Advanced Algorithmic Techniques (COMP523)

Linear Programming

Recap and plan

- **Previous 2 lectures:**
  - Polynomial time reductions
  - Computational classes: P and NP
  - NP-hardness and NP-completeness
- **This lecture:**
  - Linear Programming
  - Integer Linear Programs
  - LP-Duality
  - Total Unimodularity

A problem

- A company makes two products, X and Y, using two machines, A and B.
    - Each unit of product X requires *50min* processing time on machine A and *30min* processing time on machine B.
    - Each unit of product Y requires *24min* of processing time on machine A and *33min* of processing time on machine B.
    - At the start of the week, there are *30 units* of X and *90 units* of Y in stock.
    - The available processing time on machine A is *40 hours* and on machine B it is *35 hours*.
    - The demand for X in the week is *75 units* and for Y it is *95 units*.
- Goal: Maximise the combined sum of units of X and Y in stock at the end of the week.

A linear program

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Maximise $(x + 30 - 75) + (y + 90 - 95)$

A linear program

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$$50x + 24y \leq 2400 \qquad x \geq 75 - 30$$

$$30x + 33y \leq 2100 \qquad y \geq 95 - 90$$

A linear program

Maximise $x + y - 50$

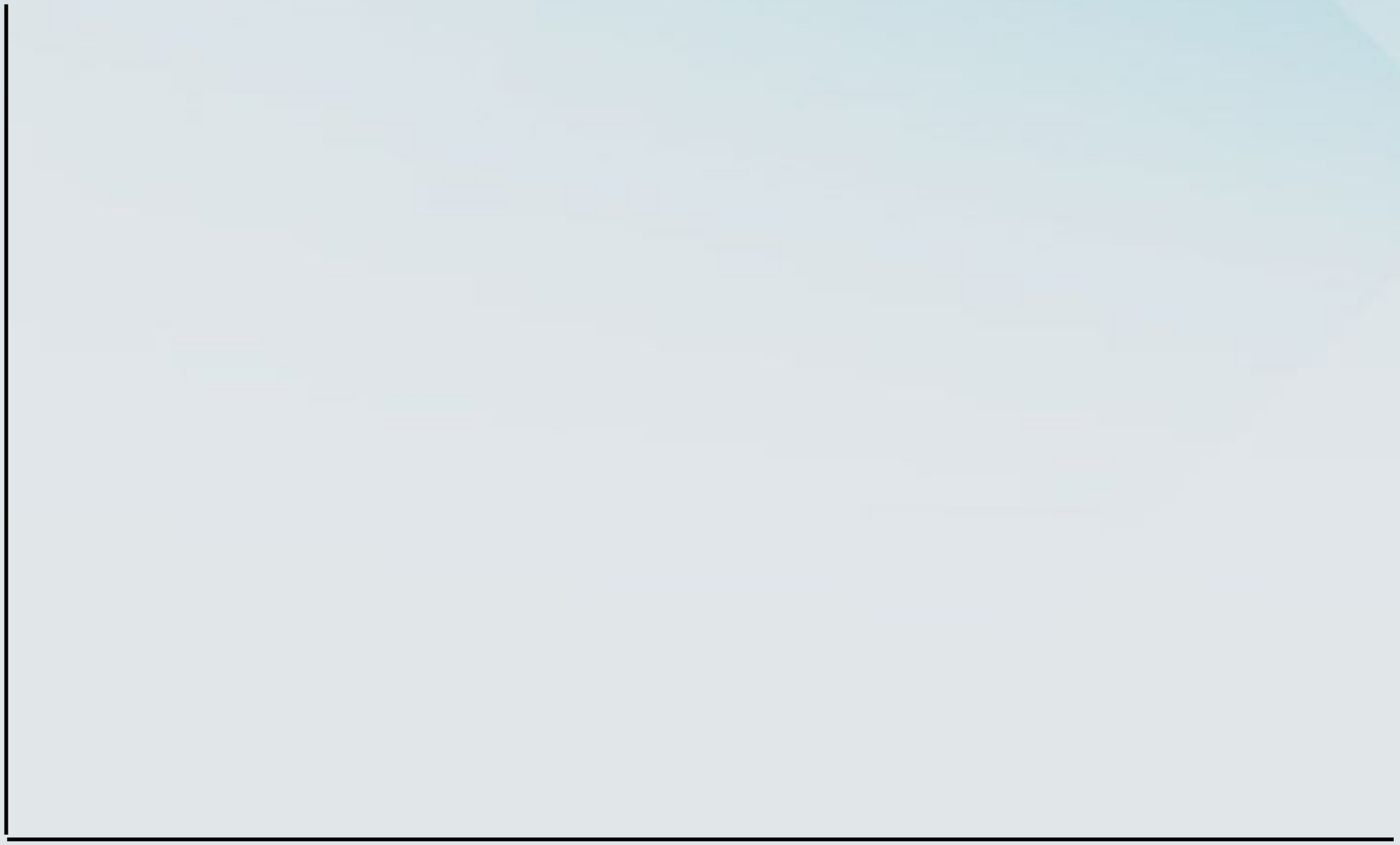
subject to $50x + 24y \leq 2400$

$30x + 33y \leq 2100$

$x \geq 45$

$y \geq 5$

Solving the linear program



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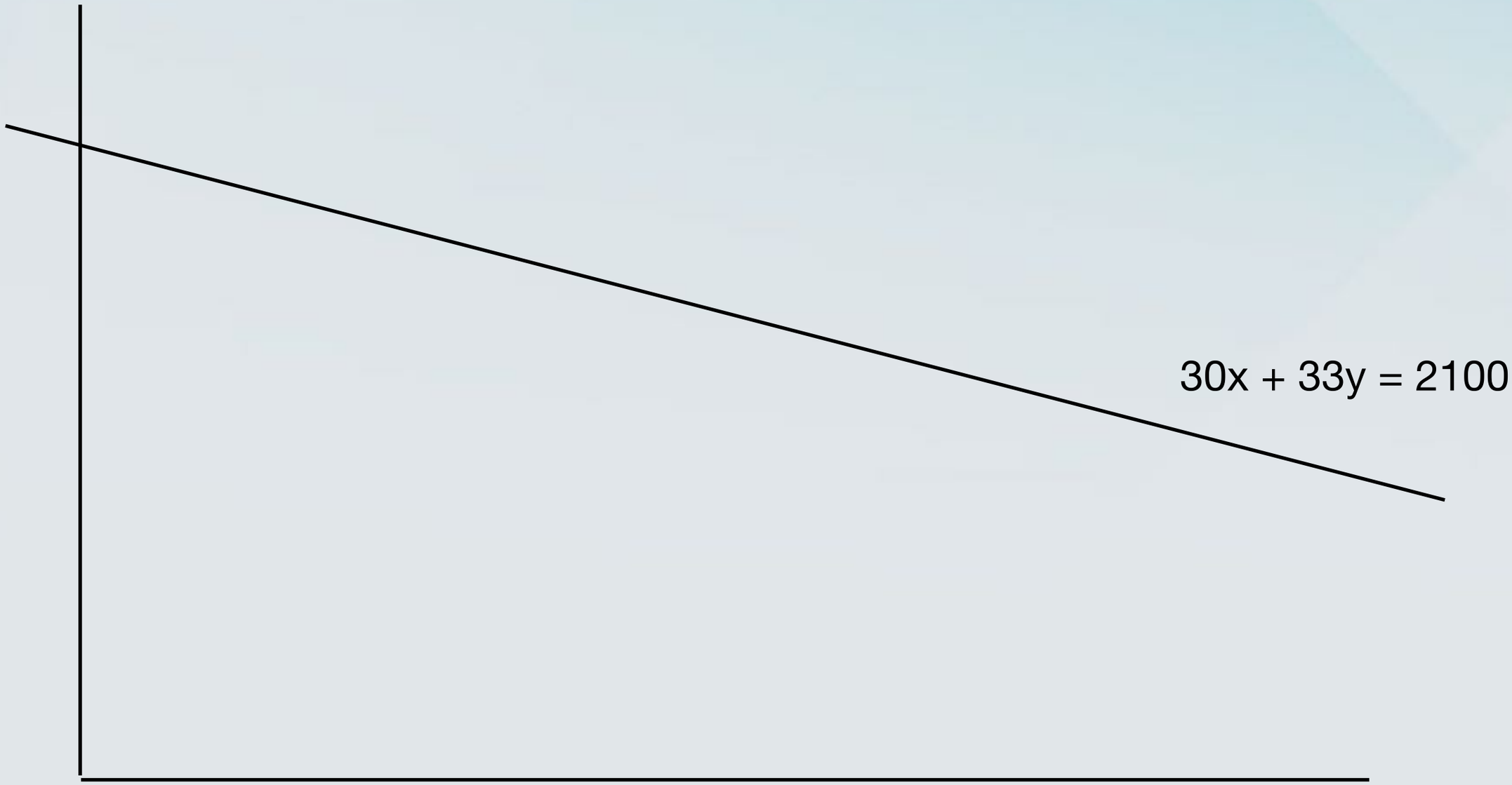

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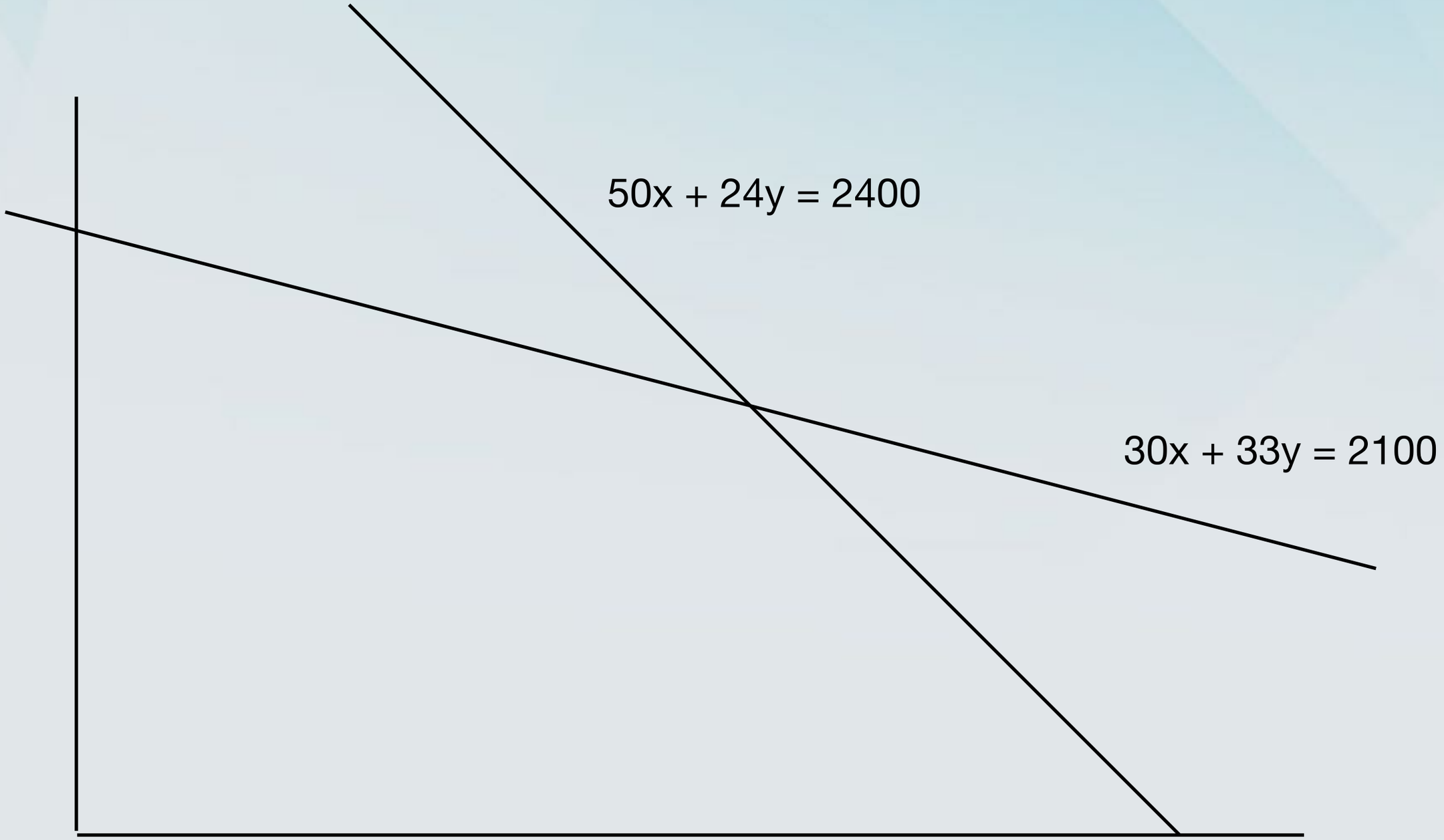

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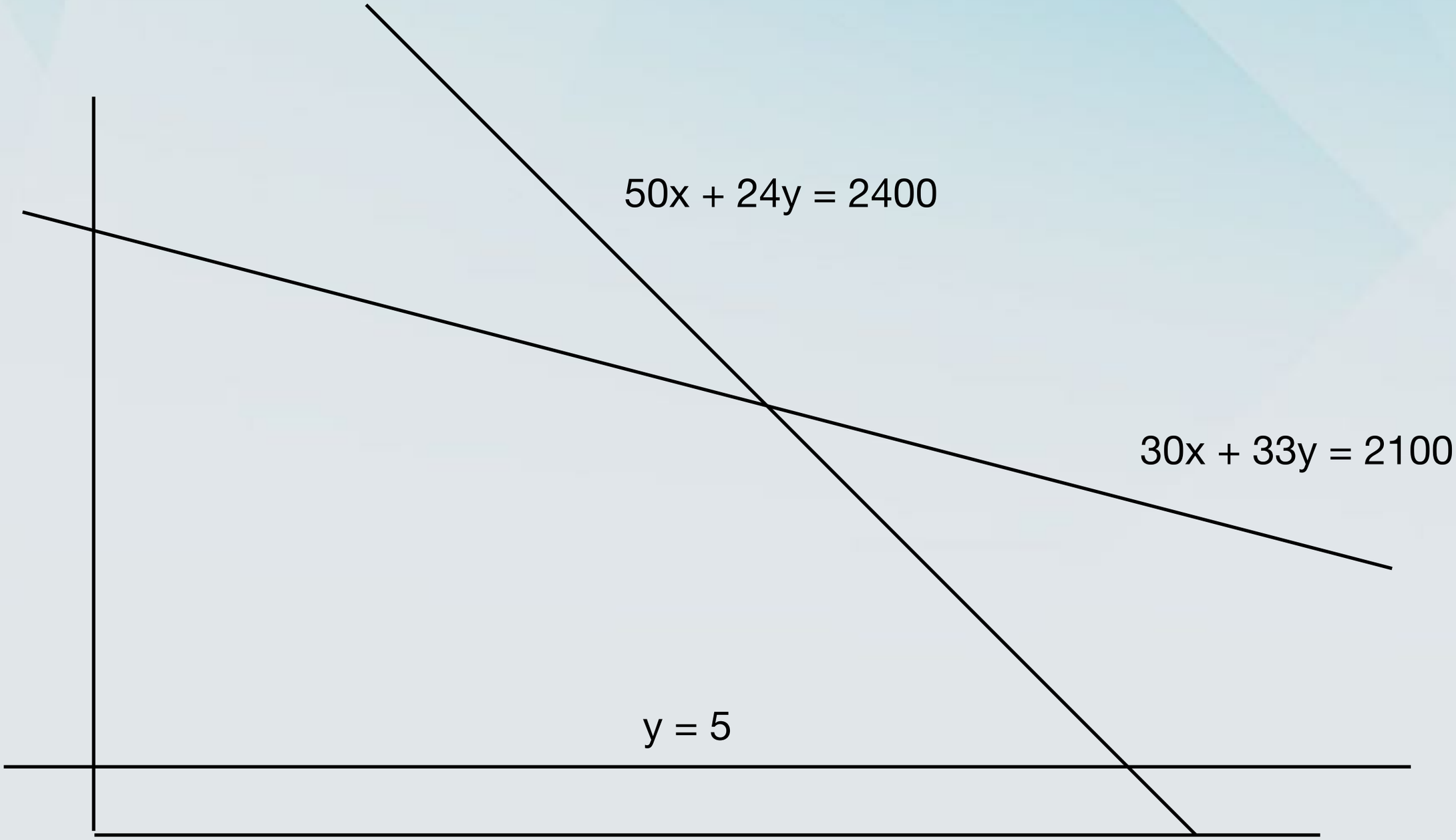

