Advanced Algorithmic Techniques (COMP523)

NP-Completeness 2

Recap and plan

- **Previous lecture:**
  - Polynomial time reductions
  - Computational classes: P and NP
  - NP-hardness and NP-completeness
  - NP-Complete problems: 3SAT and Vertex Cover
- **This lecture:**
  - Decision vs Optimisation.
  - NP-hardness for Subset Sum and Knapsack.
  - A catalogue of NP-complete problems.
  - A taxonomy of NP-complete problems.

Recall: Vertex Cover

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph G=(V, E)
  Output: A minimum vertex cover.

Vertex Cover decision version

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph G=(V, E) and a number k
  Output: Is there a vertex cover of size ≤ k?.

From optimisation to decision

- We are given an optimisation problem P (assume minimisation).
  - E.g., find the minimum vertex cover.
- We introduce a threshold k.
- The decision version P_d becomes: Given an instance of P and the threshold k as input, is there a solution to P of value at most k?
  - E.g., is there a vertex cover of size at most k?

Optimisation vs decision

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- If we can solve P in polynomial time, we can solve P_d in polynomial time. (why?)

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- Often the opposite is also true.
  - If we can solve P_d in polynomial time, we can solve P in polynomial time.

Optimisation vs decision

- Vertex Cover (Optimisation)
  Input: A graph $G=(V, E)$
  Output: A minimum vertex cover.

- Vertex Cover (Decision)
  Input: A graph $G=(V, E)$ and a number k
  Output: Is there a vertex cover of size $\leq k$?.

Optimisation vs decision

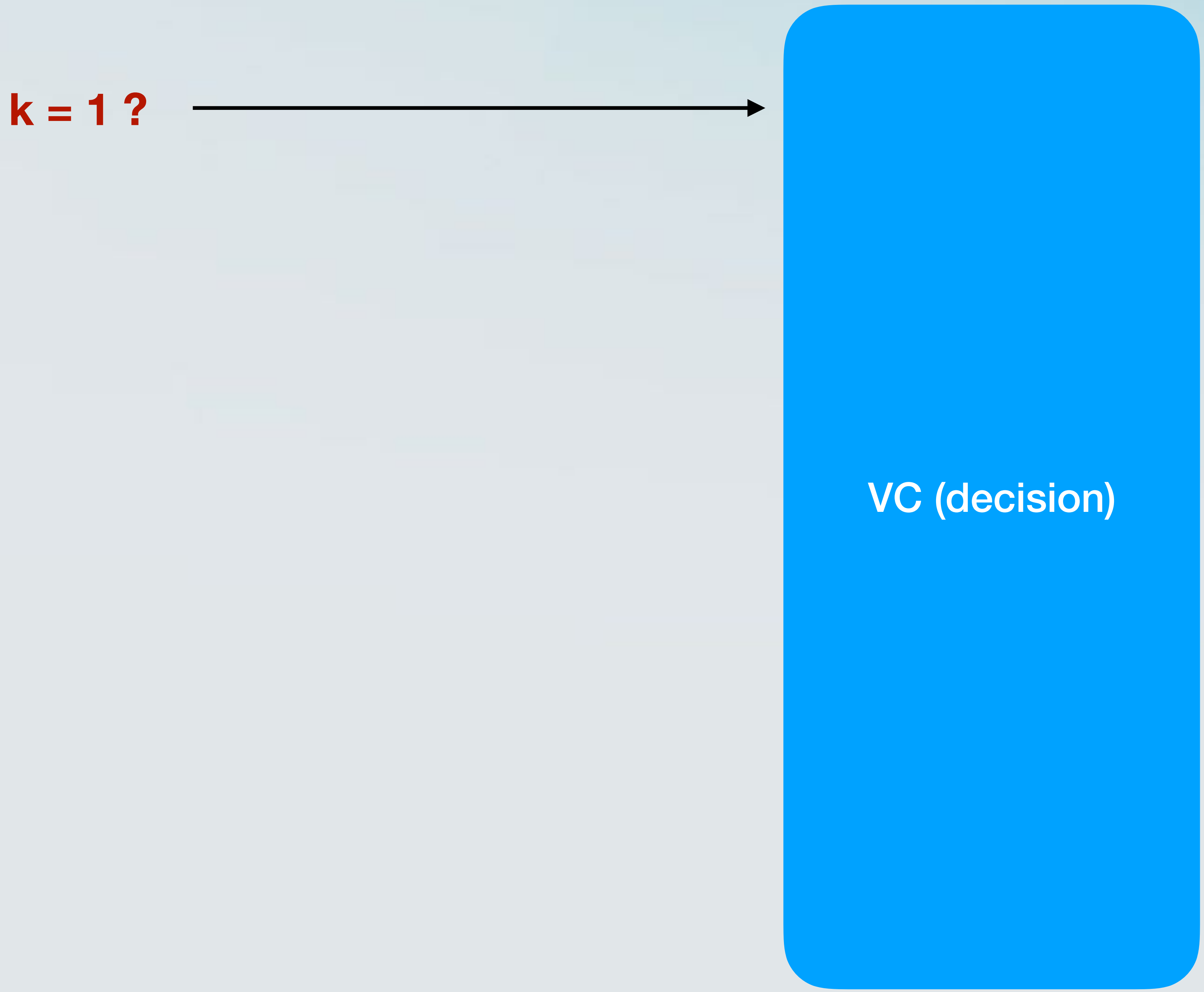
- Vertex Cover Size (Optimisation)
  Input: A graph $G=(V, E)$
  Output: **The size of** a minimum vertex cover.

- Vertex Cover (Decision)
  Input: A graph $G=(V, E)$ and a number k
  Output: Is there a vertex cover of size $\leq k$?.

Vertex Cover Size

VC (decision)

Vertex Cover Size



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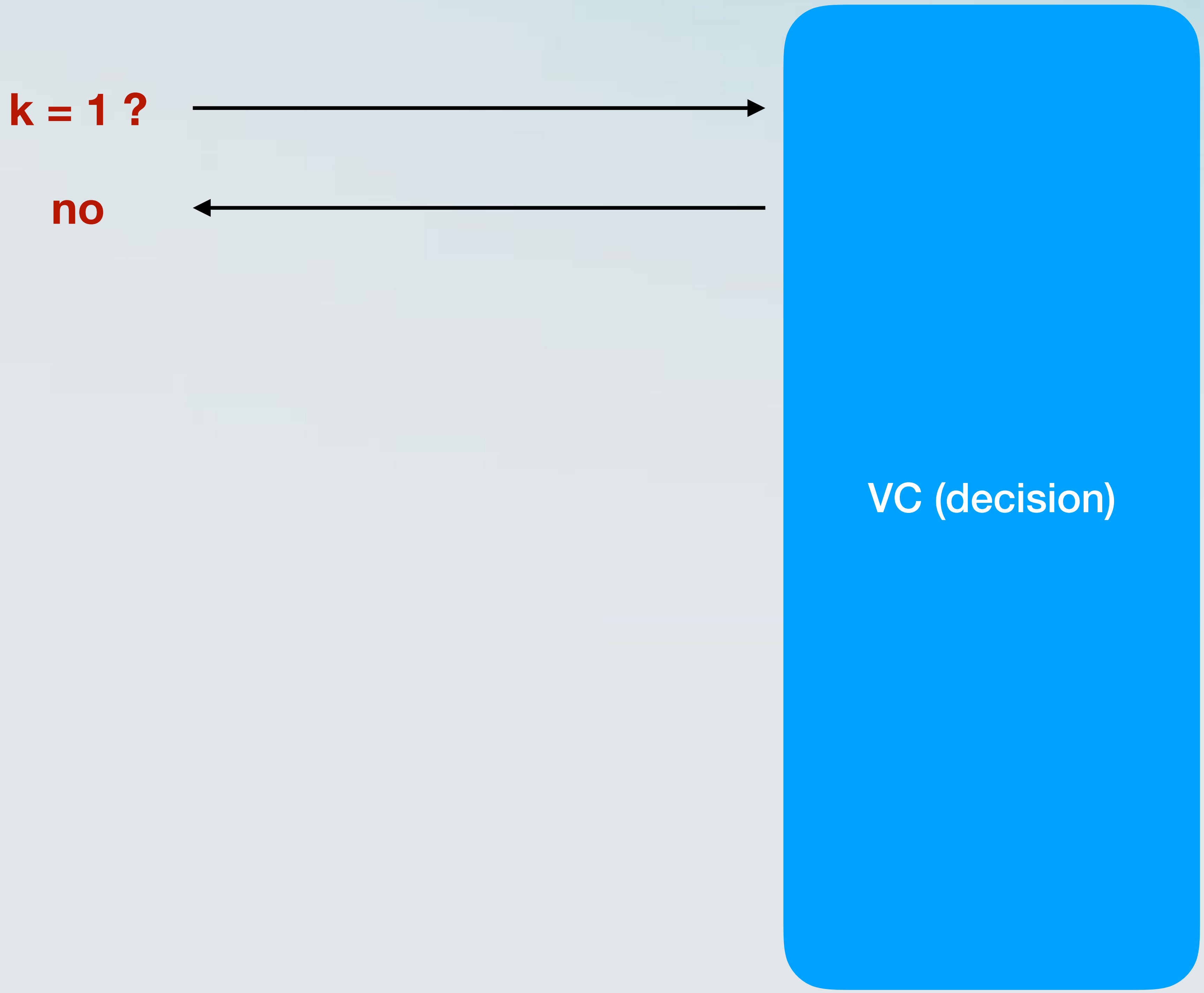

