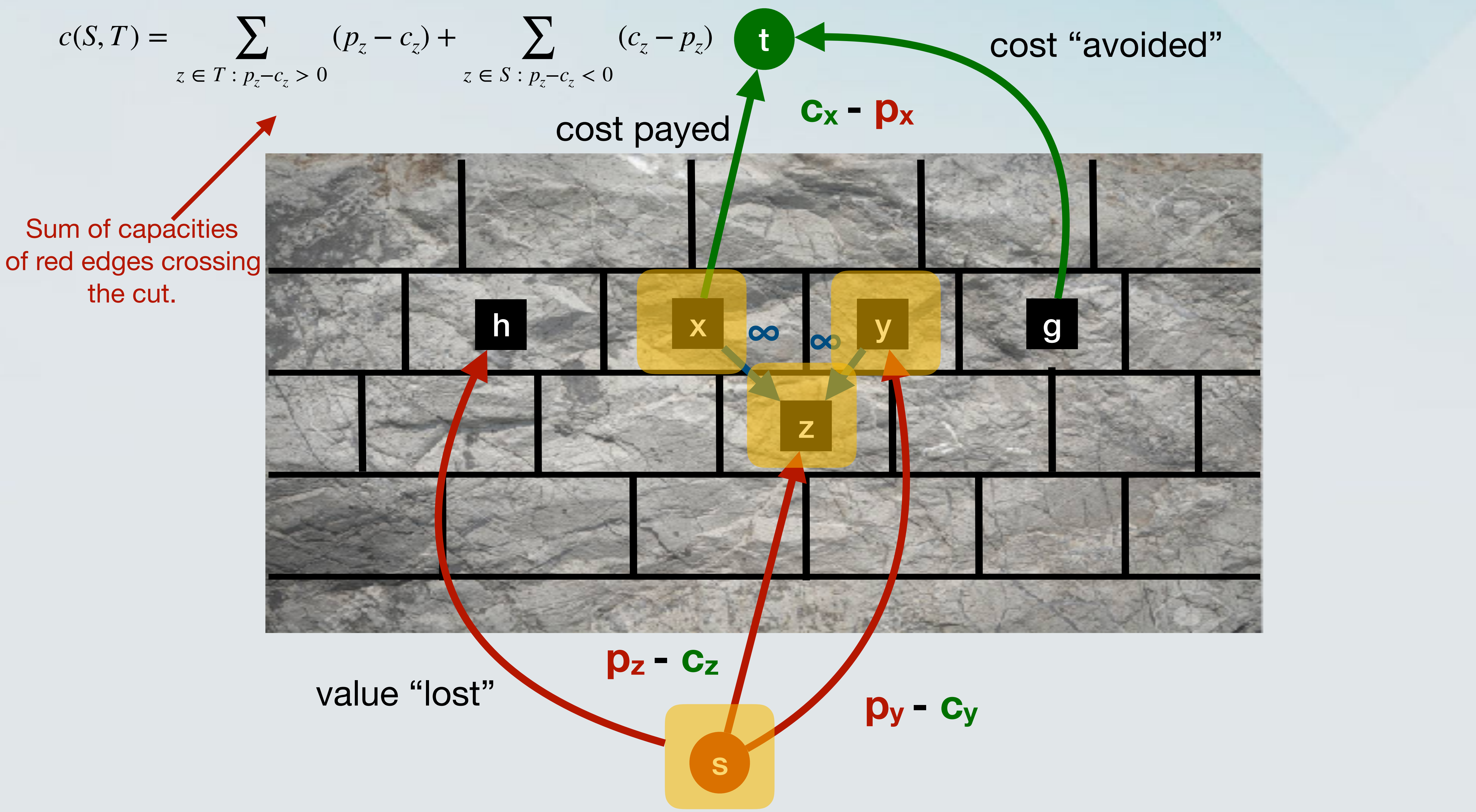

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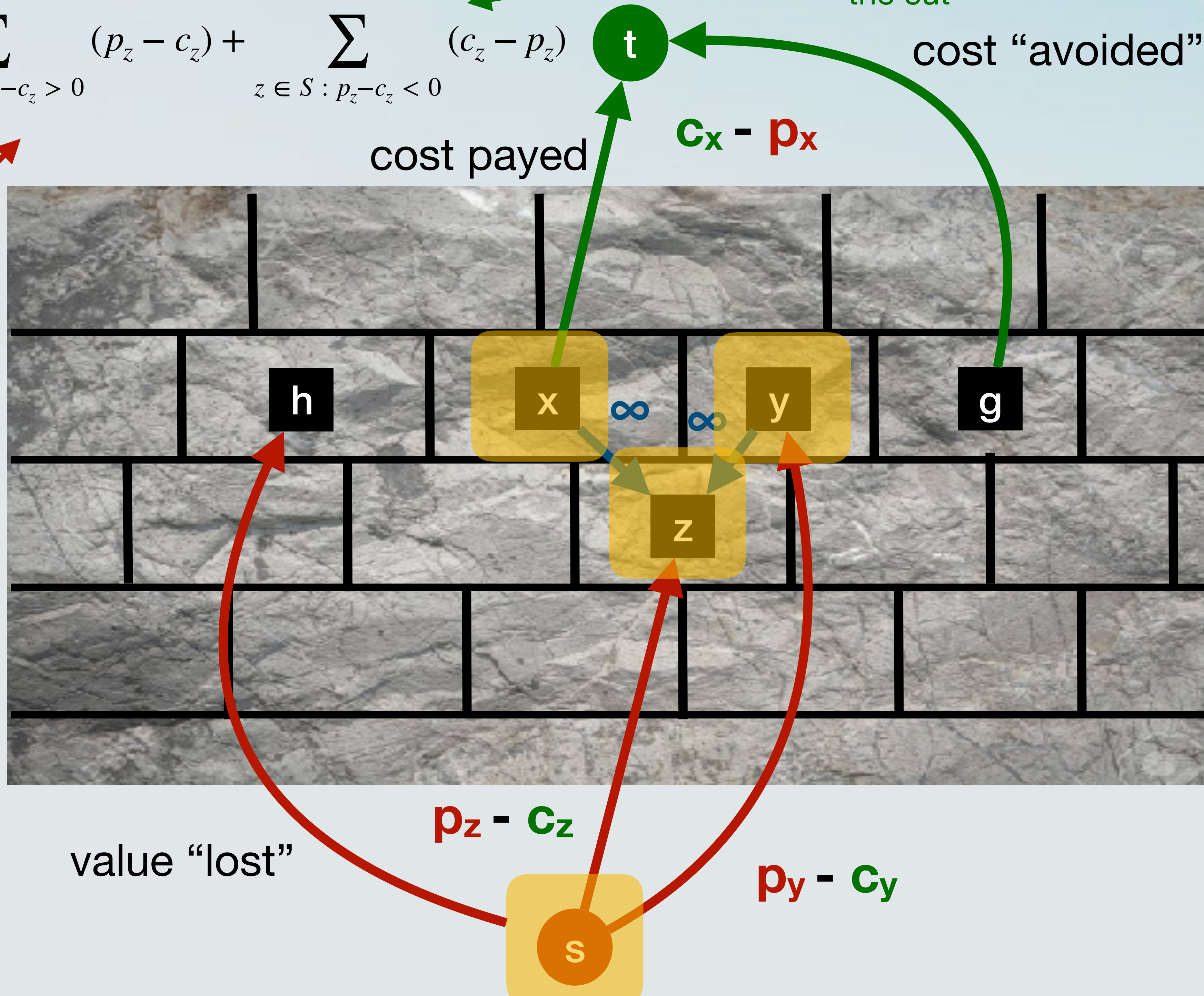
Sum of capacities of green edges crossing the cut

Sum of capacities of red edges crossing the cut.



Optimality of our mining set.

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) \;+\; \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

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Add and subtract this: $$c(S,T) = \sum_{z \in S \,:\, p_z - c_z > 0} (p_z - c_z)$$

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Add and subtract this:

$$c(S,T) = \sum_{z \in S\,:\,p_z - c_z > 0} (p_z - c_z)$$

$$c(S,T) = \sum_{z \in V\,:\,p_z - c_z > 0} (p_z - c_z) \;-\; \sum_{z \in S} (p_z - c_z)$$

Optimality of our mining set.

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) \;+ \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) \;- \sum_{z \in S \,:\, p_z - c_z < 0} (p_z - c_z)$$

Add and subtract this: $$c(S,T) = \sum_{z \in S \,:\, p_z - c_z > 0} (p_z - c_z)$$

$$c(S,T) = \sum_{z \in V \,:\, p_z - c_z > 0} (p_z - c_z) - \sum_{z \in S} (p_z - c_z)$$

constant

Optimality of our mining set.

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) + \sum_{z \in S \,:\, p_z - c_z < 0} (c_z - p_z)$$

$$c(S,T) = \sum_{z \in T \,:\, p_z - c_z > 0} (p_z - c_z) - \sum_{z \in S \,:\, p_z - c_z < 0} (p_z - c_z)$$

Add and subtract this: $$c(S,T) = \sum_{z \in S \,:\, p_z - c_z > 0} (p_z - c_z)$$

$$c(S,T) = \underbrace{\sum_{z \in V \,:\, p_z - c_z > 0} (p_z - c_z)}_{\text{constant}} - \underbrace{\sum_{z \in S} (p_z - c_z)}_{\text{Mining profit}}$$

Open-pit mining - Summarising

- Construct the flow network.
- Run Ford-Fulkerson to find a maximum flow.
- Find a minimum cut using the final residual graph.
- Mine the blocks in the S part of the cut.