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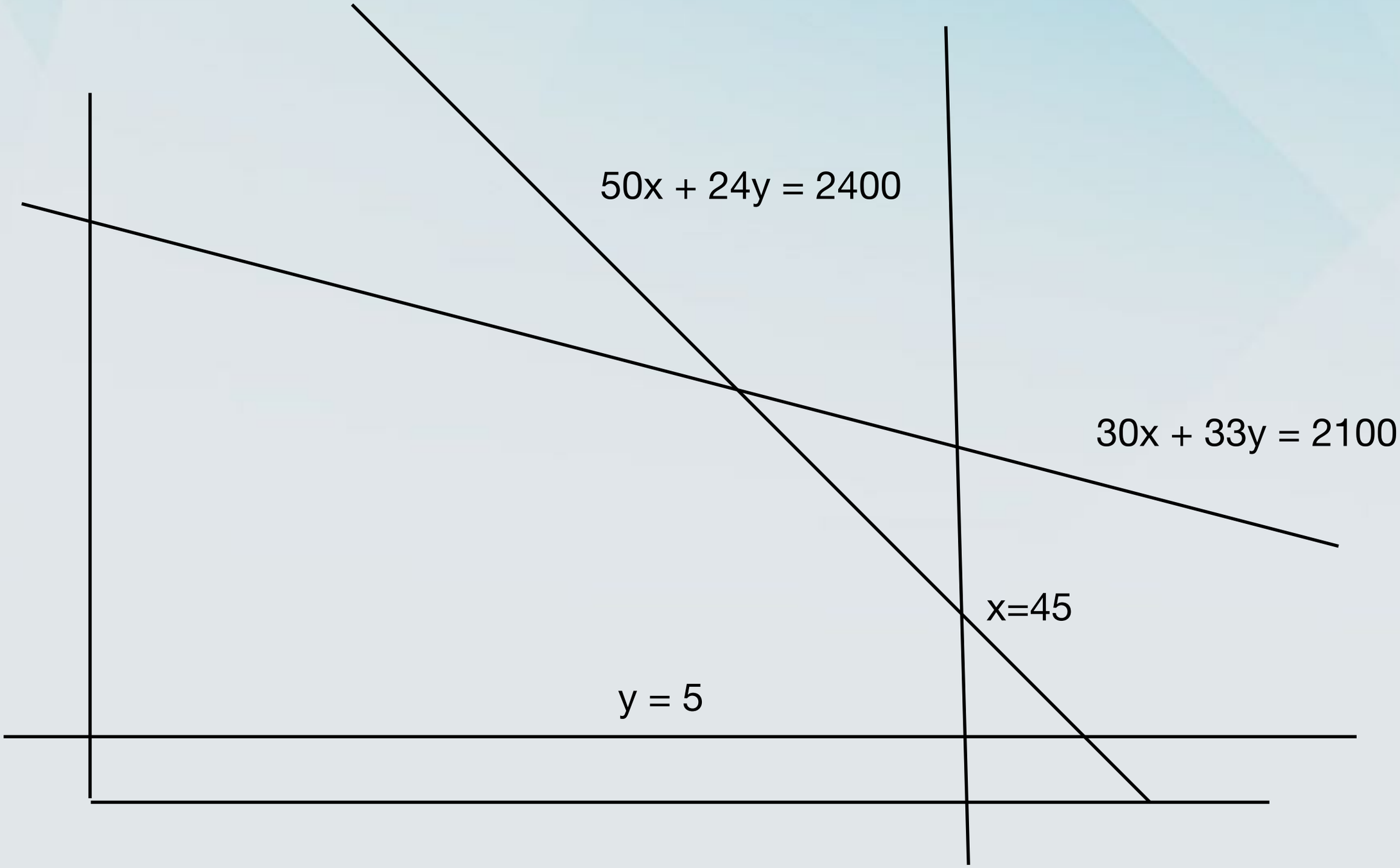

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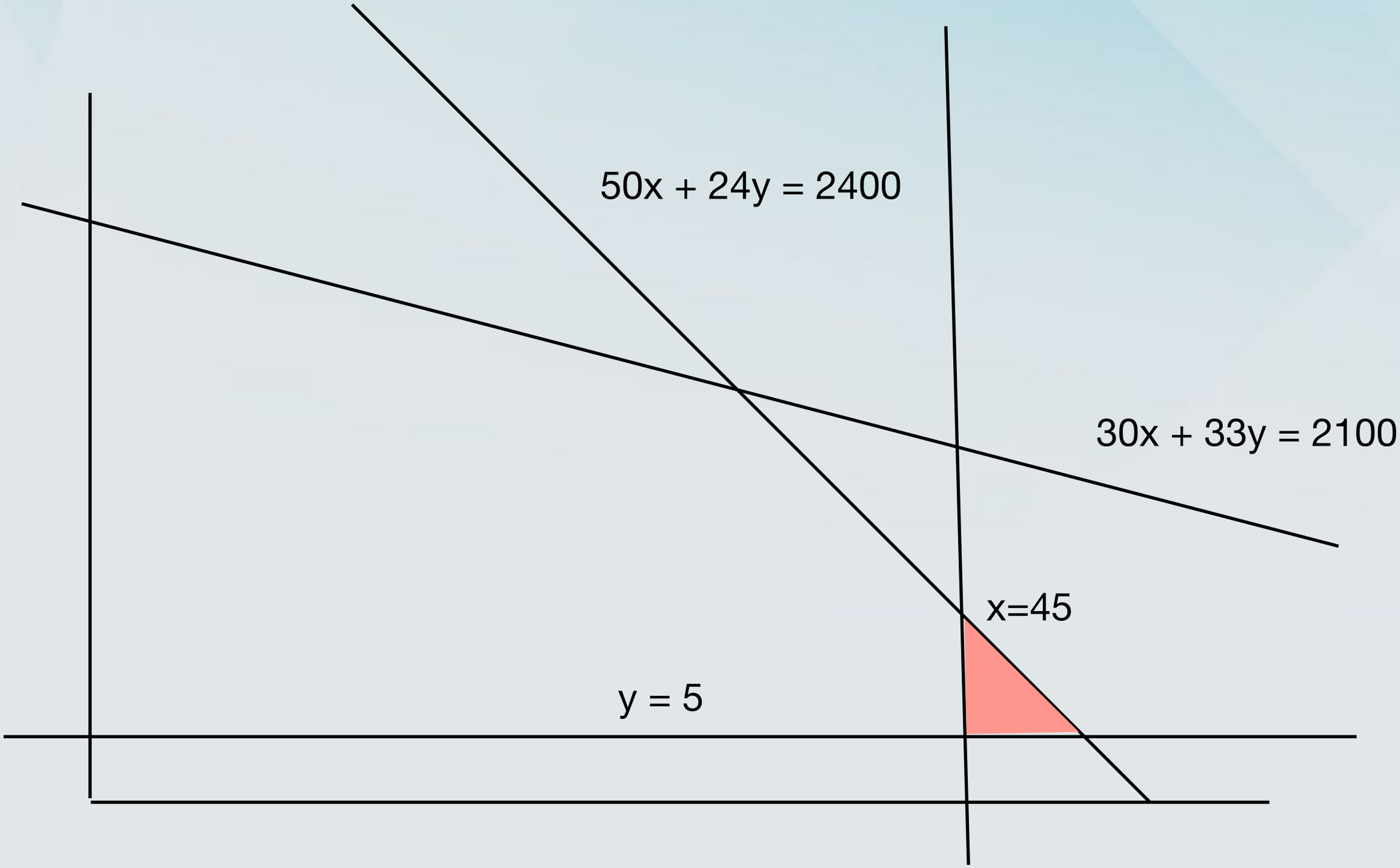

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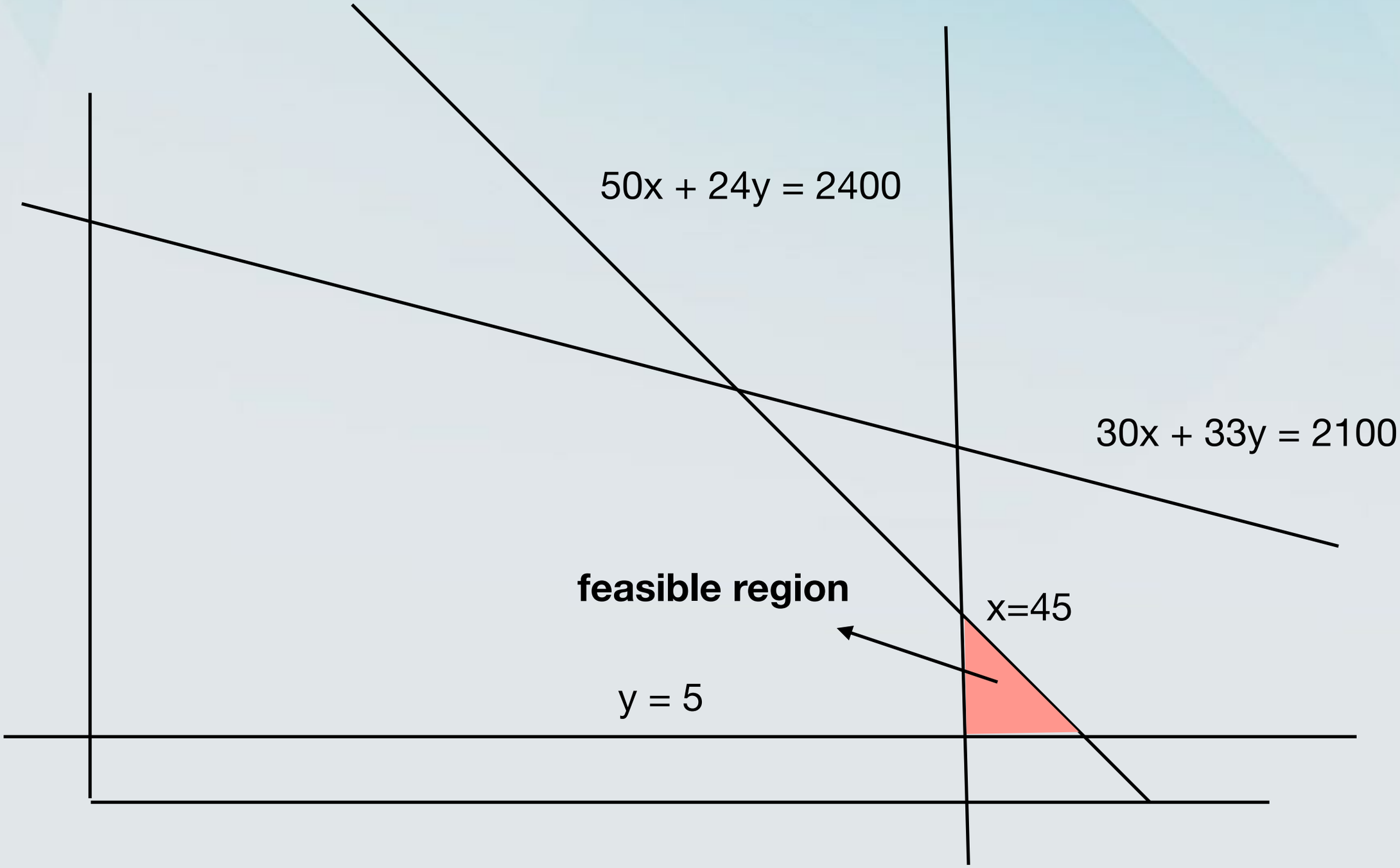

Solving the linear program

- To find the optimal solution, it suffices to examine the *corners* of the feasible region.
- These are the intersection points of the lines defined by the constraints.
  - e.g., 50x+24y - 2400 = x - 45

Linear programming

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

Linear programming (in matrix form)

$$\begin{aligned} \text{maximise} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

Feasible region

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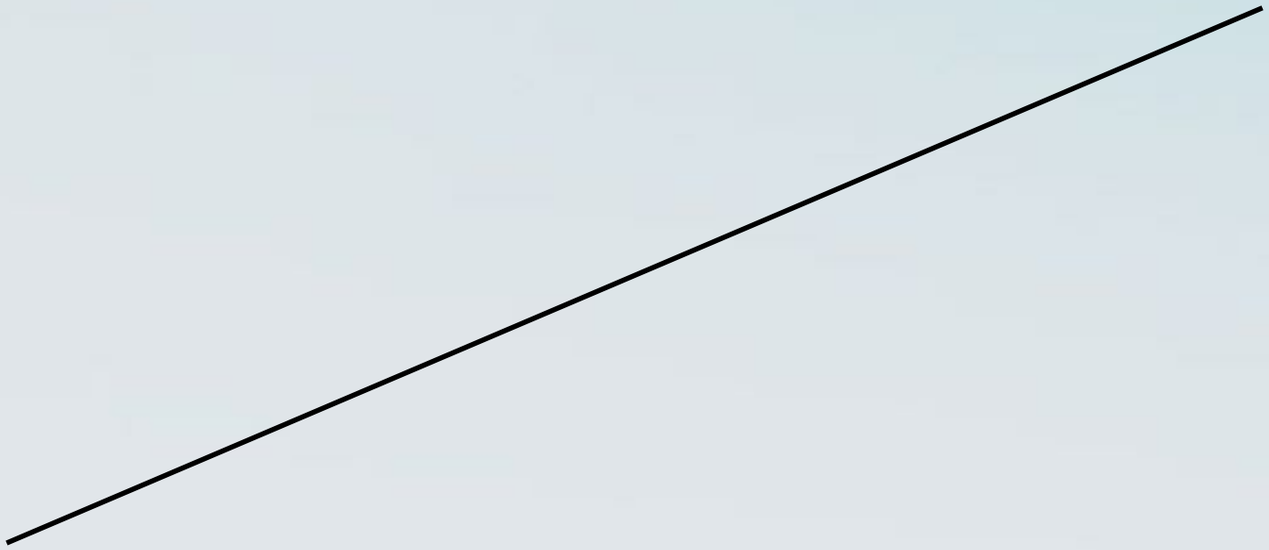

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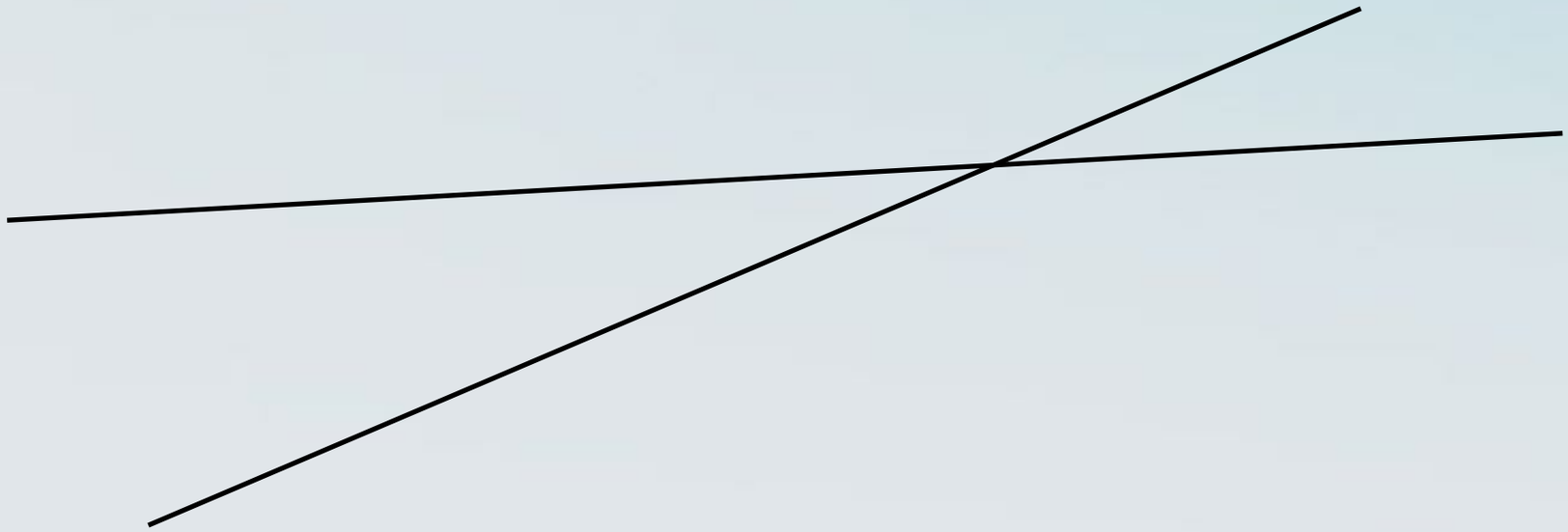