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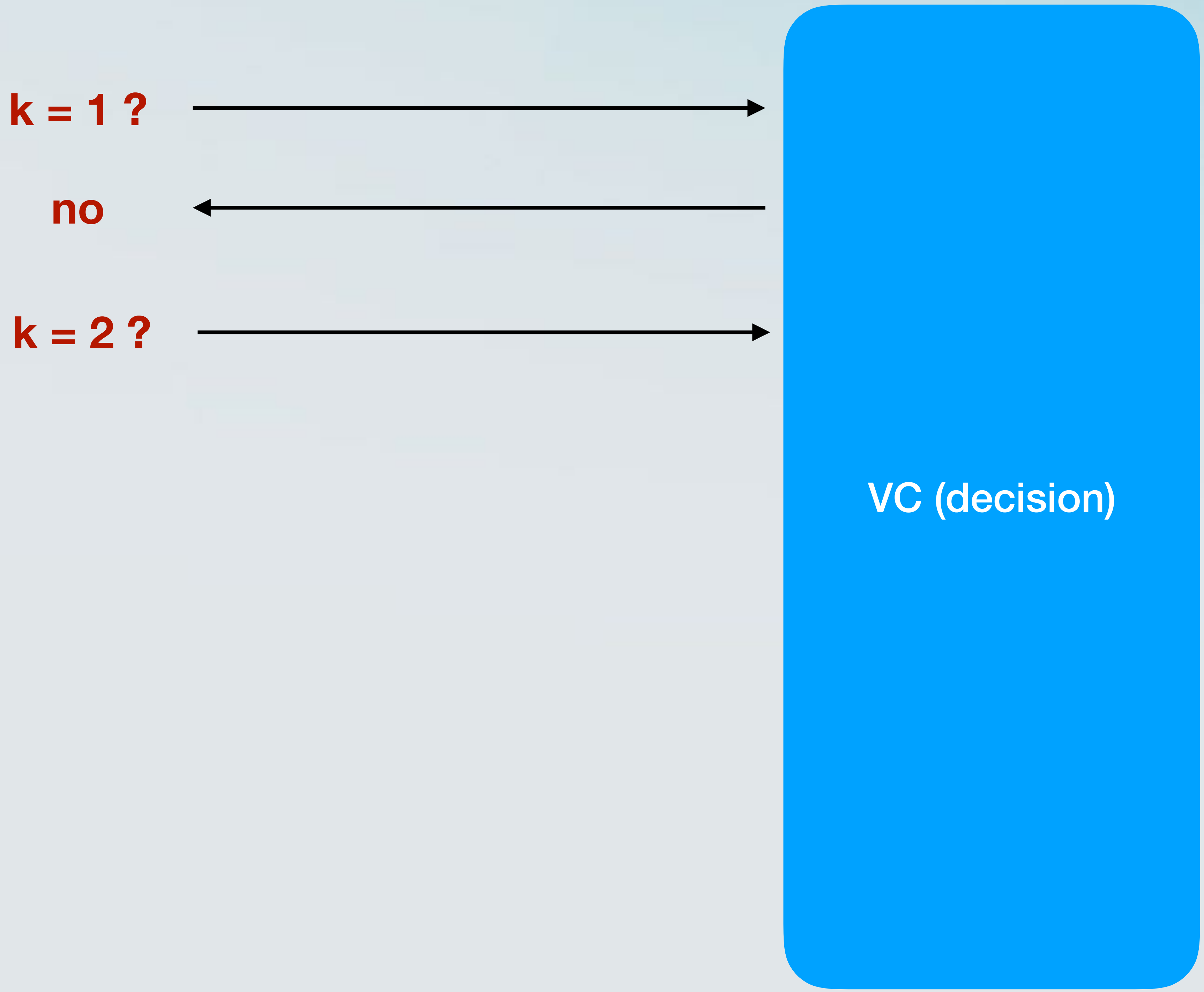

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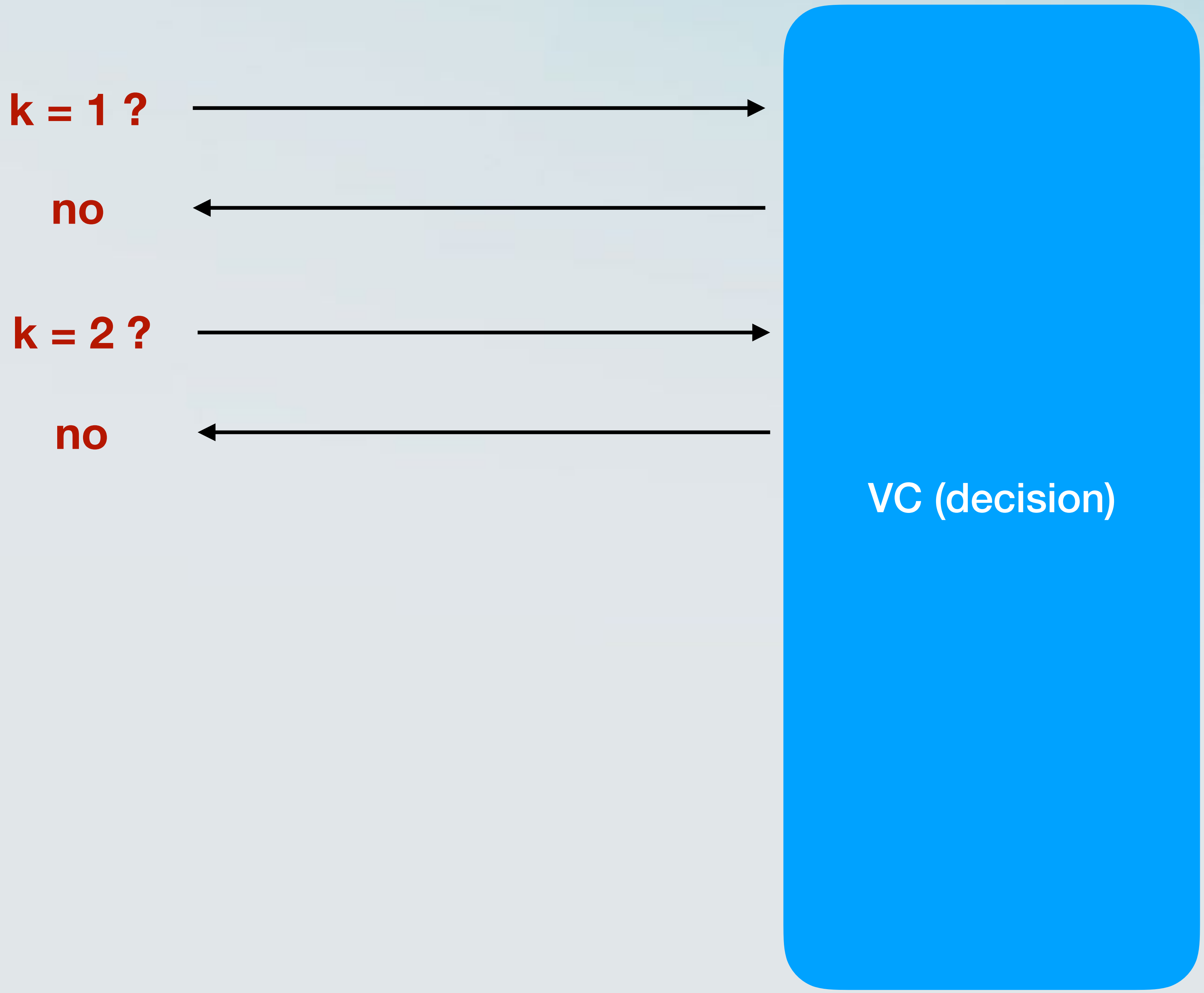

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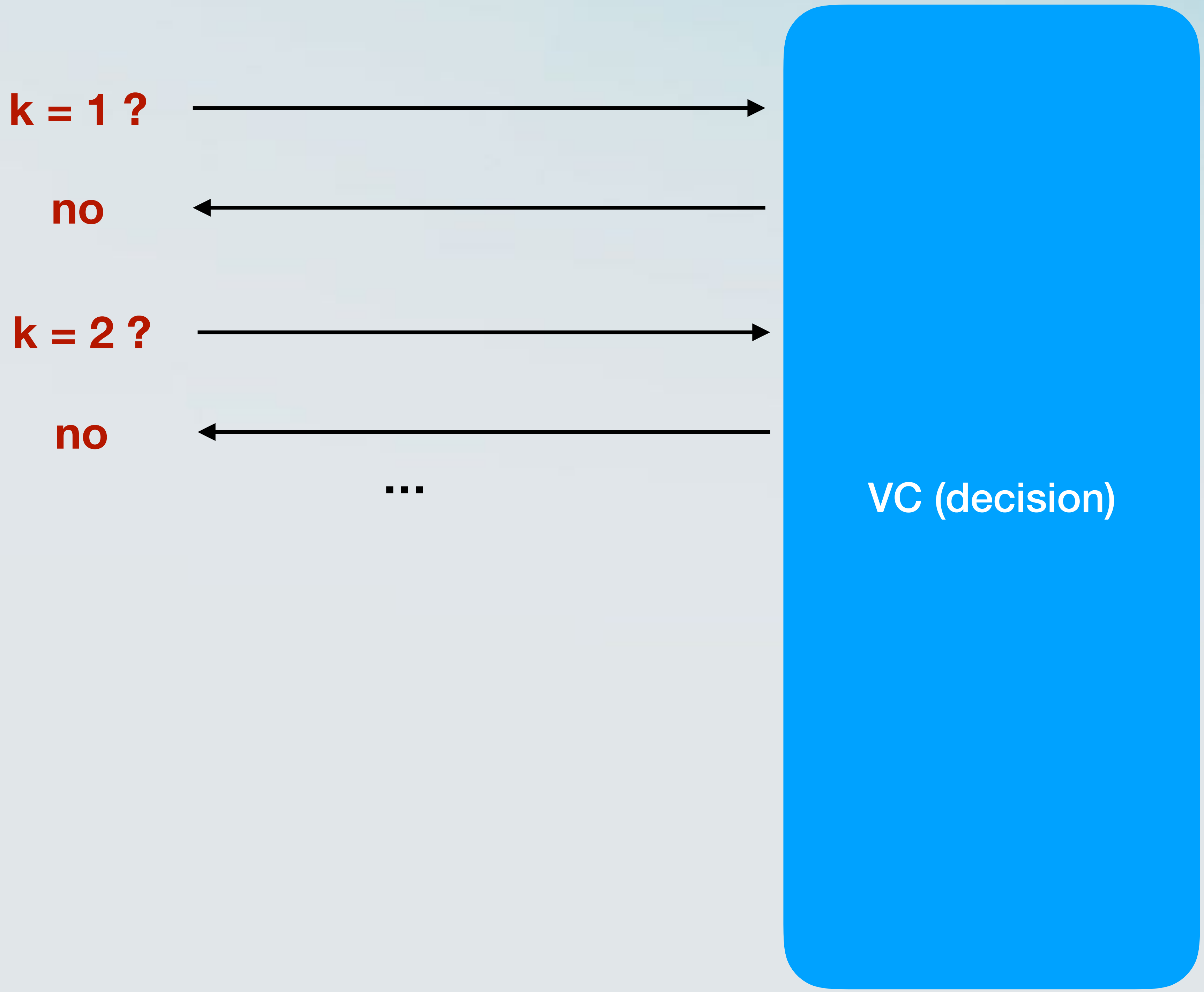