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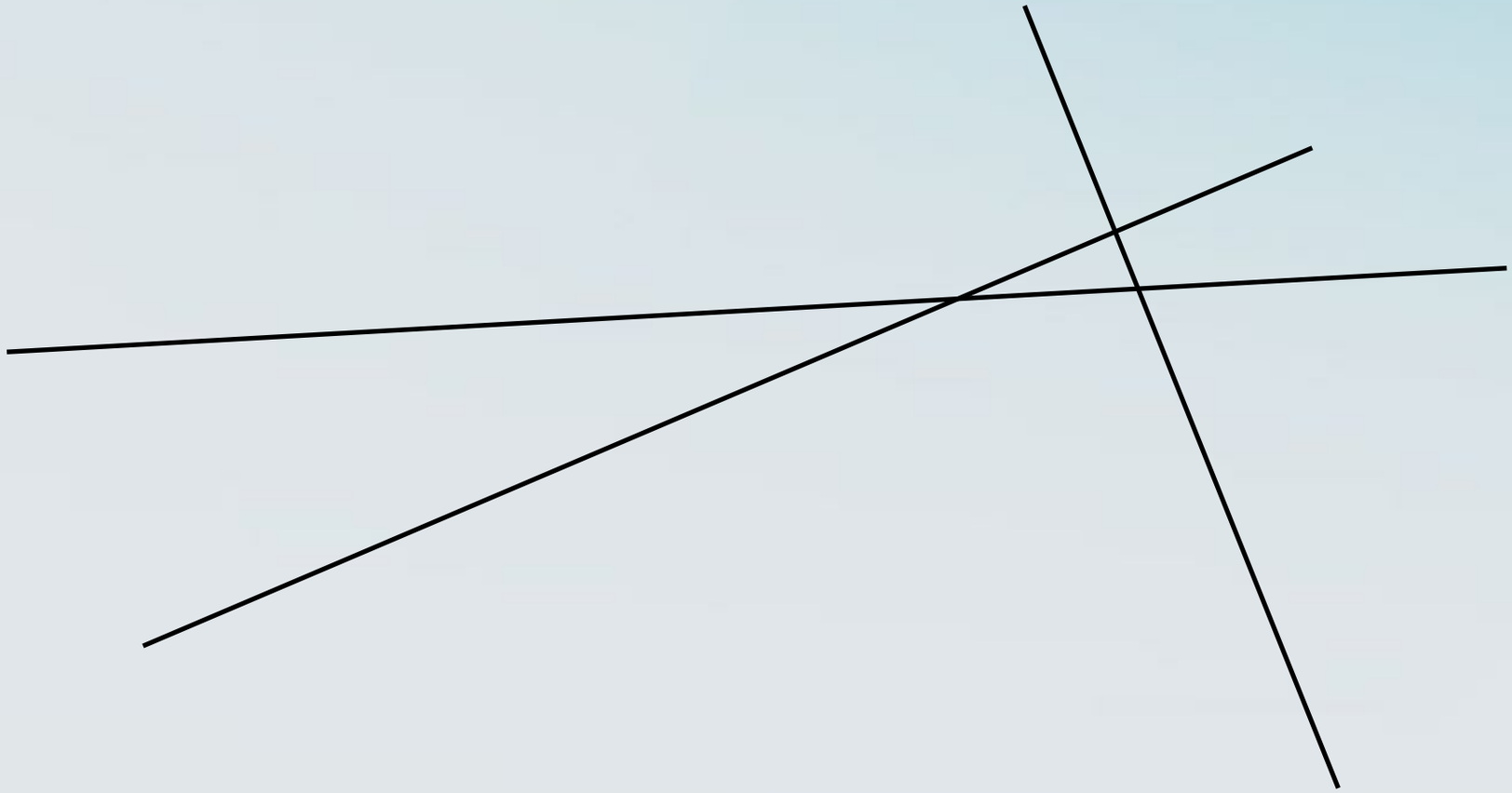

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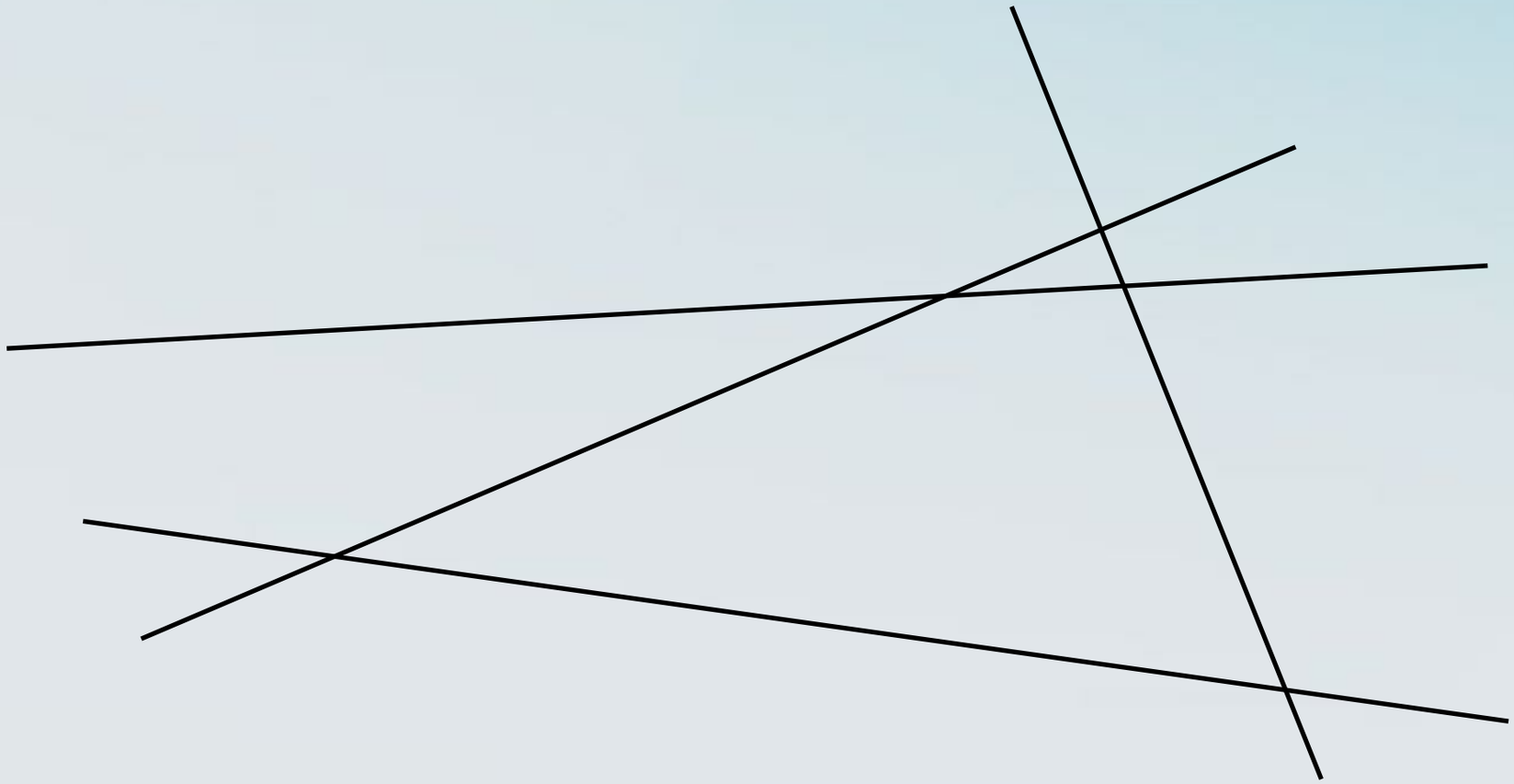

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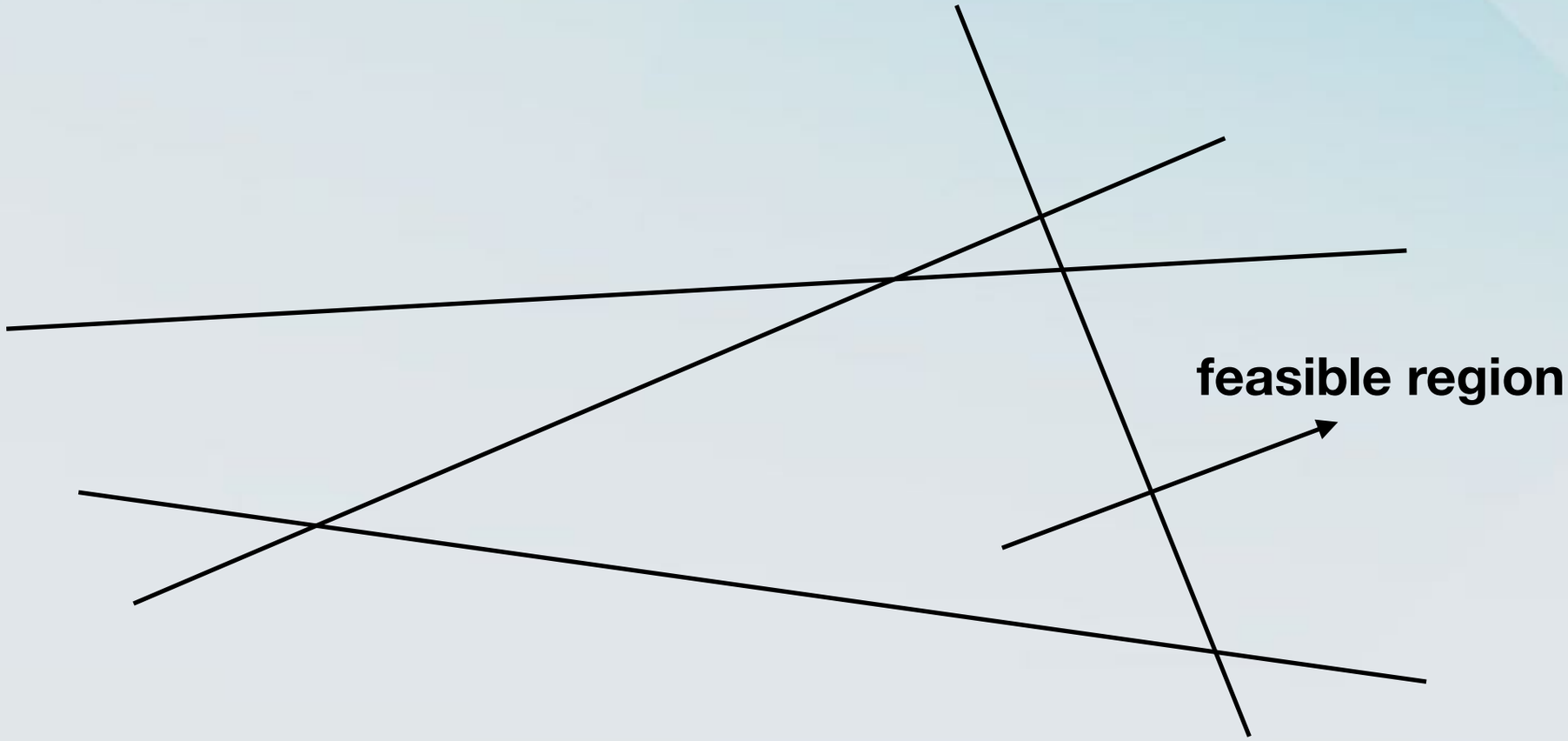

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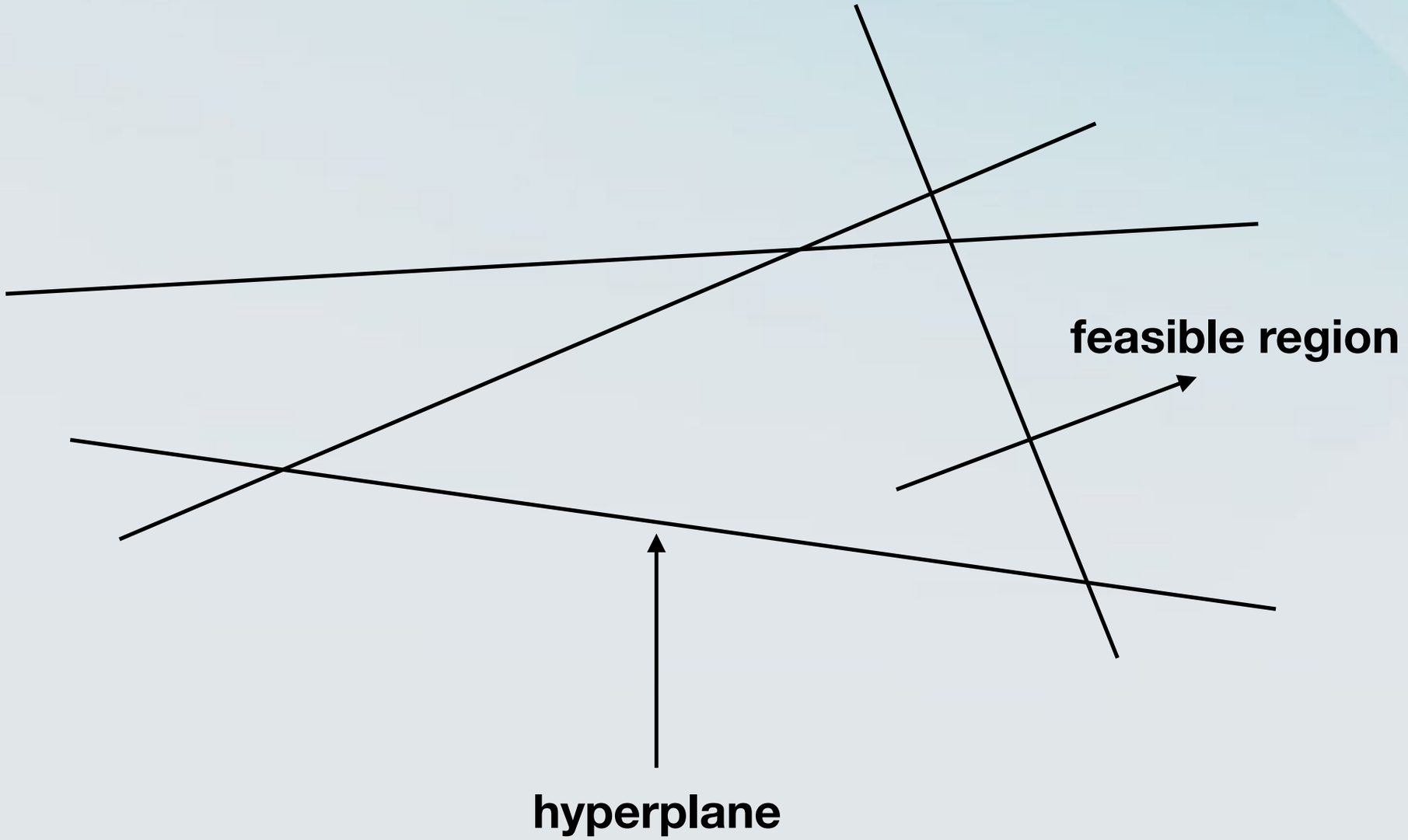

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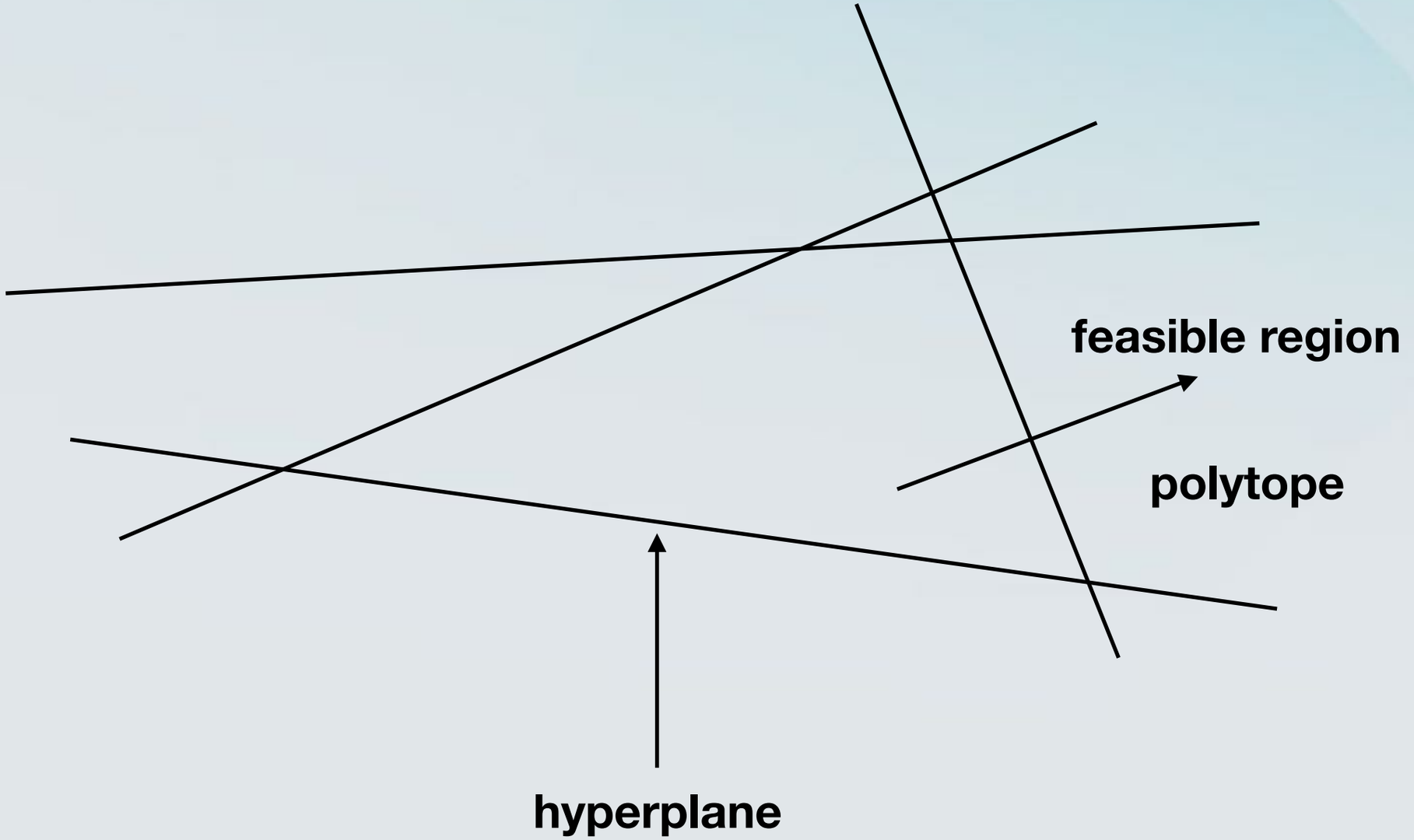

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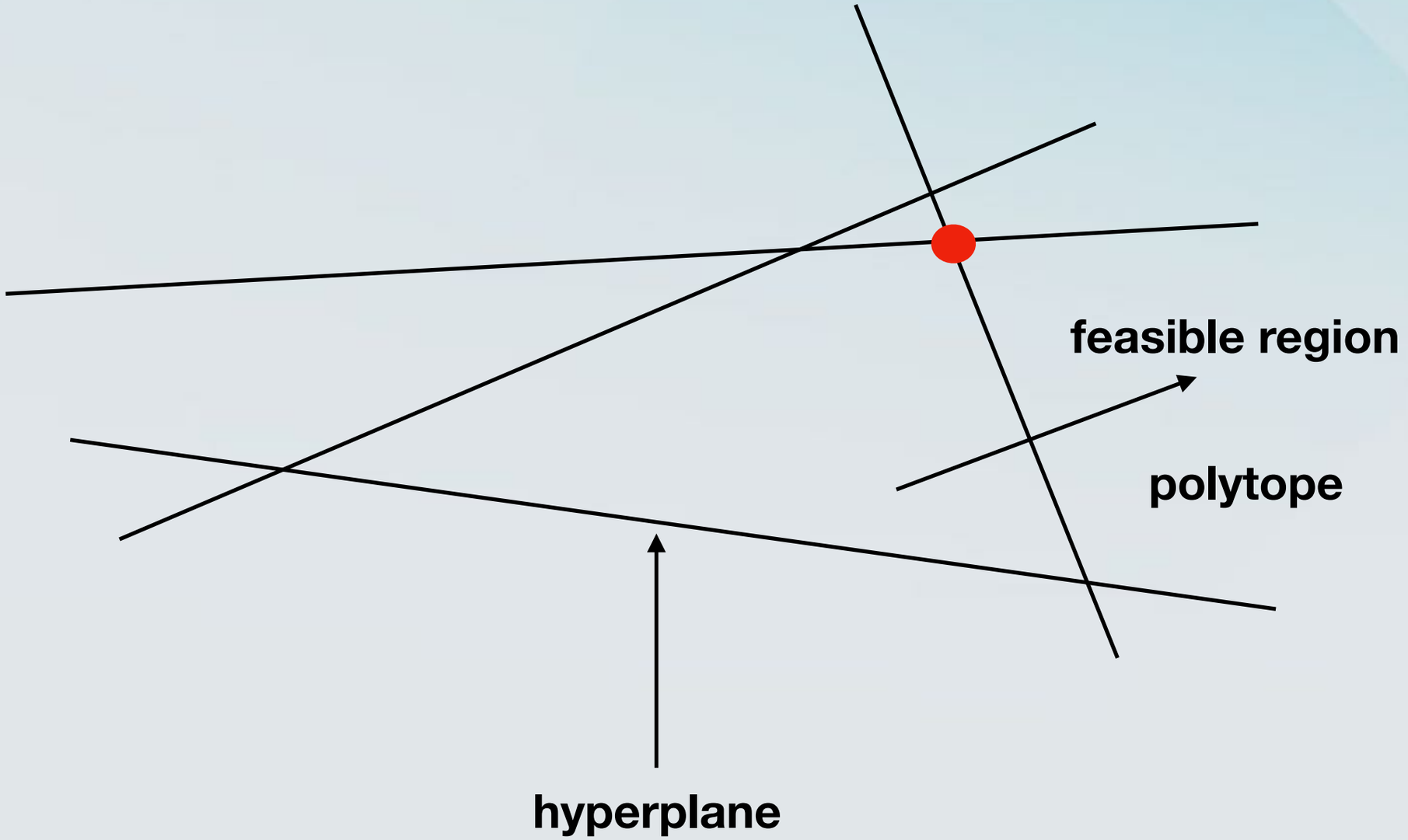

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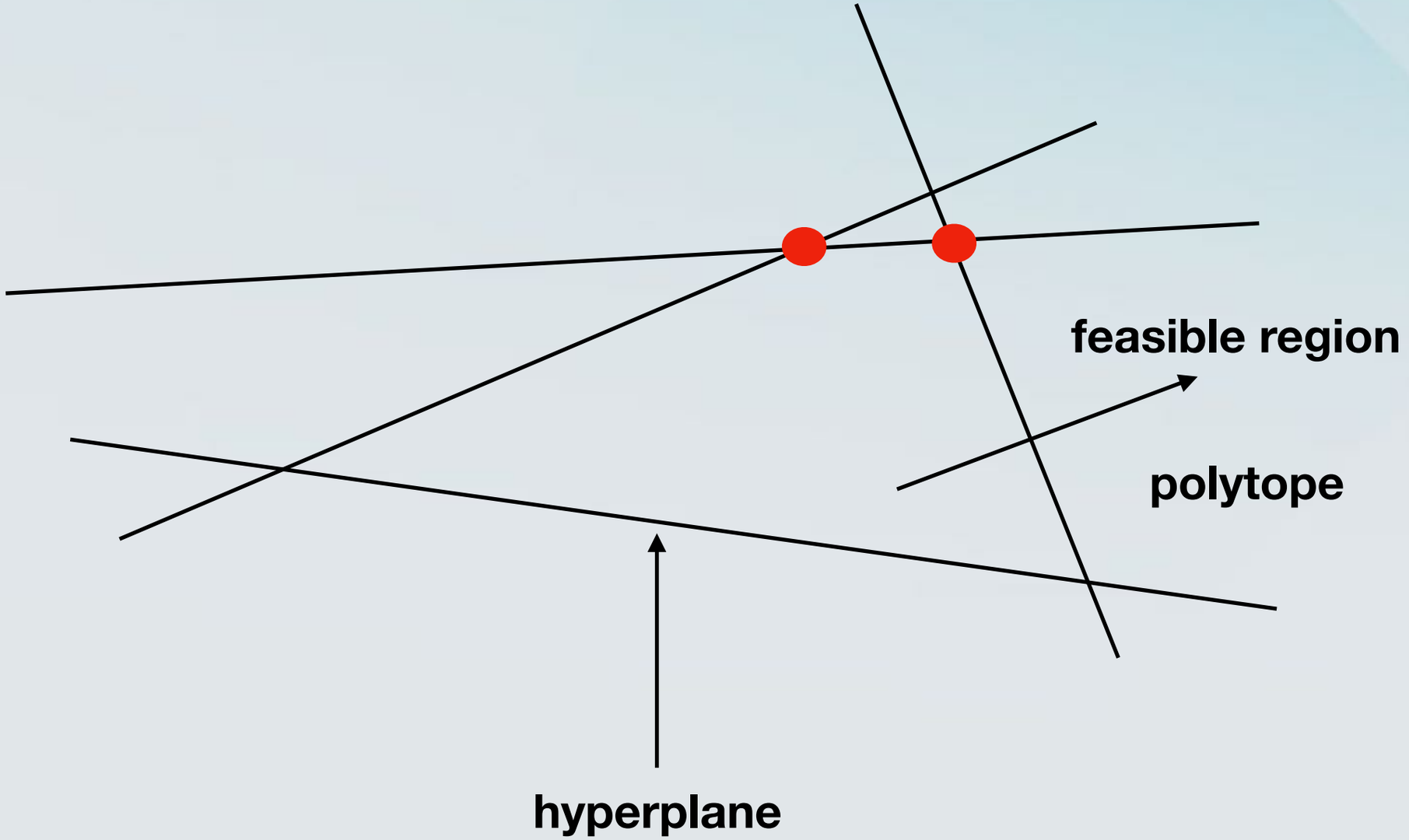

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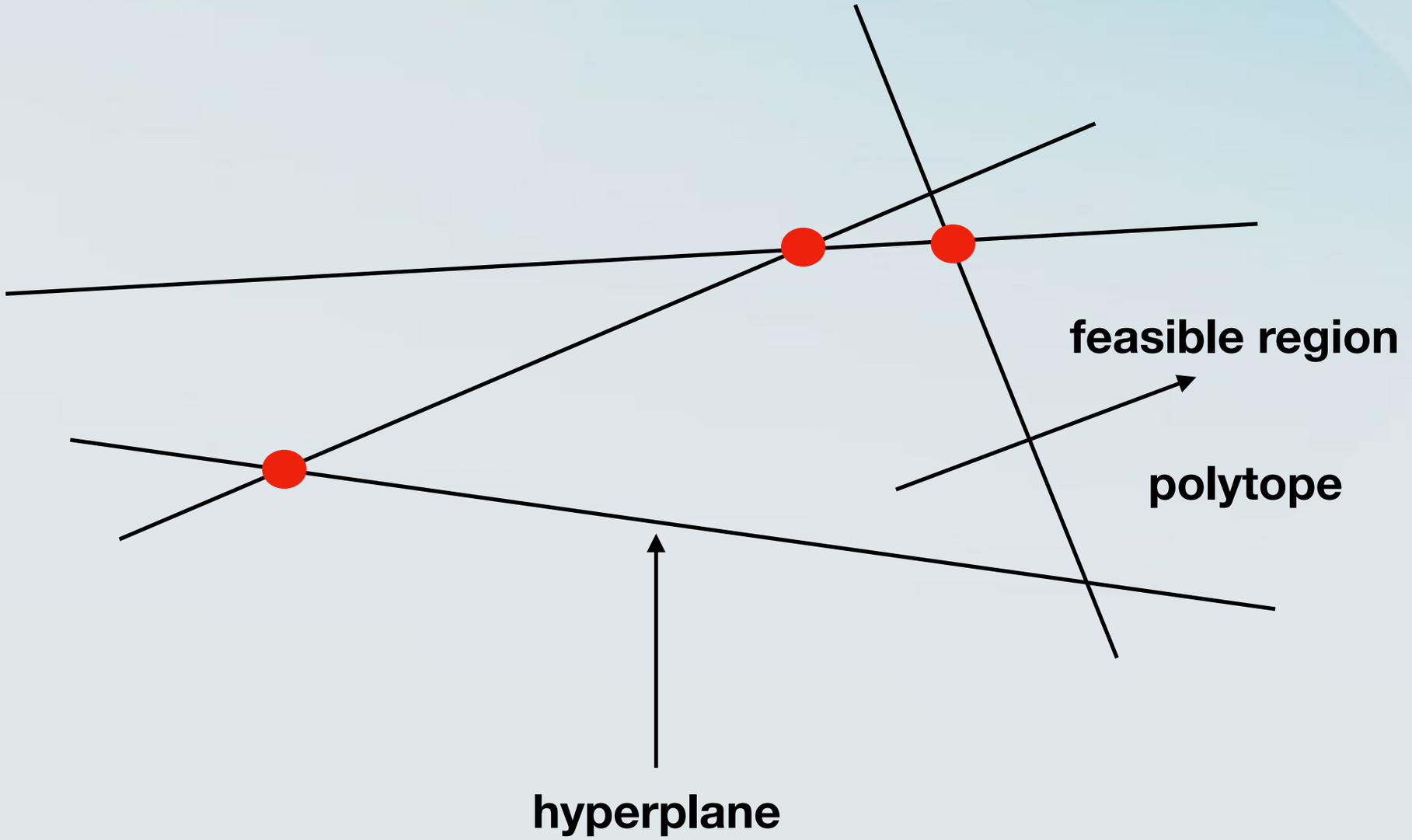

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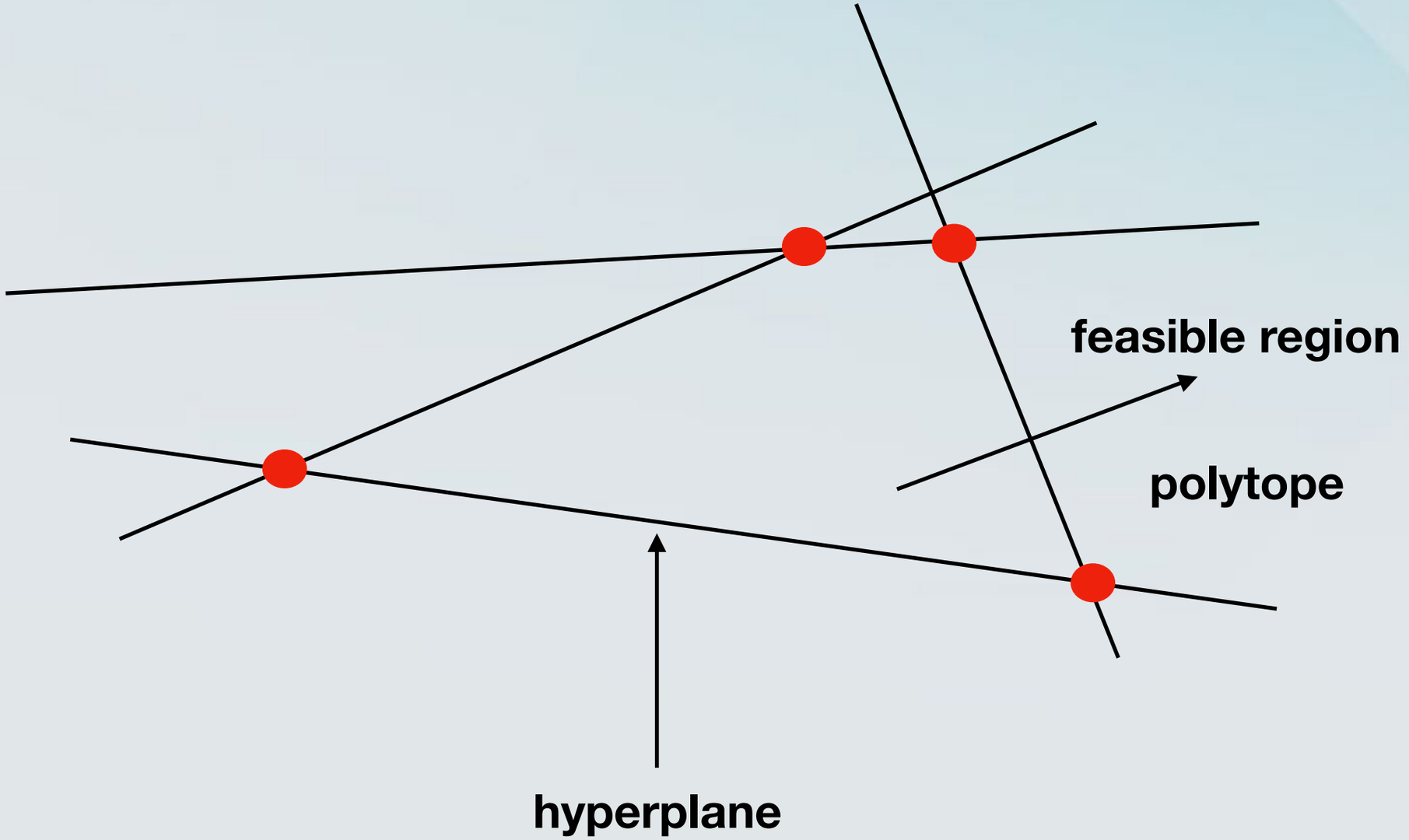