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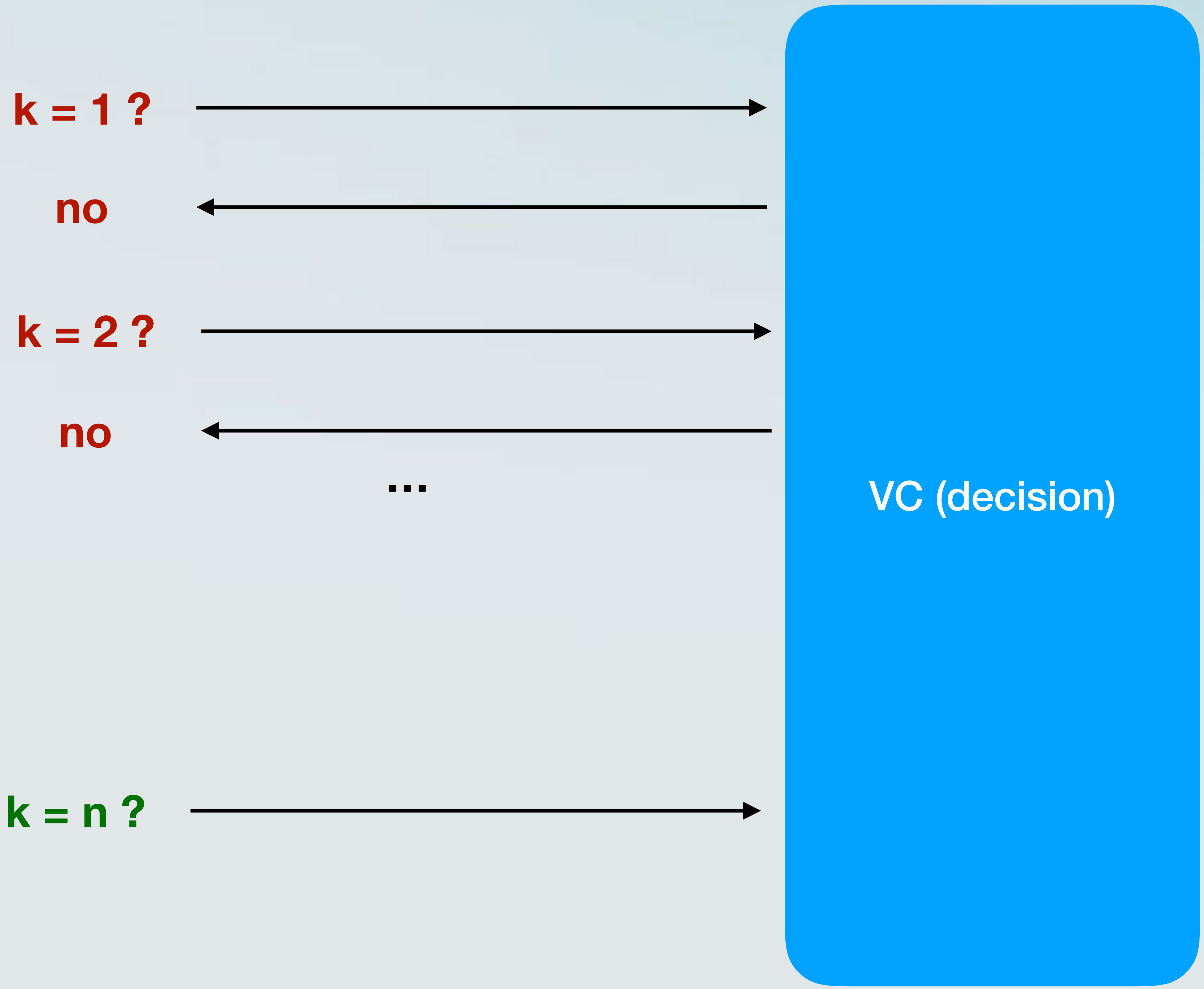

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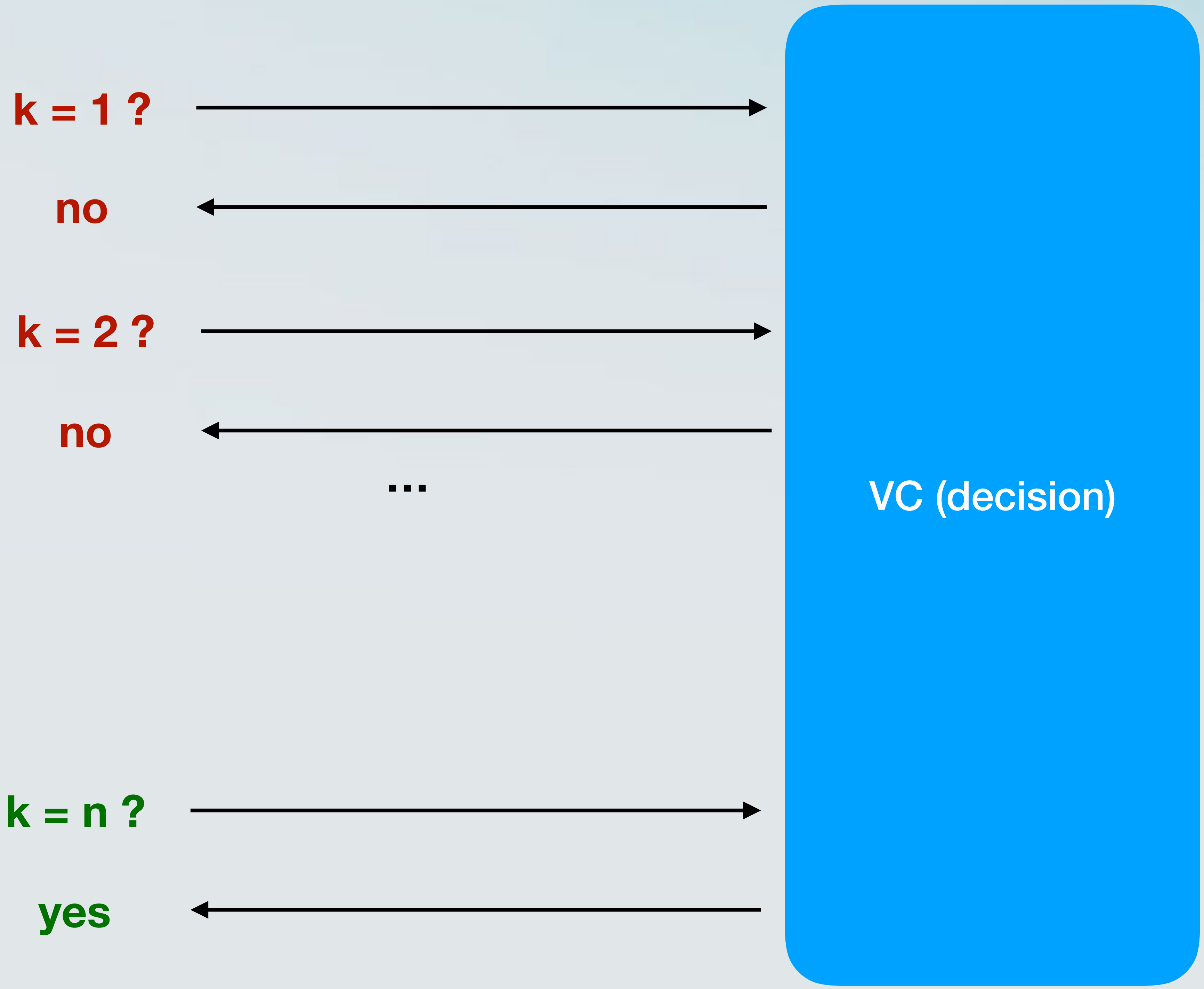