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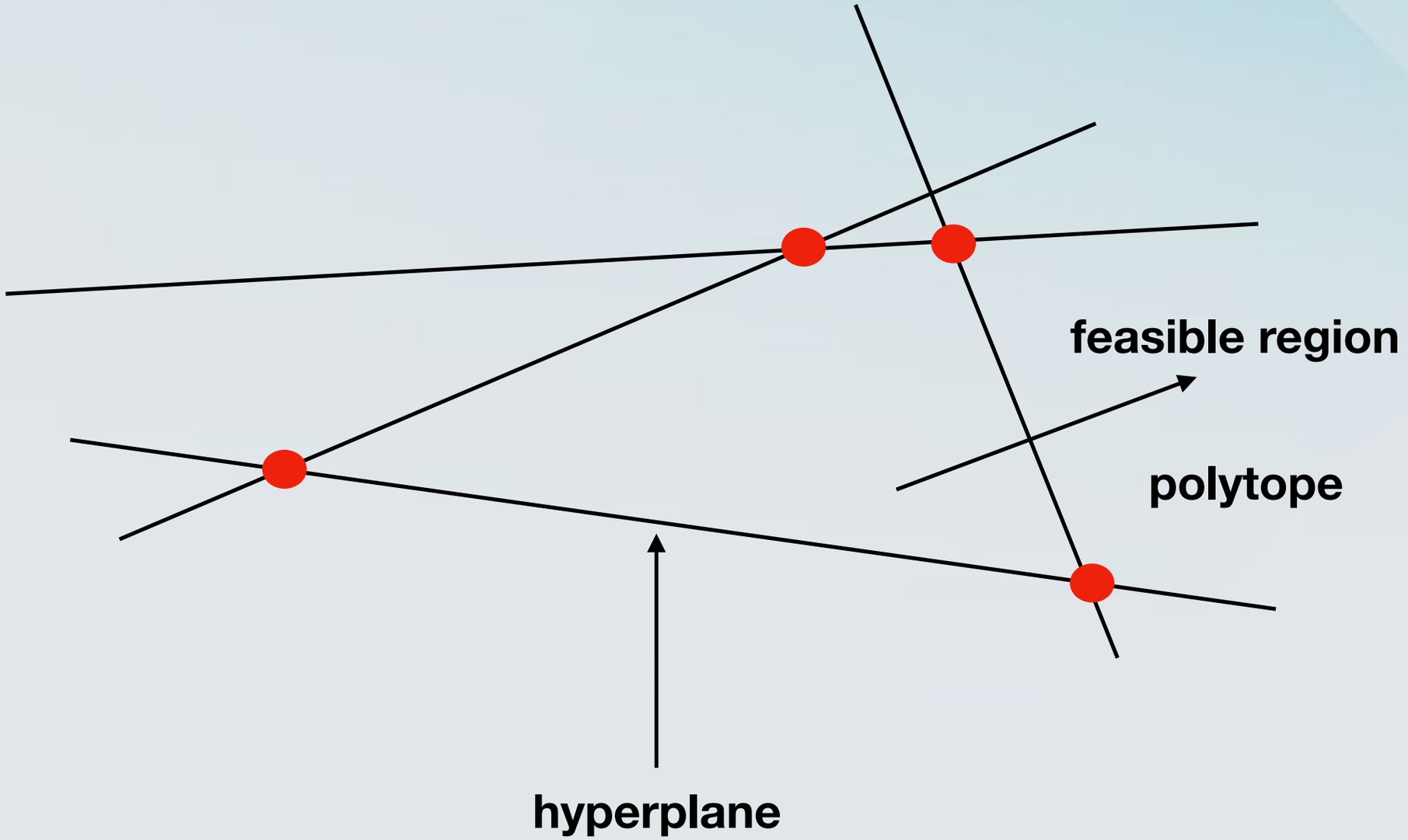

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- Other algorithms for solving LPs:
  Ellipsoid Method, Interior Point Methods

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- What we will need here:
  - If we manage to formulate a problem as a linear program, then we can solve it in polynomial time.

Linear programming

$$\text{maximise} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m$$

$$x_j \geq 0, \quad j = 1, ..., n$$

Integer Linear programming

$$\text{maximise} \qquad \sum_{j=1}^{n} c_j x_j$$

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$$x_j \geq 0, \quad j = 1, ..., n$$

$$x_j \text{ is integer}$$

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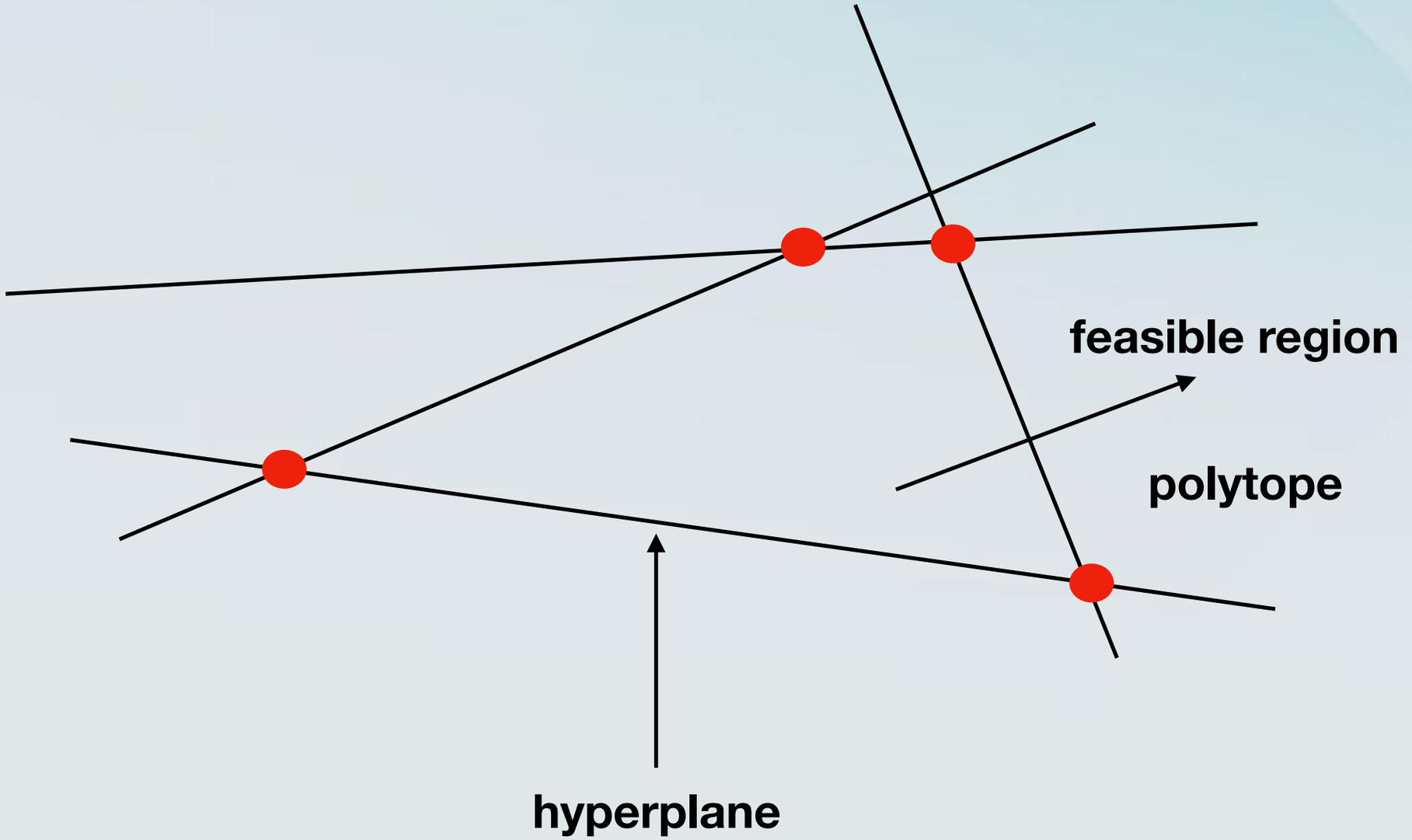

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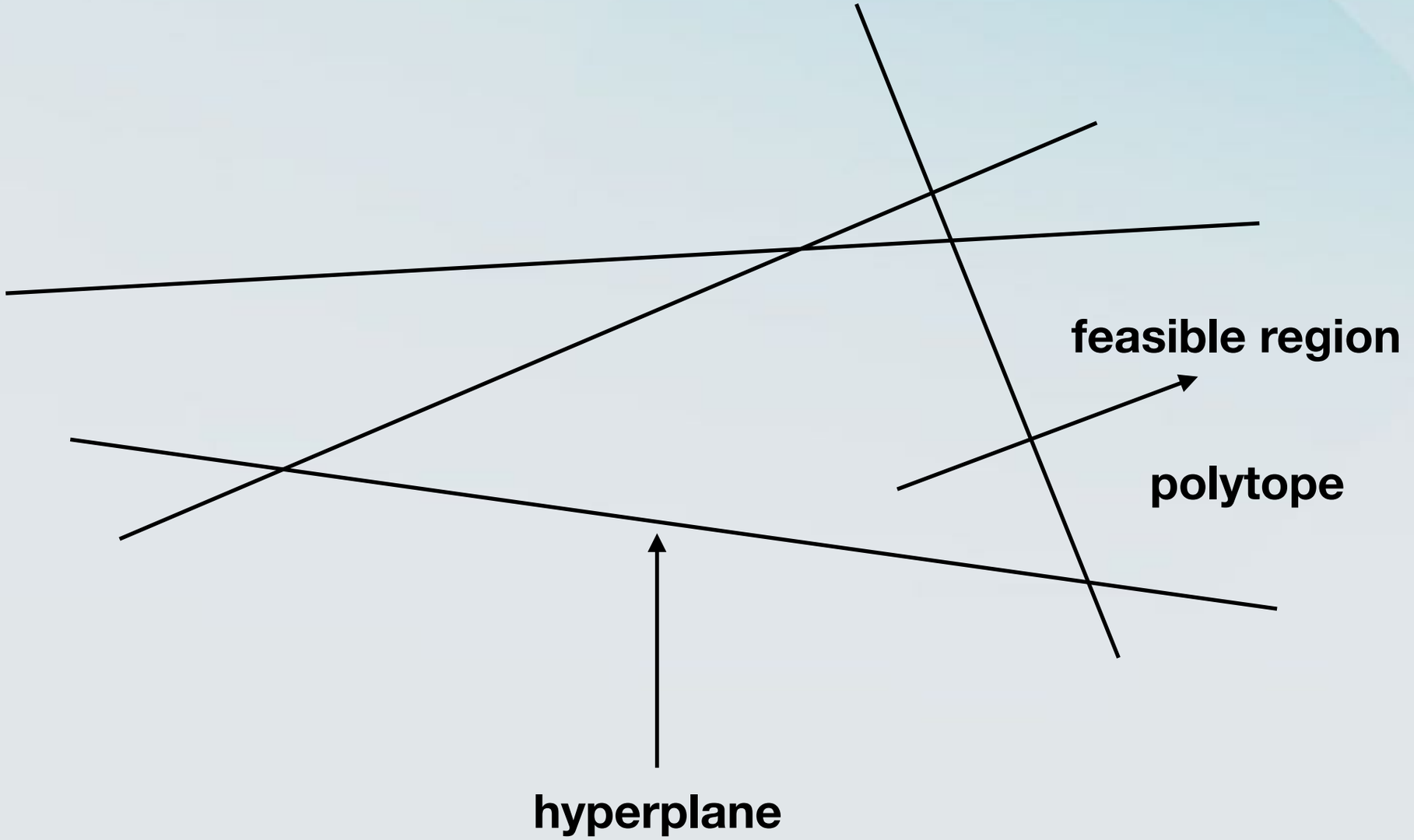

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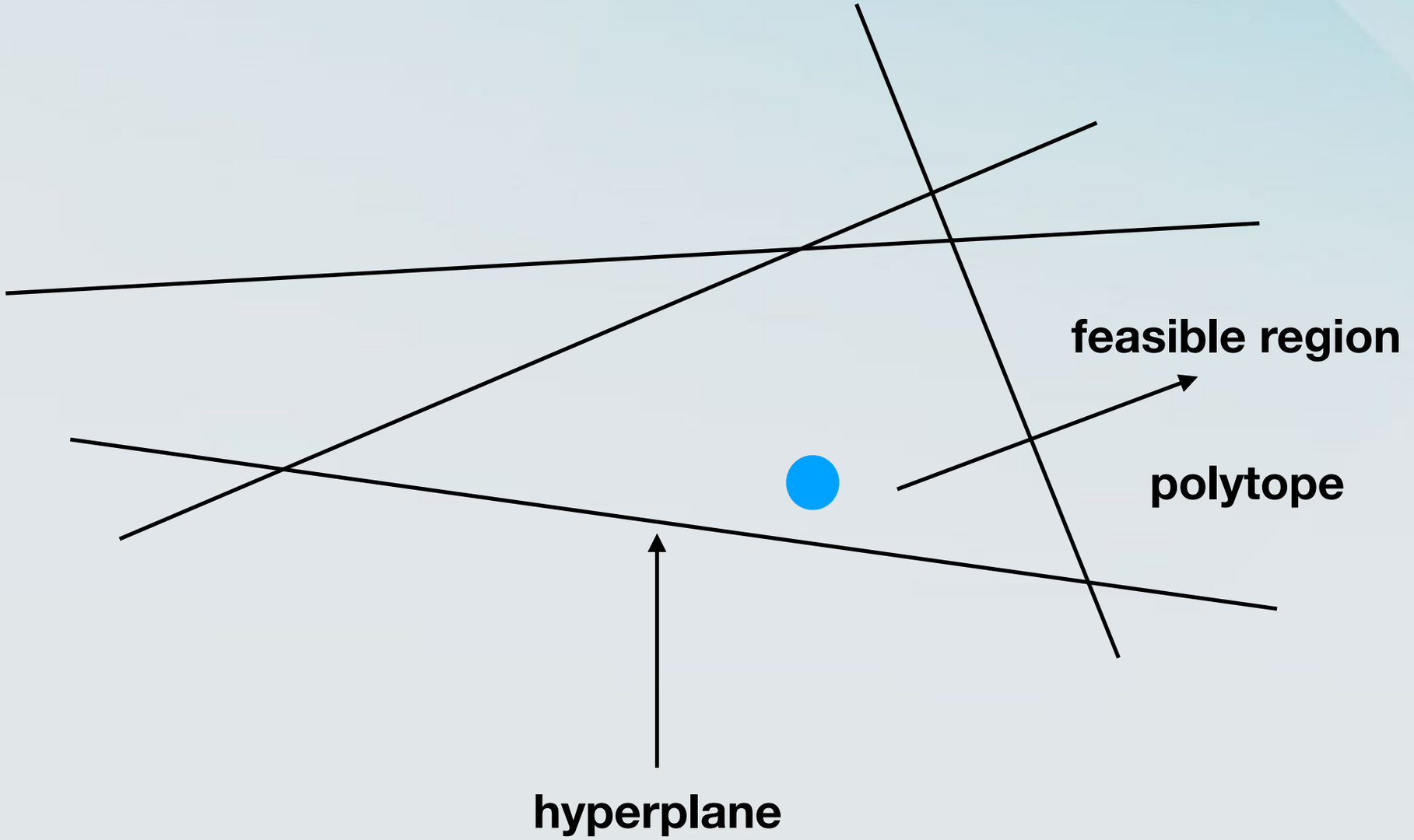