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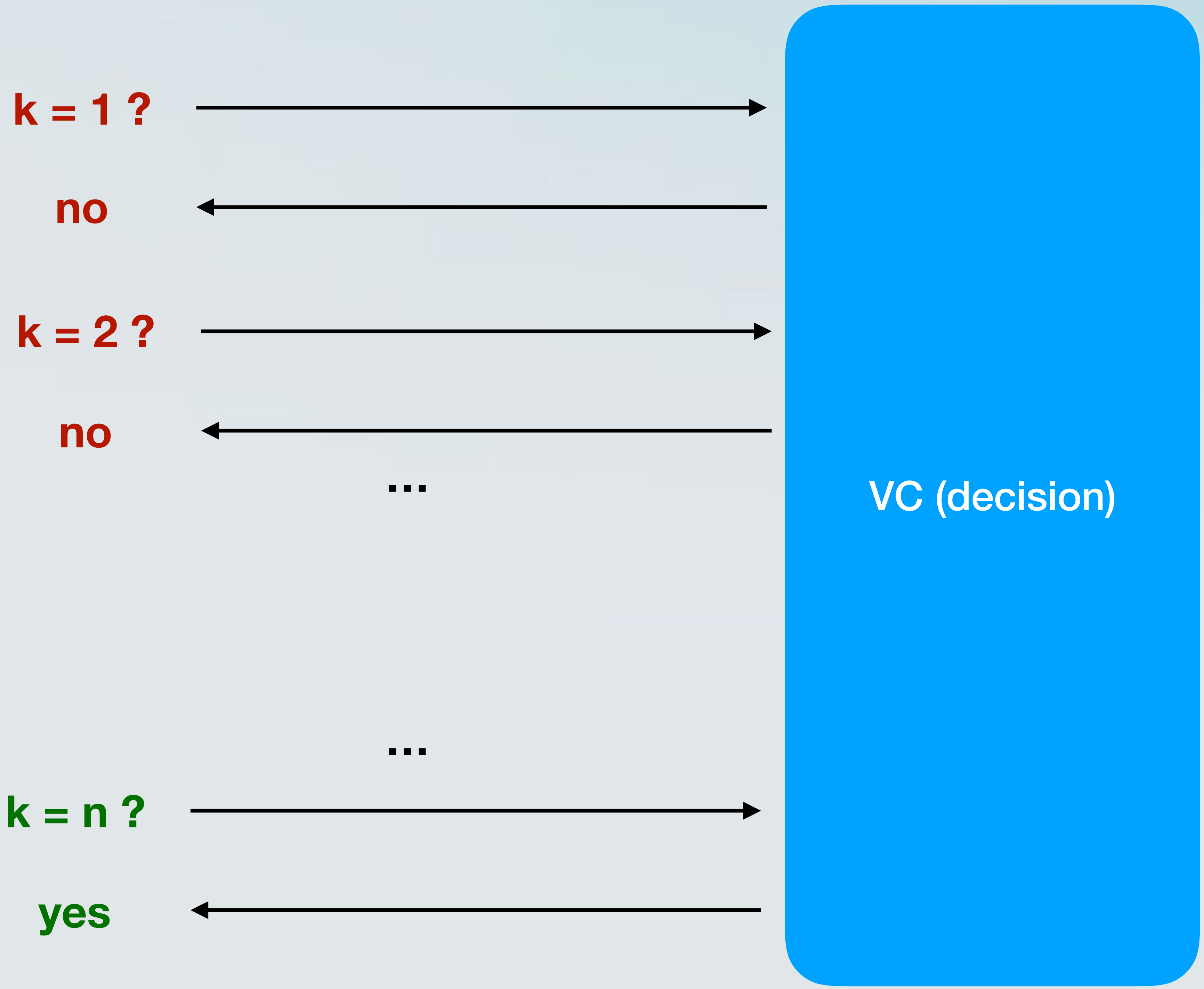

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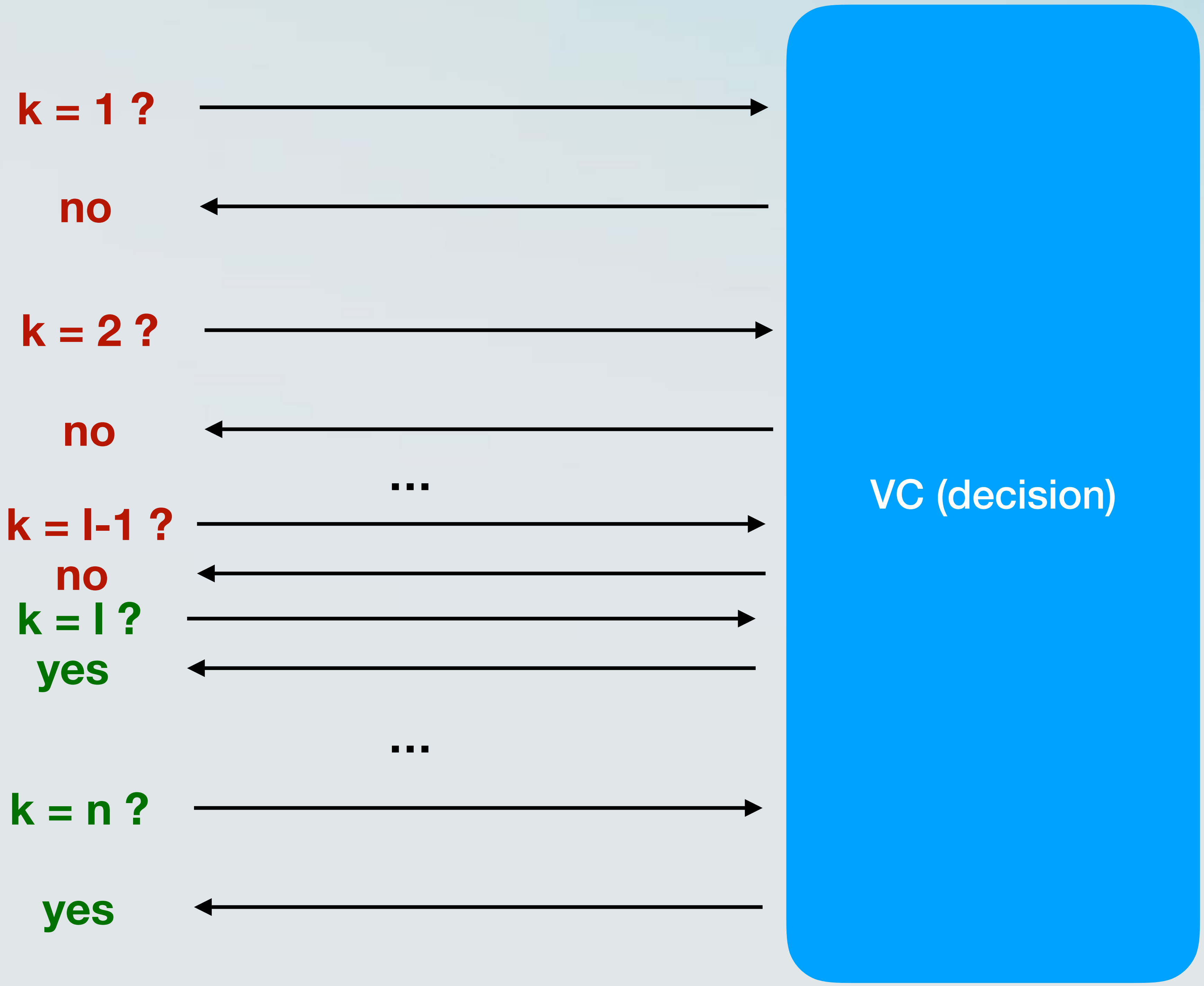

Vertex Cover Size

VC (decision)

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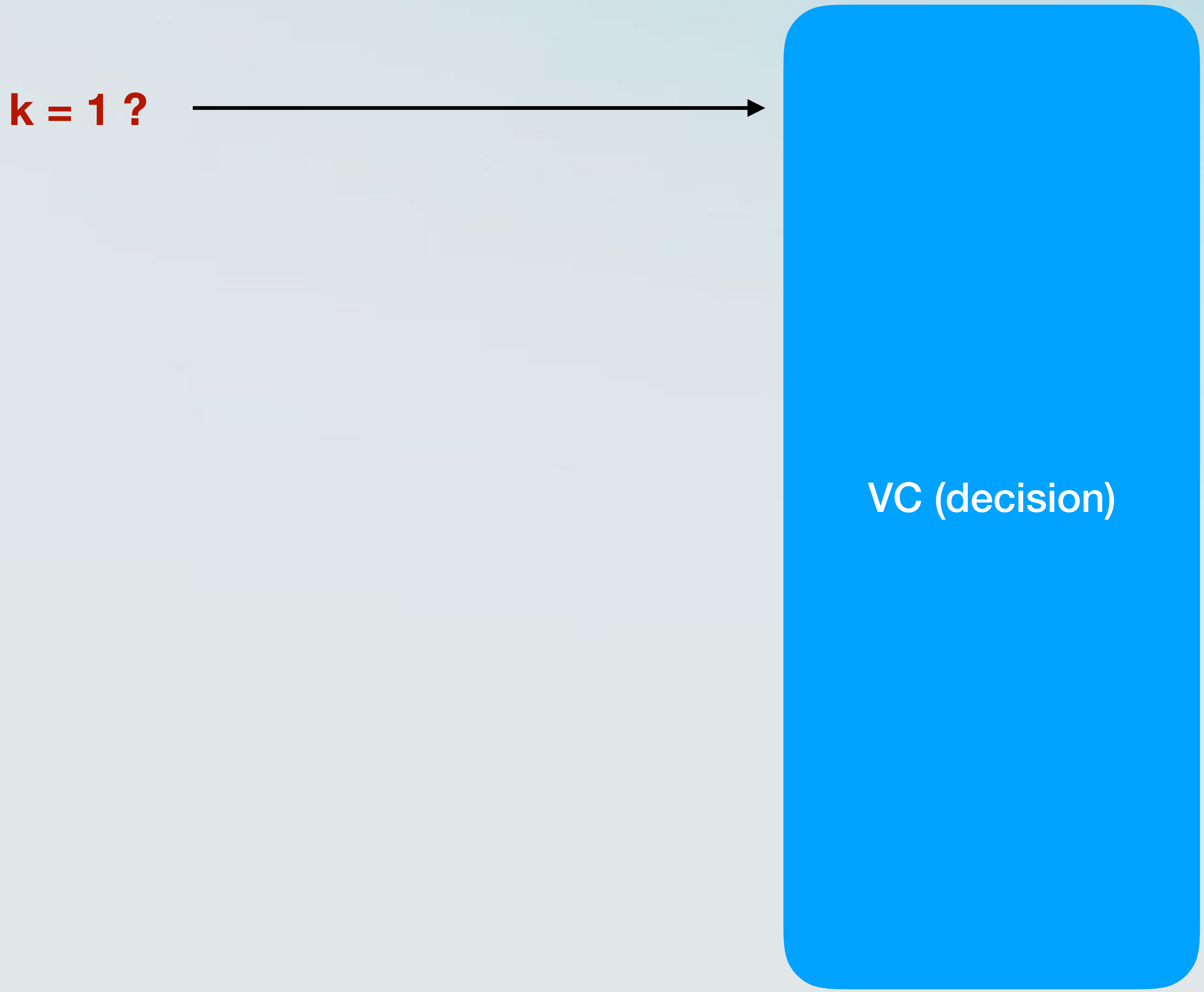