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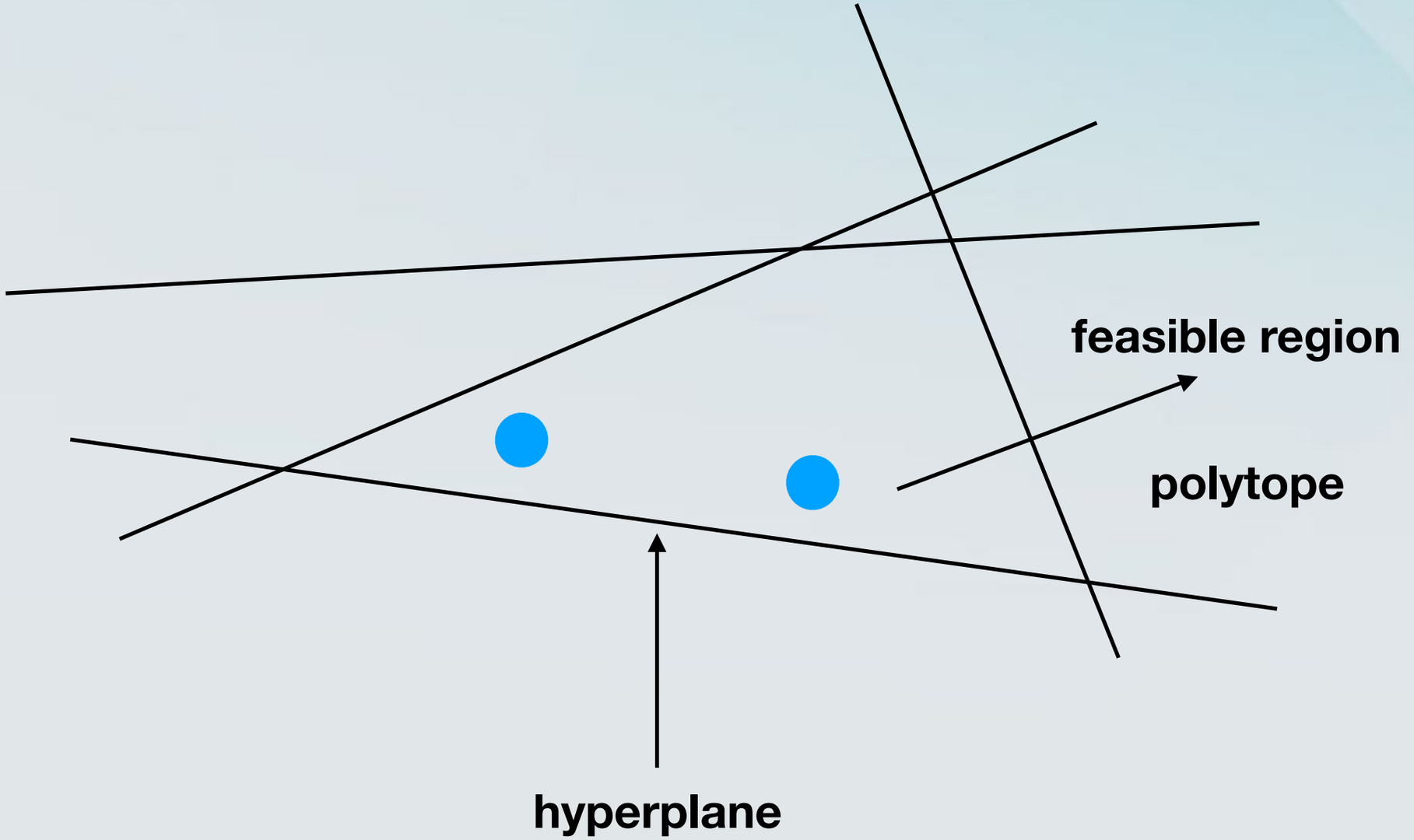

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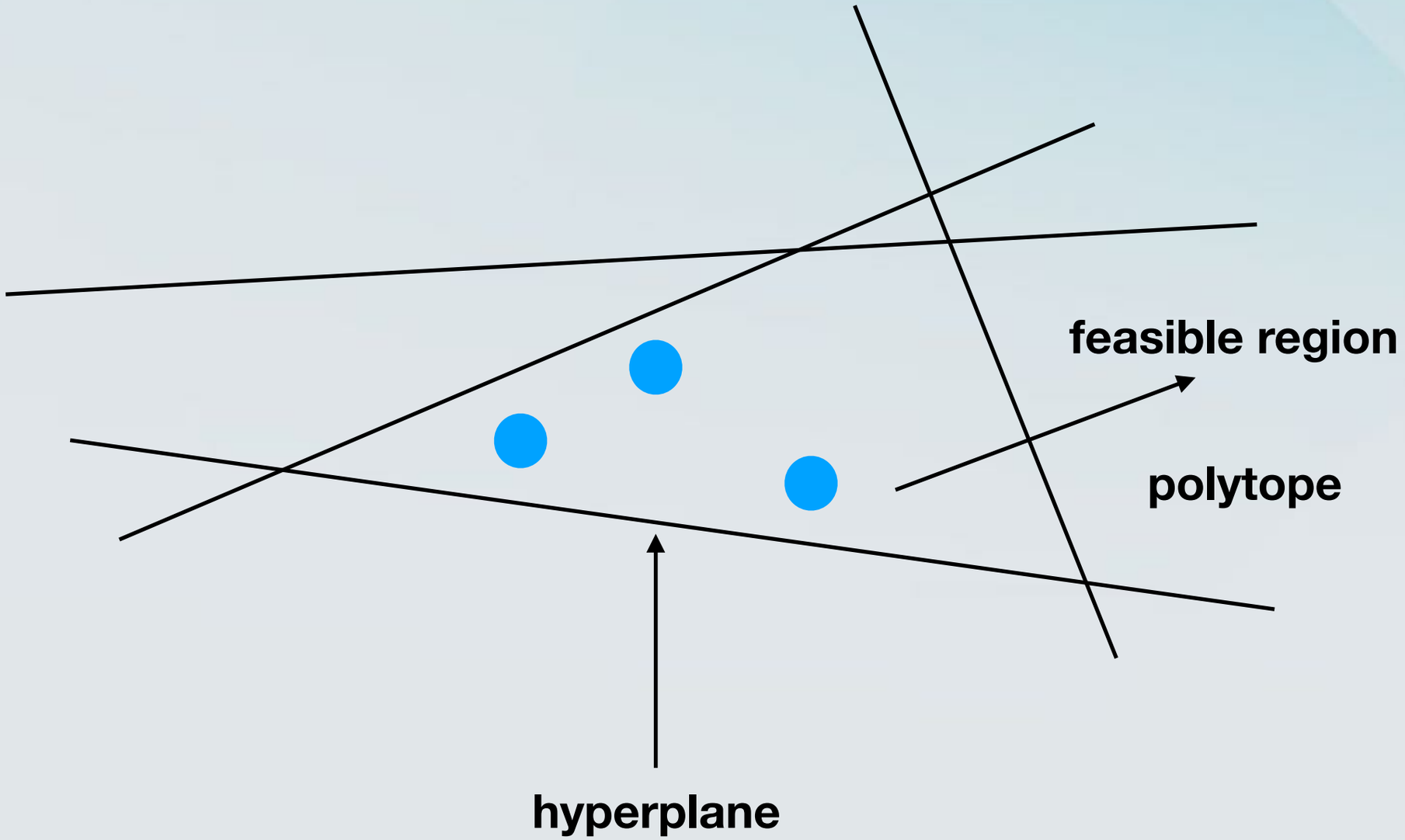

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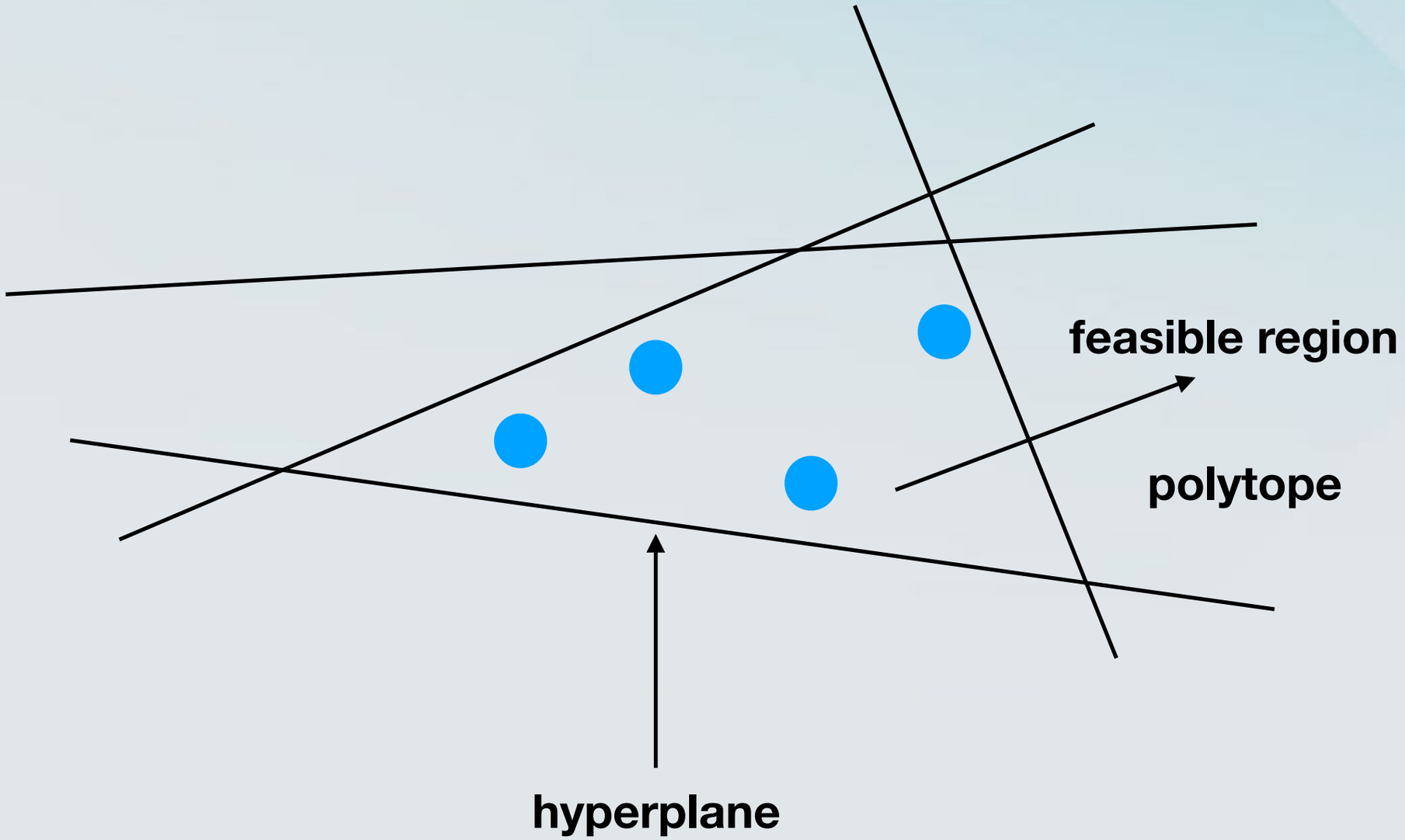

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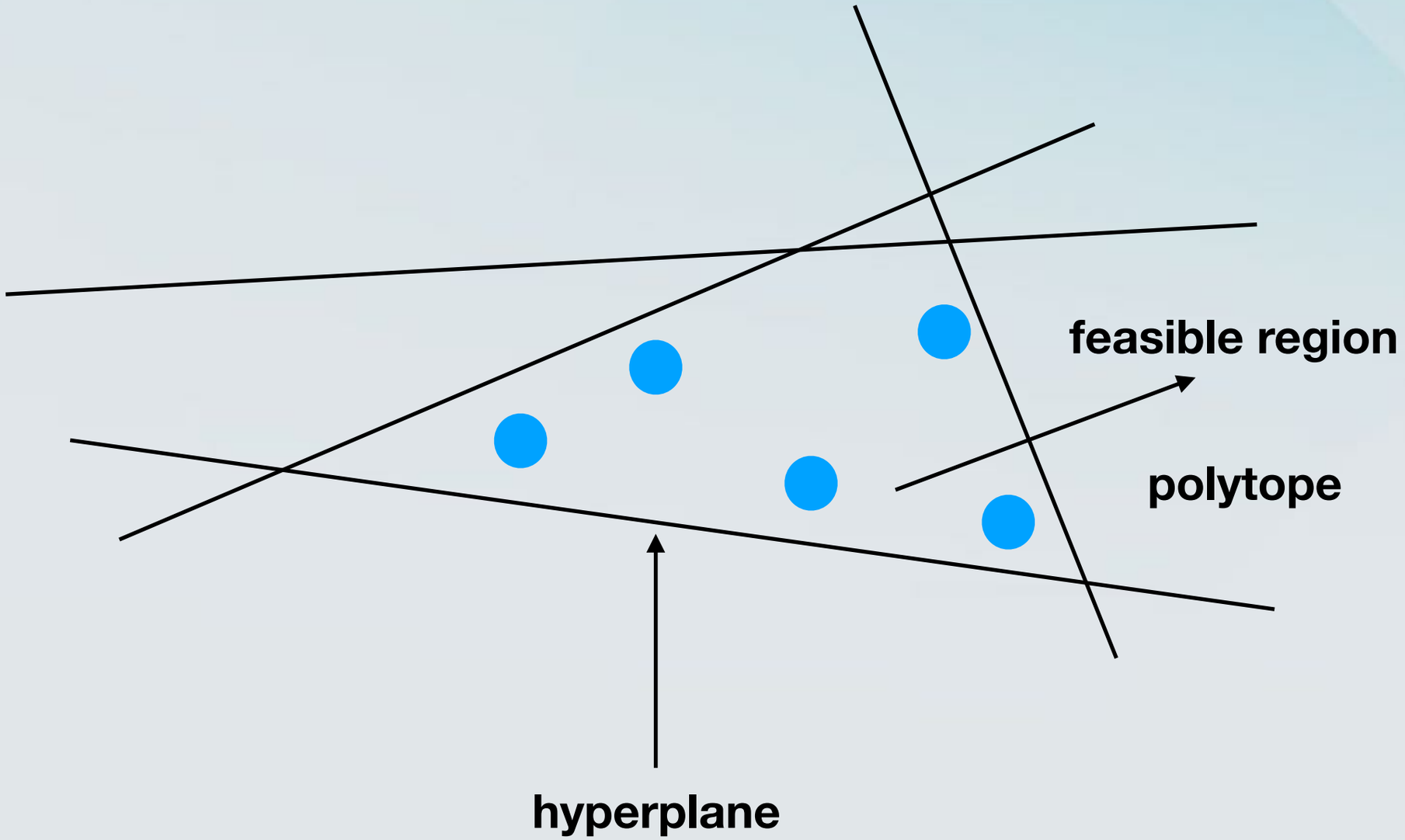