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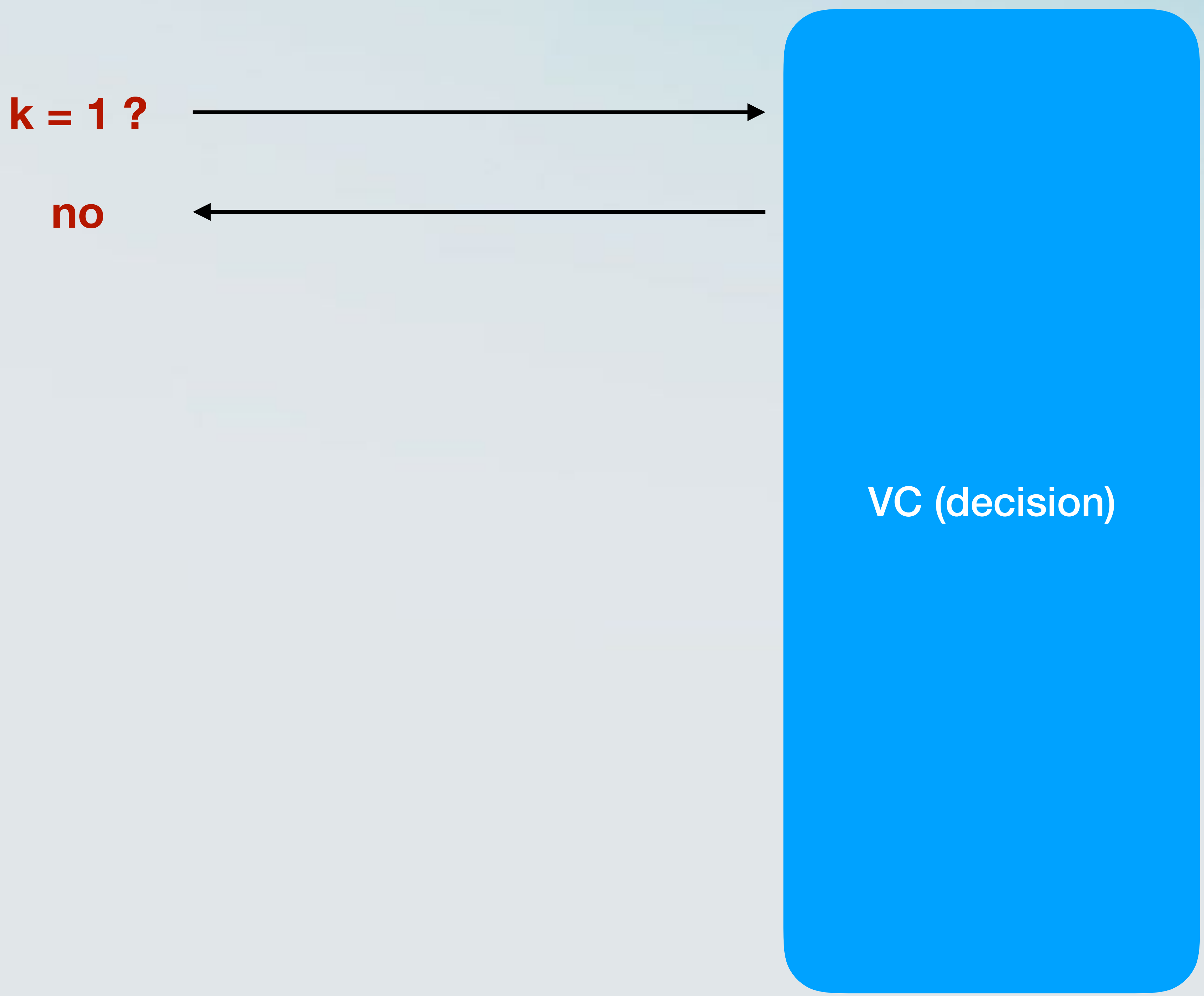

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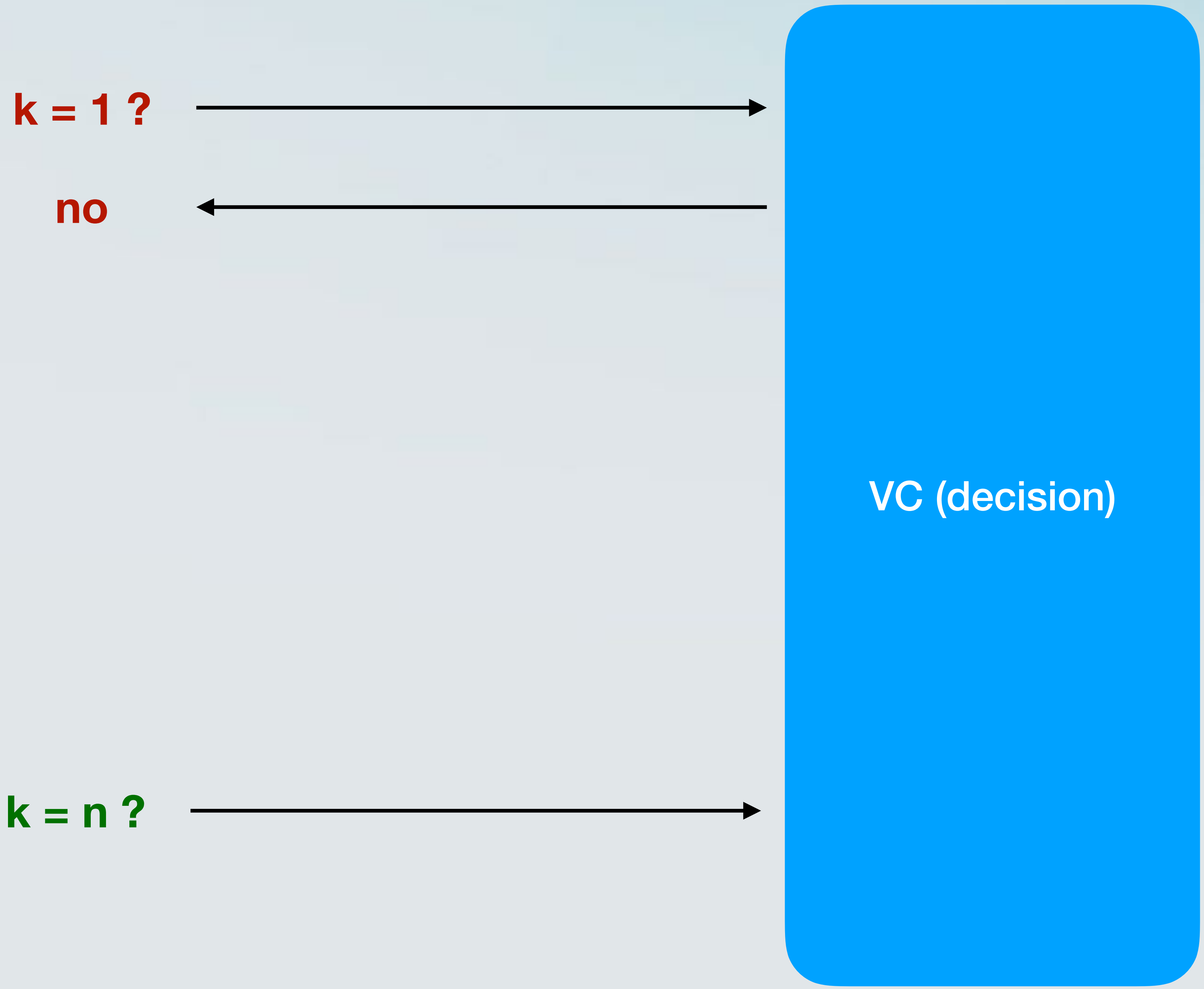

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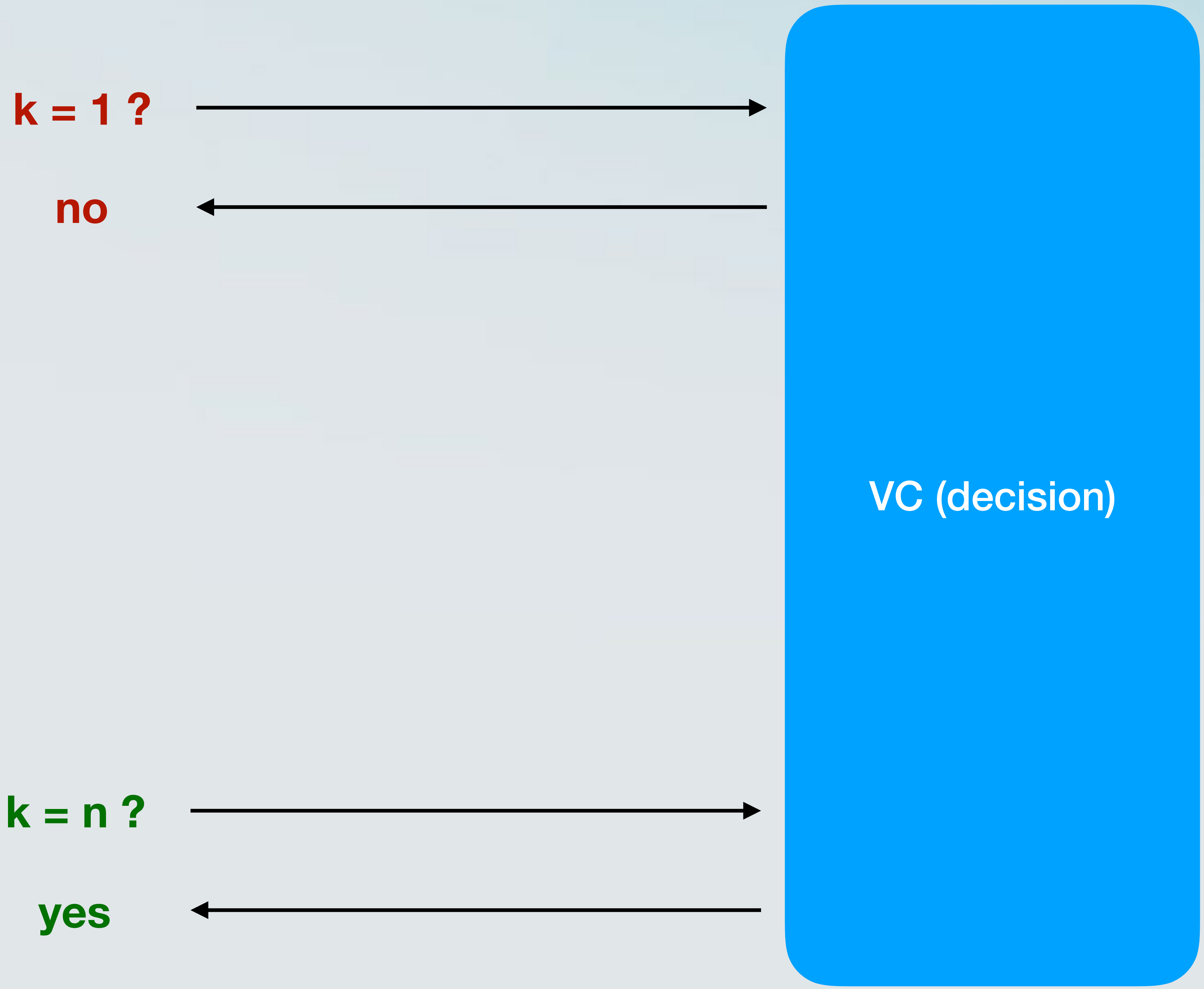

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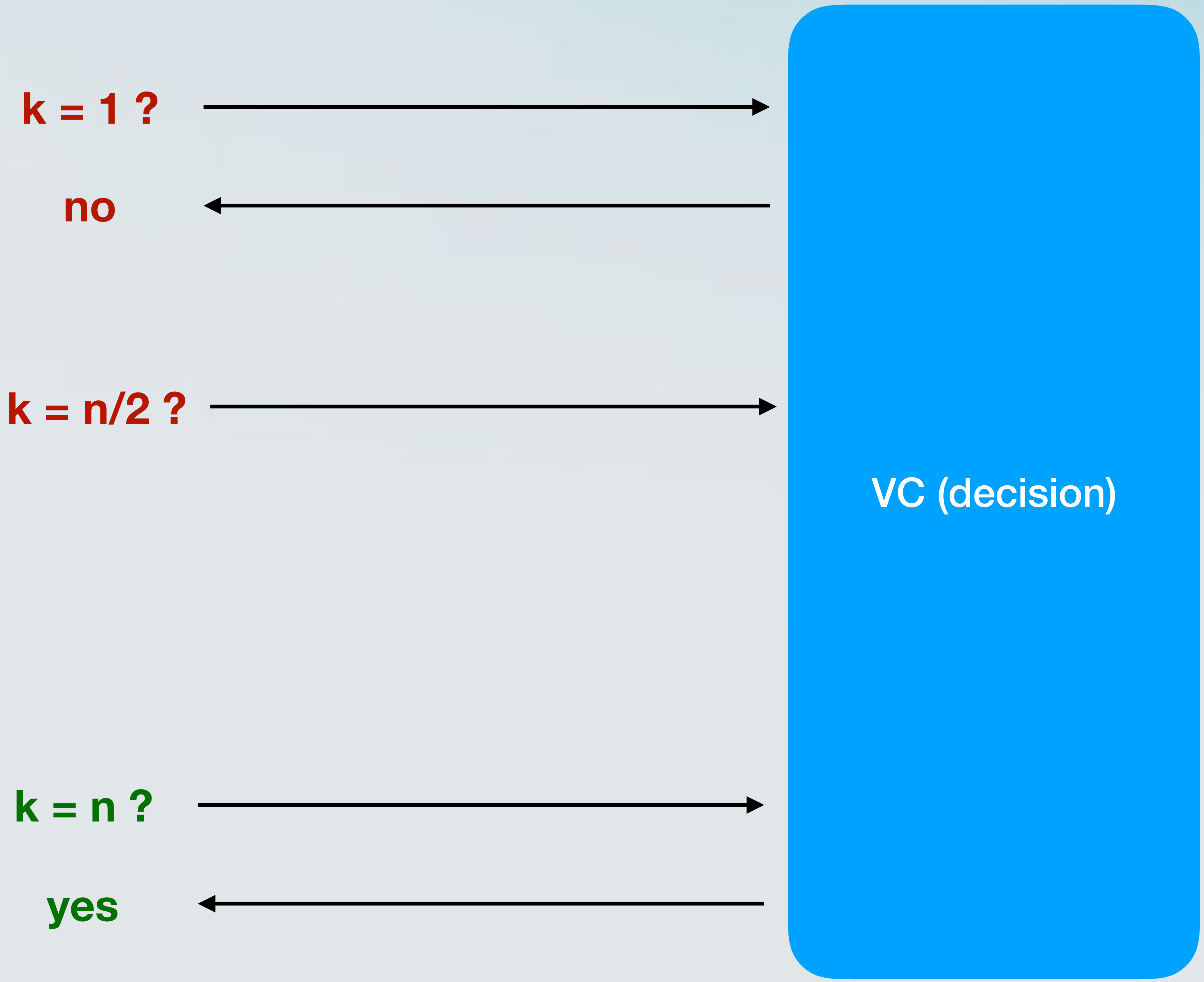

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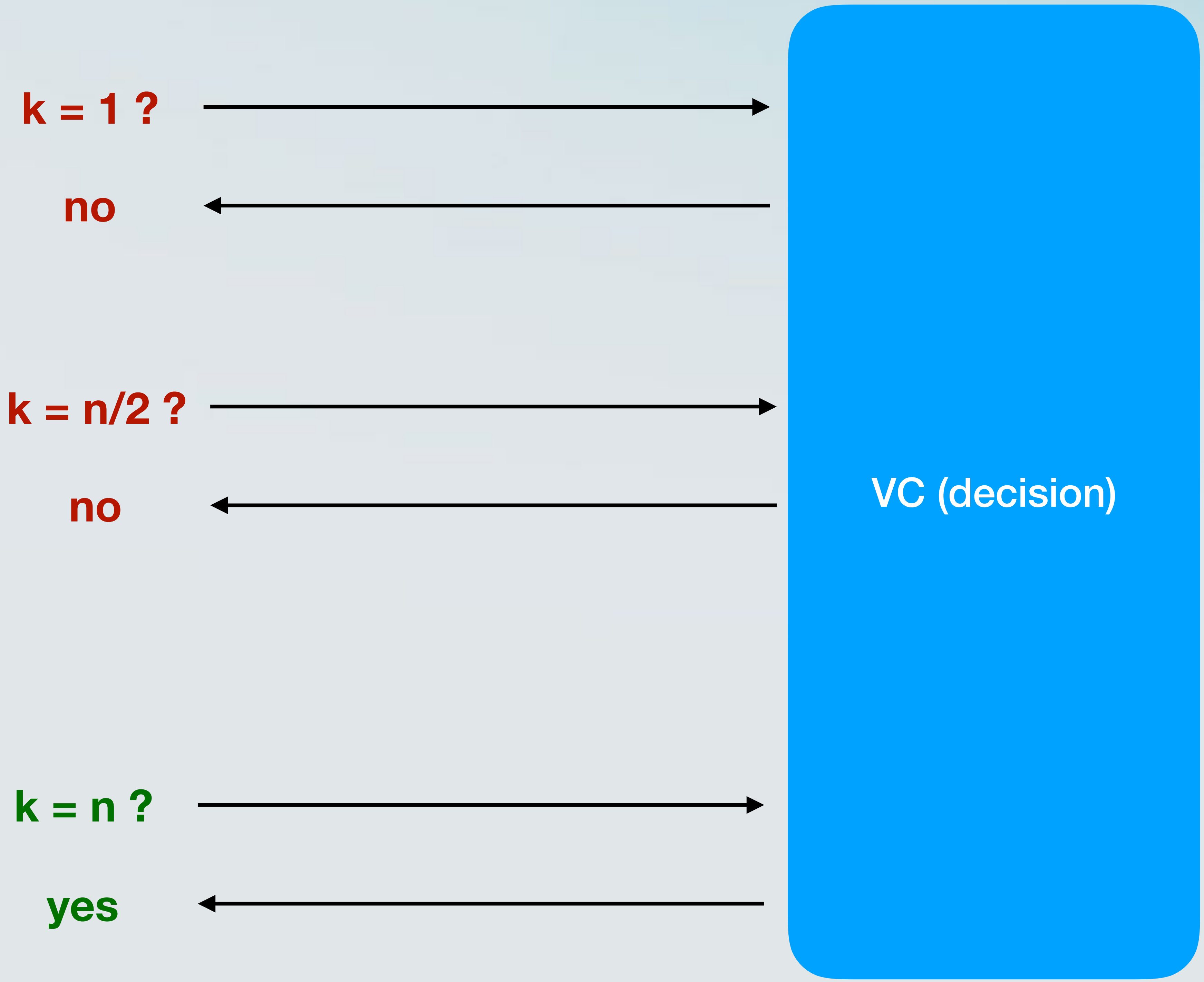