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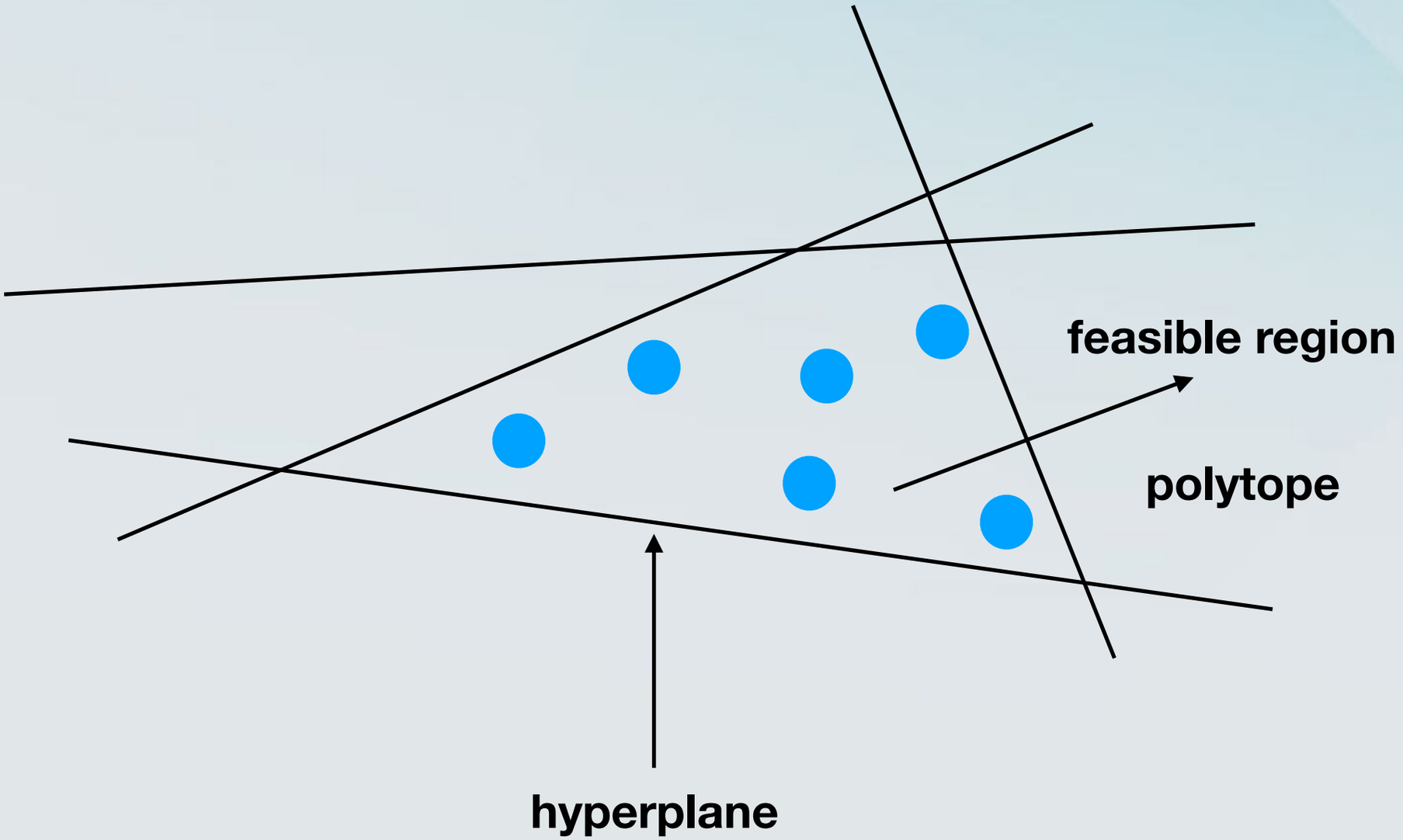

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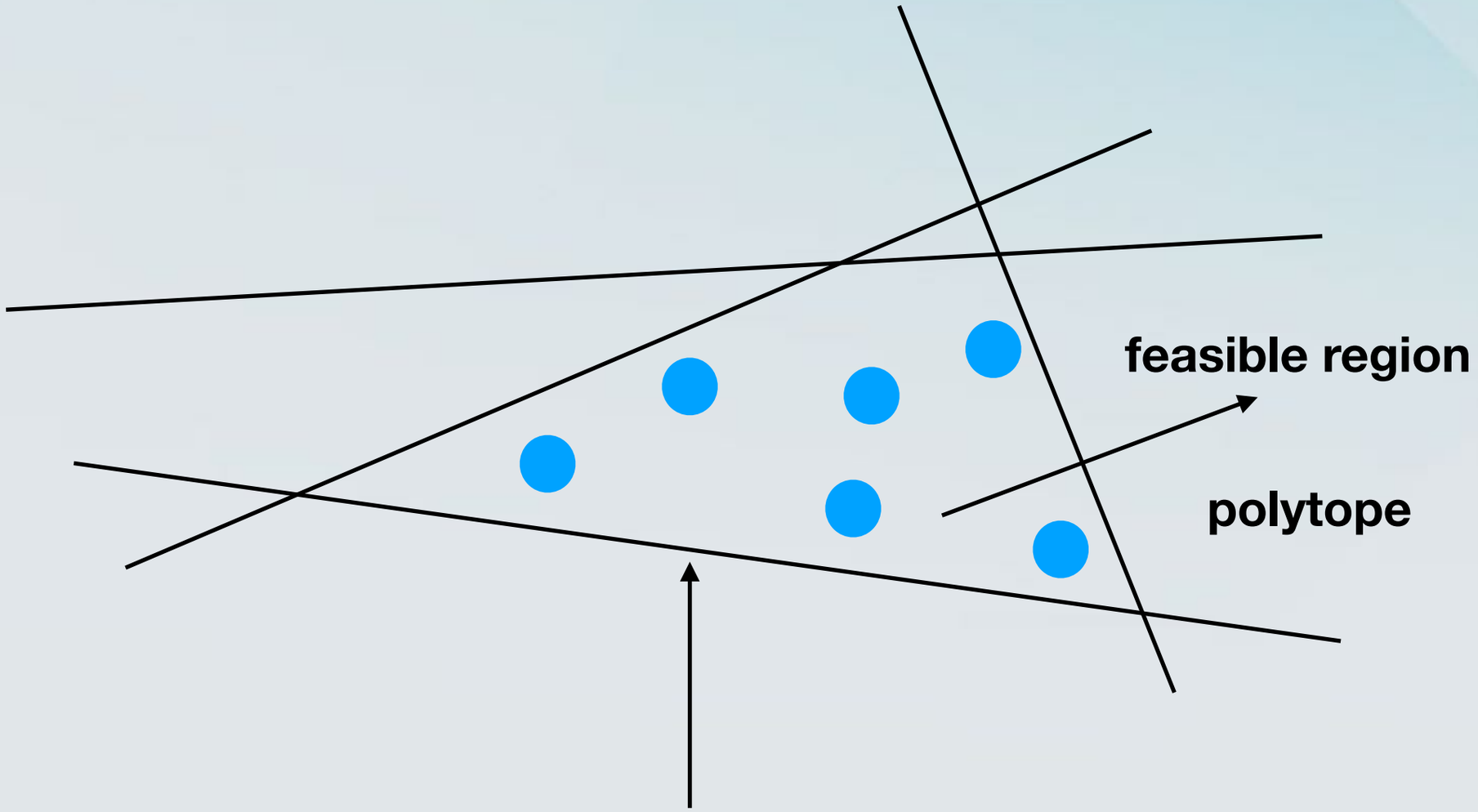

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        - Yes, but in the worst-case, it will still take exponential time in many ILPs.
    - Generally speaking, ILP solving is NP-hard.

Summarising

- Linear Programs can be solved in polynomial time.

- Integer Linear Programs generally cannot be solved in polynomial time (unless P=NP).

Duality

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- Maximisation becomes minimisation.

- The two linear programs will have a very important connection.

Duality

The Primal

$$\begin{aligned} \text{maximise} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1\in}^{n} \alpha_{ij} x_j \leq b_i, \quad i = 1, ..., m \\ & x_j \geq 0, \quad j = 1, ..., n \end{aligned}$$

The Dual

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{subject to} \quad & \sum_{i=1}^{m} \alpha_{ij} y_i \geq c_j, \quad i = 1, ..., n \\ & y_j \geq 0, \qquad j = 1, ..., m \end{aligned}$$

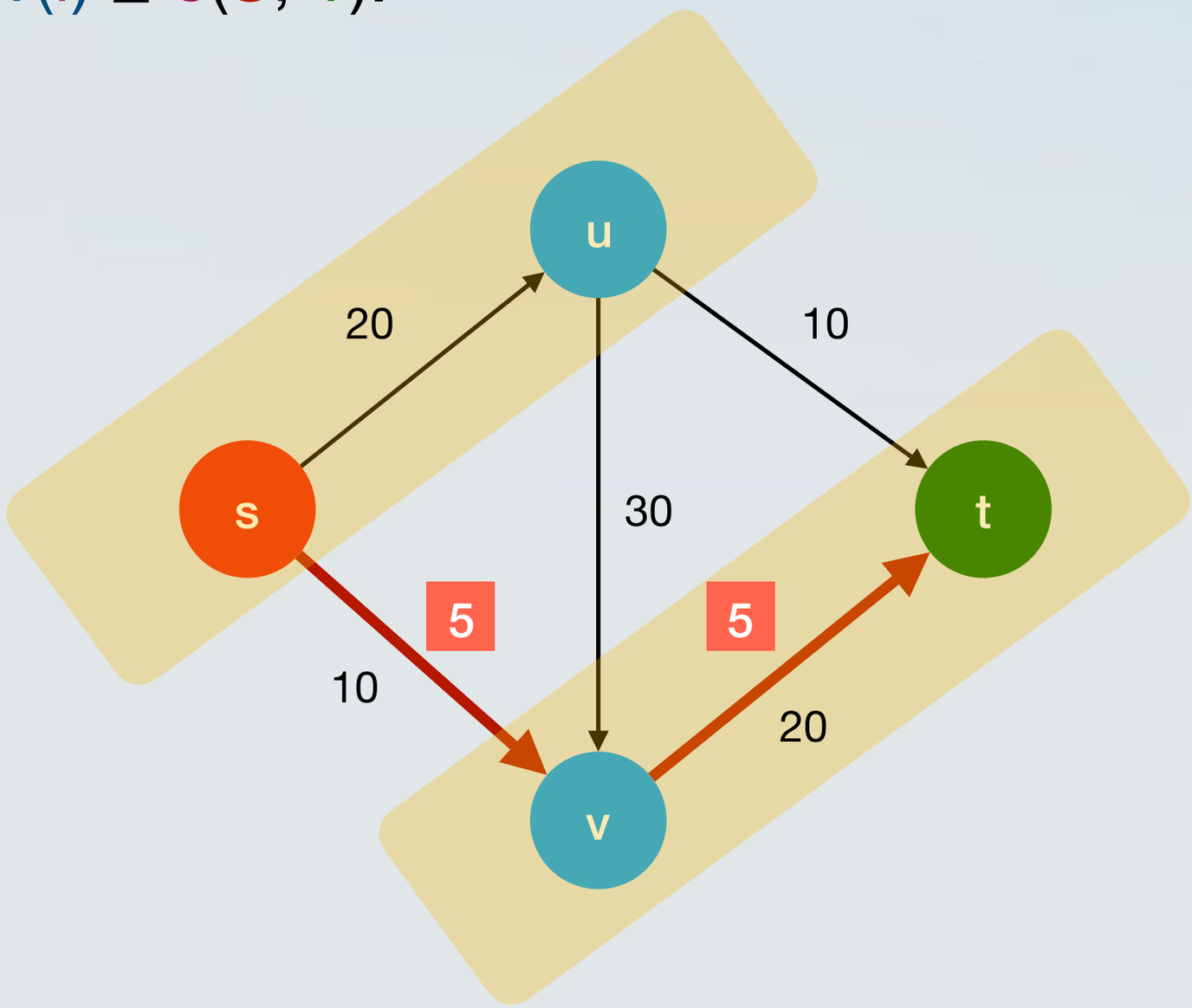
Weak Duality

- Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. Then we have that

$$\text{value}(x) \leq \text{value}(y)$$

Weak Duality

- Fact 3: Let f by any (s-t) flow and (S, T) be any (s-t) cut. Then v(f) ≤ c(S, T).



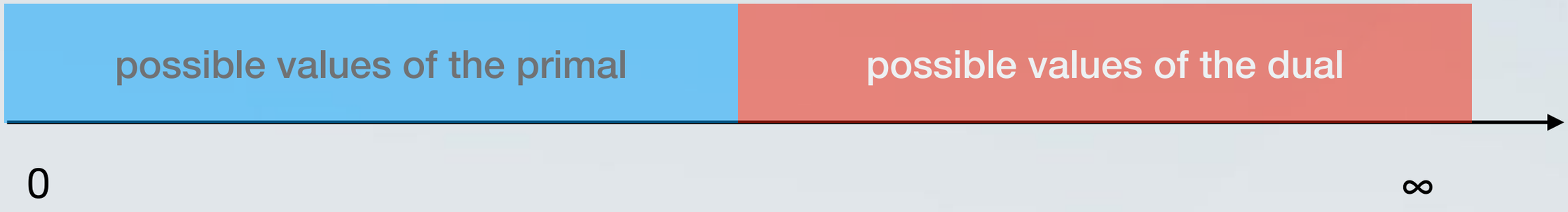
Strong Duality

- Let x be any feasible solution to the Primal and let y be any feasible solution to the Dual. If

$$\text{value}(x) = \text{value}(y)$$

then x and y are both optimal solutions.

Pictorially



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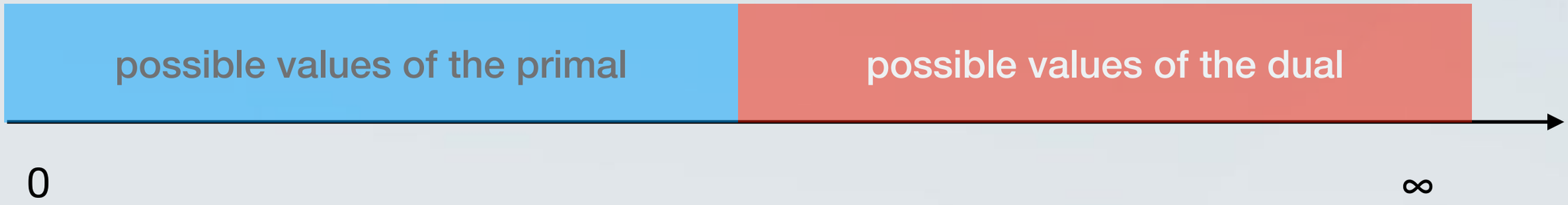

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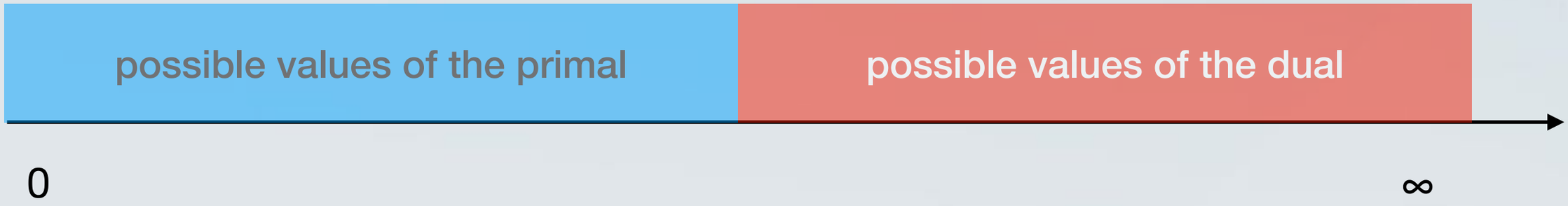

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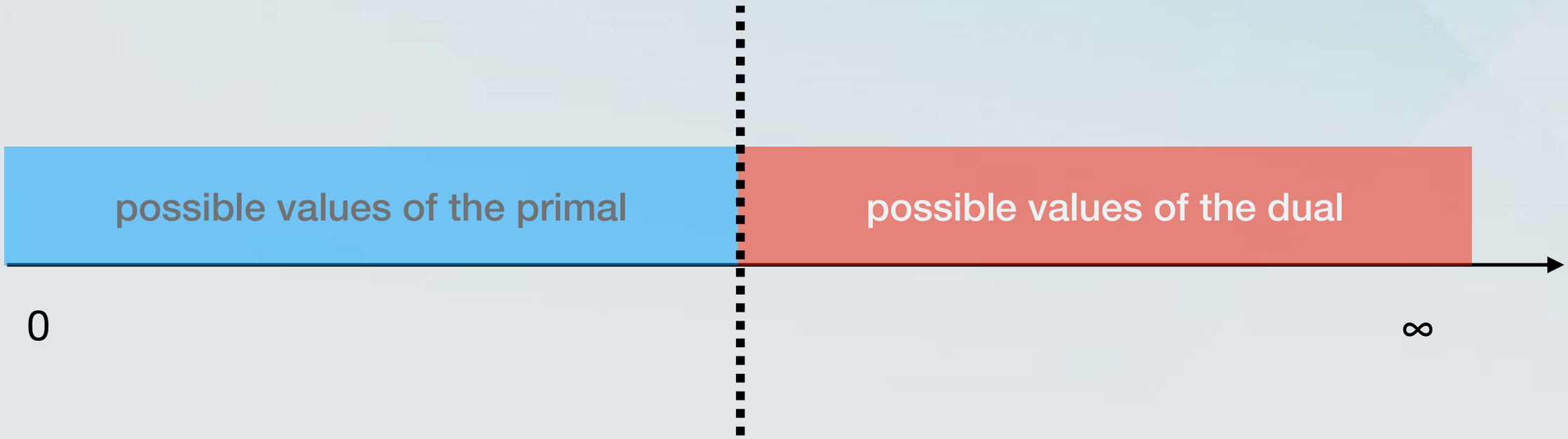

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The Max-Flow Min-Cut Theorem

- Theorem: In every flow network, the value of the maximum flow is *equal* to the capacity of the minimum cut.