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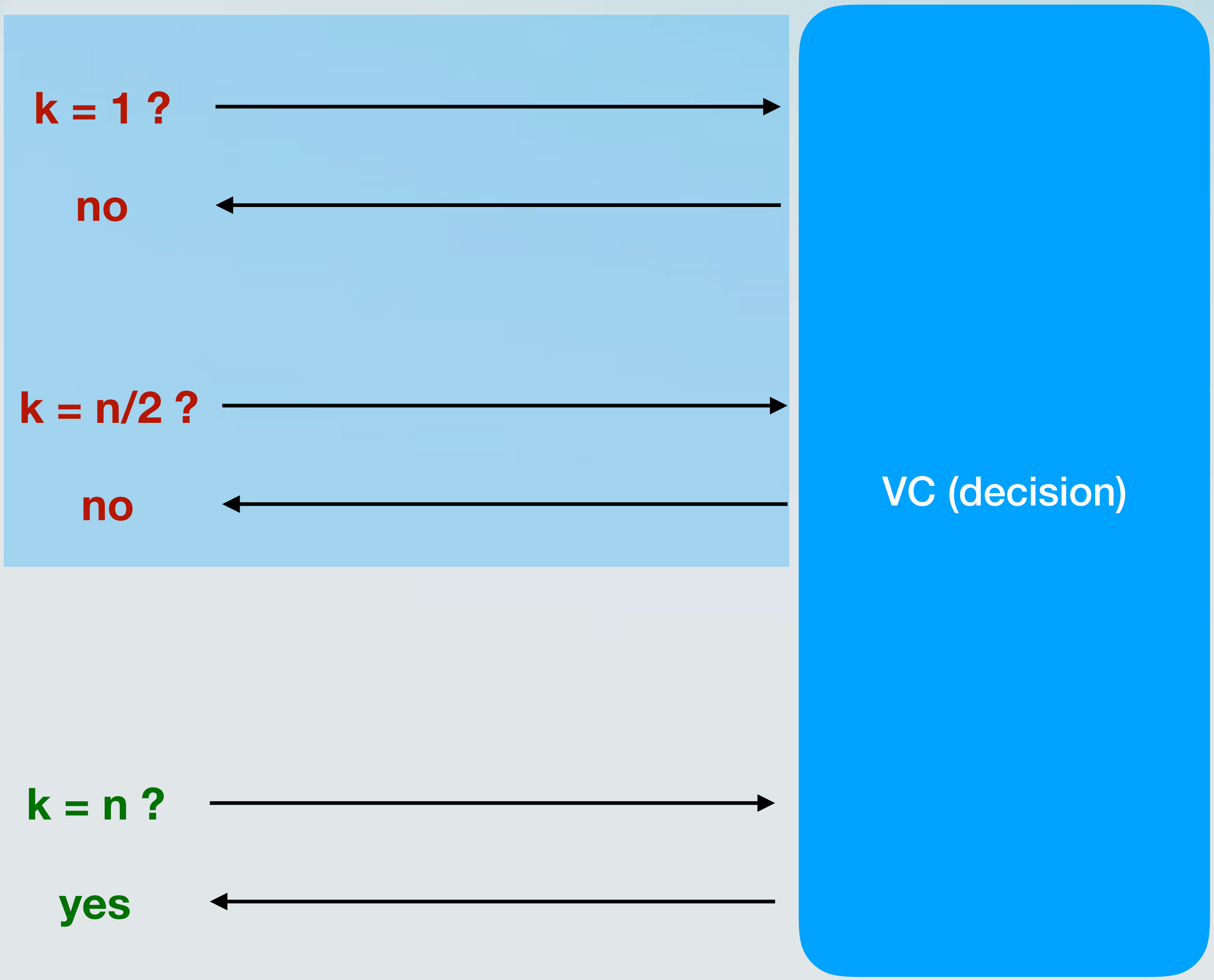

Optimisation vs decision

- Vertex Cover Size (Optimisation)
  Input: A graph $G=(V, E)$
  Output: **The size of** a minimum vertex cover.

- Vertex Cover (Decision)
  Input: A graph $G=(V, E)$ and a number k
  Output: Is there a vertex cover of size $\leq k$?.

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  Output: Is there a vertex cover of size $\leq k$?.

Vertex Cover

- First, find the value k^* of the minimum vertex cover using the algorithm for VC_d.
- Pick a vertex v in the graph.
    - Remove it (and the incident edges) to get graph $G - \{v\}$.
    - Property: If v was in any minimum vertex cover, $G - \{v\}$ has a minimum vertex cover of size k^*-1.
    - Check if the graph $G - \{v\}$ has a vertex cover of size at most k^*-1.
        - Yes: Include v in the vertex cover.
        - No: Do not include v in the vertex cover.
        - Then move to the next vertex.

The subset sum problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} w_i$ is maximised.

Equivalent formulation decision version

- We are given a set T of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative integer weight w_i.
- We are given an integer bound W.
- Goal: Decide if there exists a subset S of the items such that

$$\sum_{i \in S} w_i = W$$

Subset Sum is in NP

- If we are given a candidate solution S, we can easily check whether the following holds or not:

$$\sum_{i \in S} w_i = W$$

Subset Sum is in NP-hard

- We will reduce from 3SAT.
- Given a 3CNF formula ϕ (with m clauses and n variables) we will construct an instance <T, W> of the subset sum problem such that:
  - ϕ is satisfiable if any only if there exists a subset S of T whose sum is exactly W.
- Assumptions wlog:
  - Every variable appears in some clause.
  - A clause does not include both a variable and its negation.

The reduction

- We will create integers with $m+n$ digits that look like this:

$x_1\ x_2\ x_3 \ldots x_n \ldots C_1\ C_2 \ldots C_m$

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Variables

The reduction

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Variables Clauses

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

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- We will set W to have *1* in all “variable digits” and *0* in all “clause digits”.

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
W		1	1	1	4	4	4	4

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

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- We will set W to have *1* in all "variable digits" and *0* in all "clause digits".

- For each variable x_i, we create two integers z_i and y_i.

  - Each of them has *1* in the digit x_i and 0 in the other "variable digits".

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0				
y_1		1	0	0				
z_2		0	1	0				
y_2		0	1	0				
z_3		0	0	1				
y_3		0	0	1				
W		1	1	1	4	4	4	4

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- For each variable x_i, we create two integers z_i and y_i.
  - Each of them has *1* in the digit x_i and 0 in the other "variable digits".
  - If literal x_i appears in clause C_j, z_i contains a *1* in the corresponding "clause digit".
  - If literal $\neg x_i$ appears in clause C_j, y_i contains a *1* in the corresponding "clause digit".
  - All other "clause digits" are set to 0.

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
W		1	1	1	4	4	4	4

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  - All other "clause digits" are set to 0.

The reduction

- For each clause C_j we create two integers s_i and t_i.
  - These have 0 everywhere, except in the corresponding "clause digit".

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0		0	0	0
t_1		0	0	0		0	0	0
s_2		0	0	0	0		0	0
t_2		0	0	0	0		0	0
s_3		0	0	0	0	0		0
t_3		0	0	0	0	0		0
s_4		0	0	0	0	0	0	
t_4		0	0	0	0	0	0	
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The reduction

- For each clause C_j we create two integers s_i and t_i.
  - These have 0 everywhere, except in the corresponding "clause digit".
  - s_i has a 1 in the corresponding "clause digit".
  - t_i has a 2 in the corresponding "clause digit".

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
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Subset Sum is in NP-hard

- We will reduce from 3SAT.
- Given a 3CNF formula ϕ (with m clauses and n variables) we will construct an instance $\langle T, W \rangle$ of the subset sum problem such that:
  - ϕ is satisfiable if any only if there exists a subset S of T whose sum is exactly W.

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- We will reduce from 3SAT.
- Given a 3CNF formula ϕ (with m clauses and n variables) we will construct an instance $<T, W>$ of the subset sum problem such that:
  - ϕ is satisfiable if any only if there exists a subset S of T whose sum is exactly W.
- One direction:

  ϕ is satisfiable => there exists a subset S of T whose sum is exactly W.