- This is a consequence of the *strong duality theorem* for linear programs!

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- It improves the solution in every step (via *pivoting*) until it can no longer be improved.
- To prove that the solution is optimal, we use duality.
- Do you know of any other algorithms for which the same principle applies?

Maximum Flow as an LP

- We can write the maximum flow problem as a linear problem.
- "Maximise the flow, subject to capacity and flow conservation constraints".

$$
\begin{aligned}
\text{maximise} \quad & \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs} \\
\text{subject to} \quad & f_{uv} \leq c_{uv}, \quad \text{for each} \quad u, v \in V \\
& \sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv}, \quad \text{for each } u \in V - \{s, t\} \\
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non-negative flow

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LP-relaxation

- An *LP-relaxation* of an Integer Linear Program is a linear program which is identical to the ILP, except all the integrality constraints have been removed (“*relaxed*”), or replaced with non-integral constraints.

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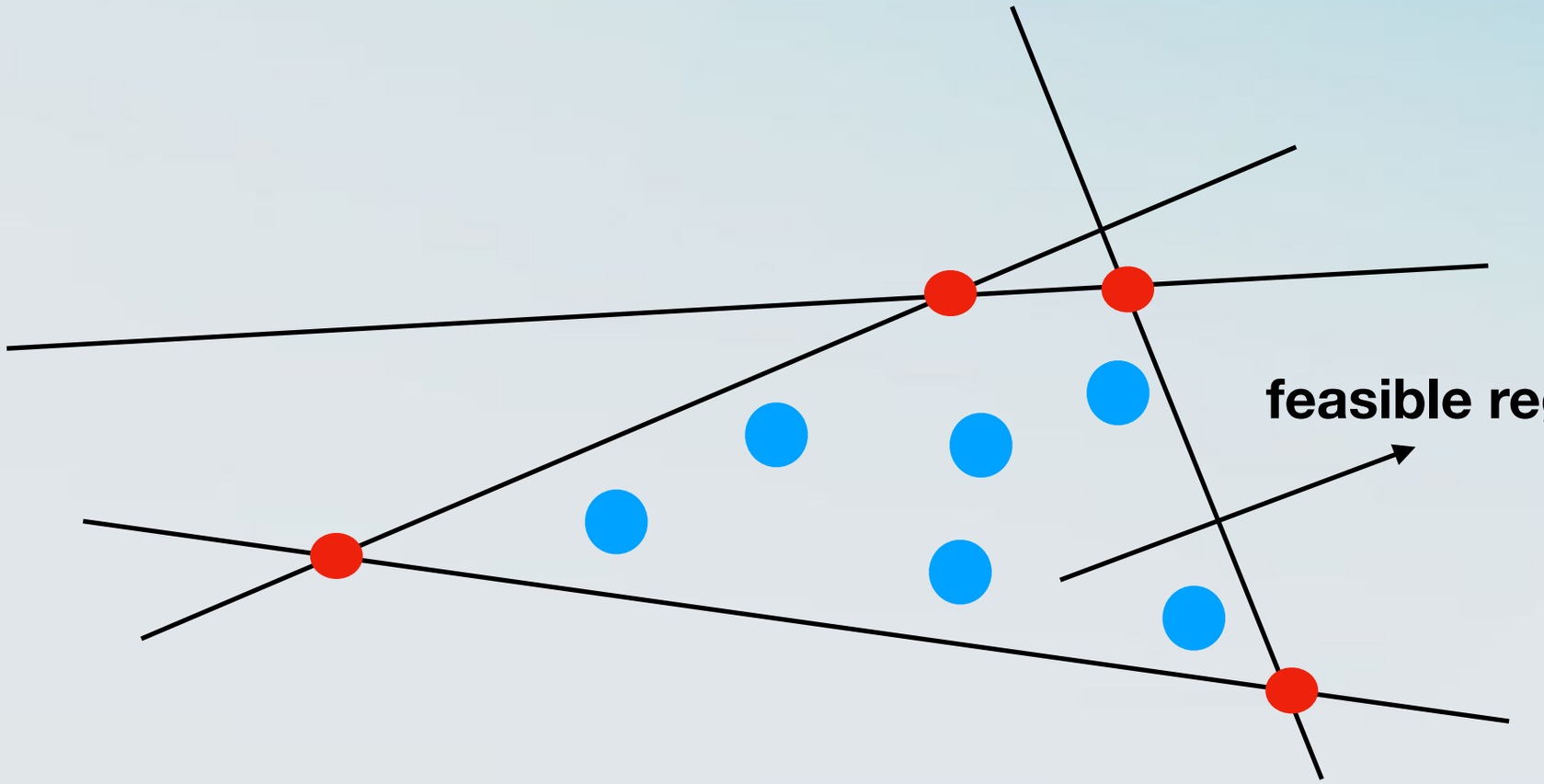
ILP vs LP-relaxation



candidate optimal solution for ILP

candidate optimal solution for LP-relaxation

ILP vs LP-relaxation

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- The ratio

  max_value(LP-relaxation) / max_value(LP)

  is called the integrality gap of the LP-formulation.

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  - In other words, the Min-Cut LP has *an integer optimal solution*.

Back to Maximum Flow

- What if we wanted an integer flow instead of any flow?

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Back to Maximum Flow

- Does the LP-relaxation of this ILP always have an integer solution?

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- If we solve those LP-relaxations, we will get *integer solutions*.

Totally Unimodular Matrices

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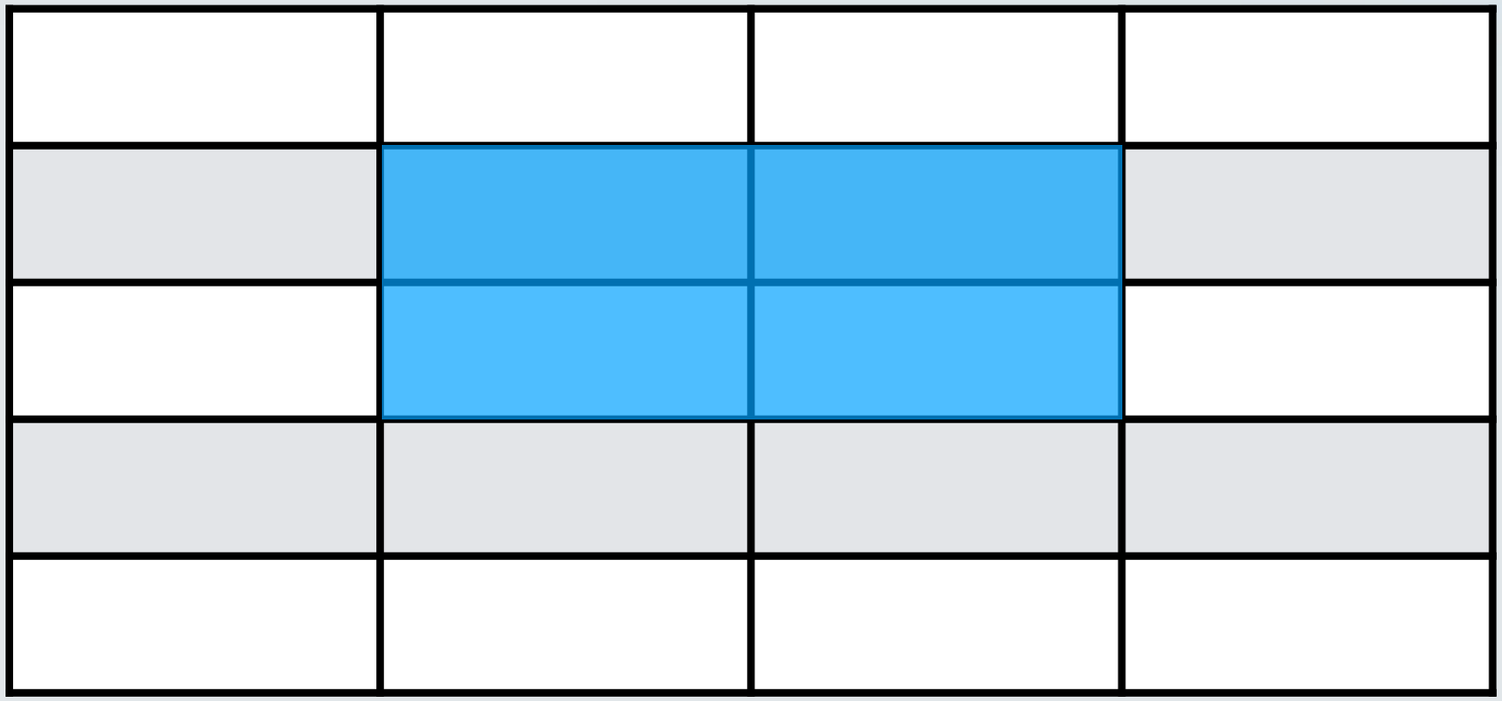

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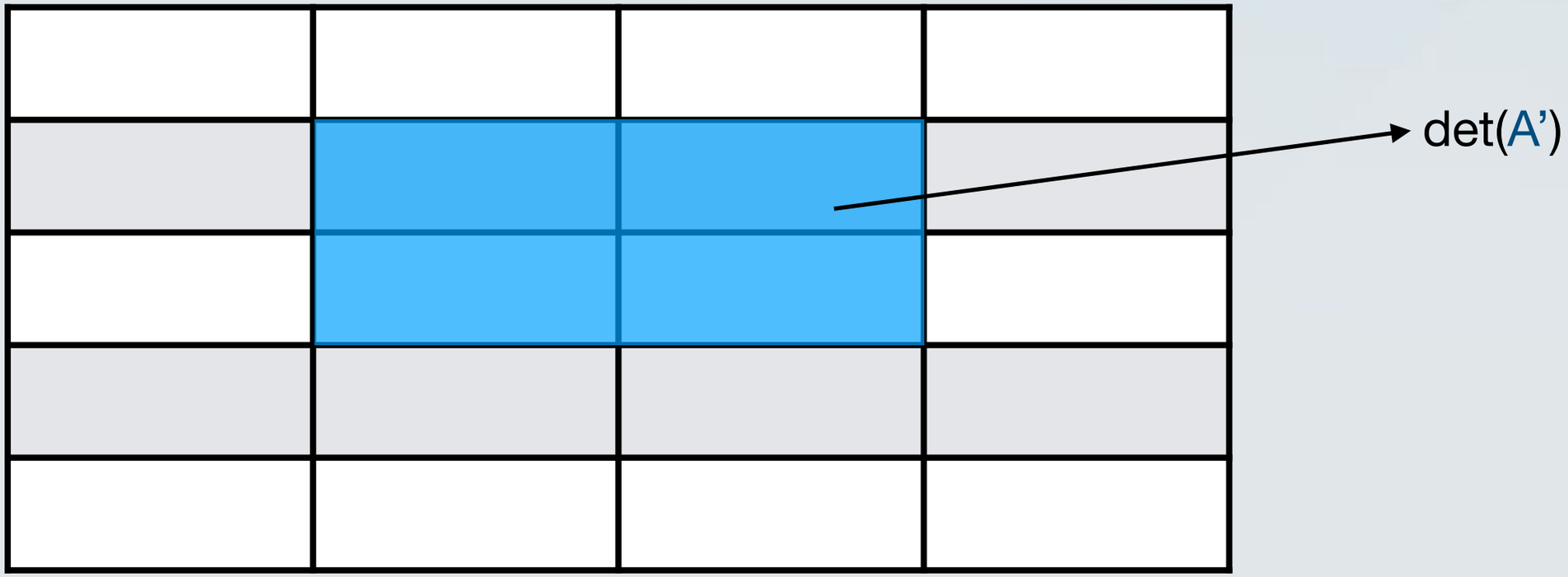

Total Unimodularity

- If the constraint matrix A is totally unimodular and b is an *integer vector*, then the LP has an *integer solution*.

- The Max-Flow and Min-Cut LP-relaxations admit integer solutions because their constraint matrices are totally unimodular.

$$\begin{aligned} \text{maximise} \quad & c^{\mathrm{T}} x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

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Back to maximum flow

maximise $\sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$

subject to $f_{uv} \leq c_{uv}, \quad \text{for each} \quad u, v \in V$

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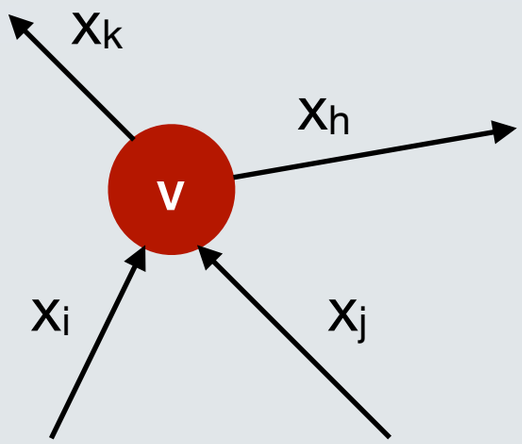

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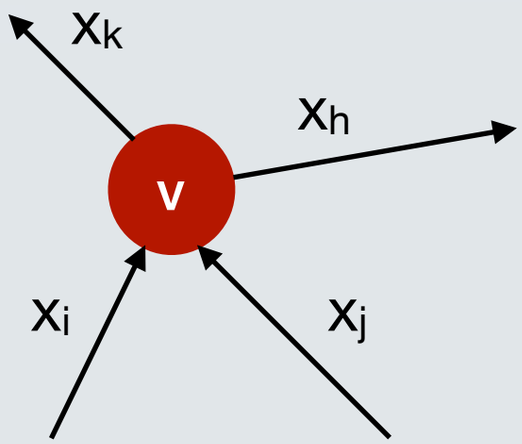

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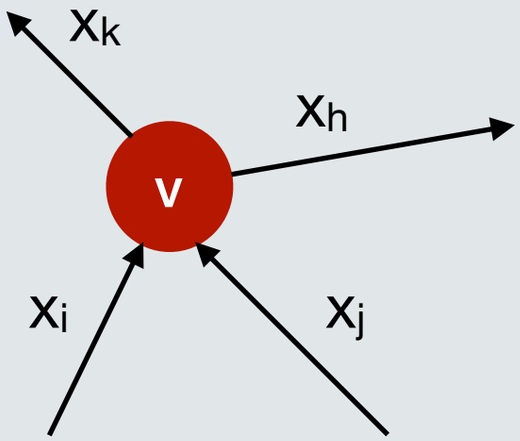

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$Ax = 0$

$A_{vi} = A_{vj} = -1$

$A_{vk} = A_{vh} = 1$

Back to maximum flow

- Consider the *incidence matrix* of the flow network (without s and t):
  - A_{ij} = *1* if edge *j* **starts** at node *i* in G_f.
  - A_{ij} = *-1* if edge *j* **ends** at node *i* in G_f.
  - A_{ij} = *0* otherwise.

Nodes/Edges		*j*	*k*
i		1	-1

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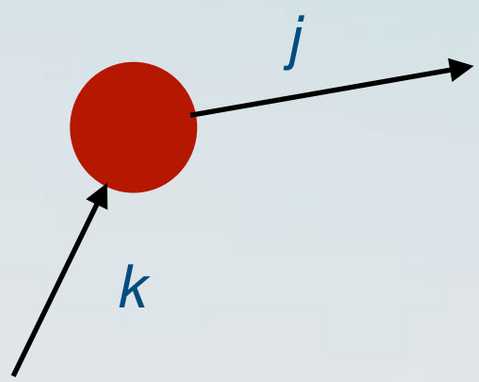

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  - This is precisely the matrix A of the max flow LP.

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  - It suffices to prove that A is totally unimodular, by Corollary.
  - Lemma: The incidence matrix of any directed graph is totally unimodular. (tutorial).

Nodes/Edges		*j*	*k*
i		1	-1