The reduction

- Let x be a satisfying assignment.
  - If $x_1 = 1$, include z_1.
  - Otherwise, include y_1.
  - Include appropriate choices of s_i and t_i.

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
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By the construction of z and y, the "variable digits" always sum up to *1111*

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y_3		0	0	1	1	1	0	0
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s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
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By the construction of z and y, the "variable digits" always sum up to *1111*

Since all clauses are satisfied, we get at least one "*1*" from one of the variables that were set to *0* in *x*.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
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z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

By the construction of z and y, the "variable digits" always sum up to *1111*

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$$\boldsymbol{\phi} = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
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y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

By the construction of z and y, the "variable digits" always sum up to *1111*

Since all clauses are satisfied, we get at least one "*1*" from one of the variables that were set to "*0*" in *x*.

If we need either
1 more "*1*" or
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to get to "*4*".

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	1	0	0	0
t_1		0	0	0	2	0	0	0
s_2		0	0	0	0	1	0	0
t_2		0	0	0	0	2	0	0
s_3		0	0	0	0	0	1	0
t_3		0	0	0	0	0	2	0
s_4		0	0	0	0	0	0	1
t_4		0	0	0	0	0	0	2
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	1	0	0	0
t_1		0	0	0	2	0	0	0
s_2		0	0	0	0	1	0	0
t_2		0	0	0	0	2	0	0
s_3		0	0	0	0	0	1	0
t_3		0	0	0	0	0	2	0
s_4		0	0	0	0	0	0	1
t_4		0	0	0	0	0	0	2
W		1	1	1	4	4	4	4

By the construction of z and y, the "variable digits" always sum up to *1111*

Since all clauses are satisfied, we get at least one "*1*" from one of the variables that were set to "*0*" in *x*.

If we need either 3 more "*1*s to get to "*4*".

We pick both s_i or t_i to make up the difference.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	1	0	0	0
t_1		0	0	0	2	0	0	0
s_2		0	0	0	0	1	0	0
t_2		0	0	0	0	2	0	0
s_3		0	0	0	0	0	1	0
t_3		0	0	0	0	0	2	0
s_4		0	0	0	0	0	0	1
t_4		0	0	0	0	0	0	2
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

By the construction of z and y, the "variable digits" always sum up to *1111*

Since all clauses are satisfied, we get at least one "*1*" from one of the variables that were set to "*0*" in x.

If we need either 3 more "*1*s to get to "*4*".

We pick both s_i or t_i to make up the difference.

$$\phi = (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_3)$$

Subset Sum is in NP-hard

- We will reduce from 3SAT.
- Given a 3CNF formula ϕ (with m clauses and n variables) we will construct an instance $<T, W>$ of the subset sum problem such that:
  - ϕ is satisfiable if any only if there exists a subset S of T whose sum is exactly W.

Subset Sum is in NP-hard

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- Given a 3CNF formula ϕ (with m clauses and n variables) we will construct an instance $\langle T, W \rangle$ of the subset sum problem such that:
  - ϕ is satisfiable if any only if there exists a subset S of T whose sum is exactly W.
- Other direction:

  there exists a subset S of T whose sum is exactly $W \Rightarrow \phi$ is satisfiable.

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

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The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

Set x_i = *1* if S contains z_i and x_i = *0* otherwise.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

Set $x_i = 1$ if S contains z_i and $x_i = 0$ otherwise.

Consider an arbitrary clause C_j.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

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Set $x_i = 1$ if S contains z_i and $x_i = 0$ otherwise.

Consider an arbitrary clause C_j.

Consider the corresponding “clause digits”.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

Set x_i = *1* if S contains z_i and x_i = *0* otherwise.

Consider an arbitrary clause C_j.

Consider the corresponding "clause digits".

Since these add up to "*4*", it must receive at least "*1*" from the z or y numbers.

$$\boldsymbol{\phi} = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

Set $x_i = 1$ if S contains z_i and $x_i = 0$ otherwise.

Consider an arbitrary clause C_j.

Consider the corresponding "clause digits".

Since these add up to "*4*", it must receive at least "*1*" from the *chosen* z or y numbers.

$$\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

The reduction

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
z_1		1	0	0	1	0	0	1
y_1		1	0	0	0	1	1	0
z_2		0	1	0	0	0	0	1
y_2		0	1	0	1	1	1	0
z_3		0	0	1	0	0	1	1
y_3		0	0	1	1	1	0	0
s_1		0	0	0	**1**	0	0	0
t_1		0	0	0	**2**	0	0	0
s_2		0	0	0	0	**1**	0	0
t_2		0	0	0	0	**2**	0	0
s_3		0	0	0	0	0	**1**	0
t_3		0	0	0	0	0	**2**	0
s_4		0	0	0	0	0	0	**1**
t_4		0	0	0	0	0	0	**2**
W		1	1	1	4	4	4	4

S must contain *exactly one* of z and y for each index *i*, otherwise the sum is not W.

Set $x_i = 1$ if S contains z_i and $x_i = 0$ otherwise.

Consider an arbitrary clause C_j.

Consider the corresponding "clause digits".

Since these add up to "*4*", it must receive at least "*1*" from the *chosen* z or y numbers.

This implies that the clause is satisfied.

$$\phi = (x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_3)$$

Knapsack

- 0/1-Knapsack is also NP-complete.
  - Define the decision problem, containment is easy to see.
  - How do we prove hardness?
  - Which problem should we reduce from?

NP-complete problems

NP-complete problems

- Independent Set in graph G: A set of nodes in the graph, such that there is no edge between any two nodes in the set.

NP-complete problems

- Independent Set in graph G: A set of nodes in the graph, such that there is no edge between any two nodes in the set.

- Maximum Independent Set
  Given a graph G, find an independent set of maximum size.

NP-complete problems

- Independent Set in graph G: A set of nodes in the graph, such that there is no edge between any two nodes in the set.

- Maximum Independent Set
Given a graph G, find an independent set of maximum size.

- Maximum Independent Set (decision version)
Given a graph G, and an integer k, is there an independent set of size at least k?

NP-complete problems

NP-complete problems

- Set Packing
  Given a set U of elements, a collection $S_1, \dots , S_m$ of subsets of U and a number k, does there exist a collection of at least k of these sets such that no two of them intersect?

NP-complete problems

- Set Packing
  Given a set U of elements, a collection $S_1, \ldots, S_m$ of subsets of U and a number k, does there exist a collection of at least k of these sets such that no two of them intersect?

- Set Cover
  Given a set U of elements, a collection $S_1, \ldots, S_m$ of subsets of U and a number k, does there exist a collection of at most k of these sets whose union is equal to U?

NP-complete problems

NP-complete problems

- 3-Dimensional Matching
  Given disjoint sets X, Y and Z each of size n, and given a set T (which is a subset of $X \times Y \times Z$) of ordered triples, does there exist a set of n triples in T, so that each element of $X \cup Y \cup Z$ is contained in exactly in one of these triples?

NP-complete problems

NP-complete problems

- k-Colouring of a graph G: A function f: V → {*1*, …, *k*} so that for every edge (u, v) we have that f(u) ≠ f(v).

NP-complete problems

- k-Colouring of a graph G: A function f: V → {*1*, …, *k*} so that for every edge (u, v) we have that f(u) ≠ f(v).
- 3-Colouring
  Given a graph G, does it have a 3-Colouring?

Interlude

- k-Colouring of a graph G: A function f: $V \rightarrow \{1, \ldots, k\}$ so that for every edge (u, v) we have that $f(u) \neq f(v)$.
- 3-Colouring
  Given a graph G, does it have a 3-Colouring?
- What about 2-Colouring? Is it NP-complete?

NP-complete problems

NP-complete problems

- Hamiltonian cycle in a directed graph G: A cycle in a directed graph that visits each vertex *exactly once*.

NP-complete problems

- Hamiltonian cycle in a directed graph G: A cycle in a directed graph that visits each vertex *exactly once*.
- Hamiltonian path in a directed graph G: A path in a directed graph that contains each vertex *exactly once*.

NP-complete problems

- Hamiltonian cycle in a directed graph G: A cycle in a directed graph that visits each vertex *exactly once*.

- Hamiltonian path in a directed graph G: A path in a directed graph that contains each vertex *exactly once*.

- Hamiltonian Cycle
  Given a directed graph G, does it have a Hamiltonian Cycle?

NP-complete problems

- Hamiltonian cycle in a directed graph G: A cycle in a directed graph that visits each vertex *exactly once*.

- Hamiltonian path in a directed graph G: A path in a directed graph that contains each vertex *exactly once*.

- Hamiltonian Cycle
  Given a directed graph G, does it have a Hamiltonian Cycle?

- Hamiltonian Path
  Given a directed graph G, does it have a Hamiltonian Path?

NP-complete problems

- Hamiltonian cycle in a directed graph G: A cycle in a directed graph that visits each vertex *exactly once*.
- Hamiltonian path in a directed graph G: A path in a directed graph that contains each vertex *exactly once*.
- Hamiltonian Cycle
  Given a directed graph G, does it have a Hamiltonian Cycle?
- Hamiltonian Path
  Given a directed graph G, does it have a Hamiltonian Path?
- Traveling Salesman
  (def Kleinberg and Tardos, p. 474).

NP-completeness, a taxonomy

Packing problems

- Independent Set
- Set Packing

Covering problems

- Vertex Cover
- Set Cover

Partitioning problems

- 3D-Matching
- Graph Colouring

Sequencing problems

- Hamiltonian Cycle
- Hamiltonian Path
- Traveling Salesman

Numerical problems

- Subset Sum
- Knapsack

Constraint Satisfaction problems

- 3 SAT