Advanced Algorithmic Techniques (COMP523)

Approximation Algorithms

Recap and plan

- **Previously on COMP523:**
  - We designed polynomial-time algorithms for several problems.
  - We saw that some problems do not have polynomial time algorithms (NP-hard problems).
- **Next 4 lectures:**
  - Approximation algorithms.
- **This lecture:**
  - Greedy approximation algorithms.
  - Load balancing on identical machines.

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- For some problems (e.g., knapsack), we do not expect to find polynomial time algorithms.
- How should we approach these problems?
- We can design approximation algorithms, which
  - Run in polynomial time.
  - Compute a solution that is "*close*" to the optimal.

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- What does “*close*” to the optimal mean? How do we measure that?
- How do we make such an argument, if we cannot really find the optimal?
- How do we know if our algorithm is the best possible? Can we get “*closer*” to the optimal?

Methods for approximation algorithms

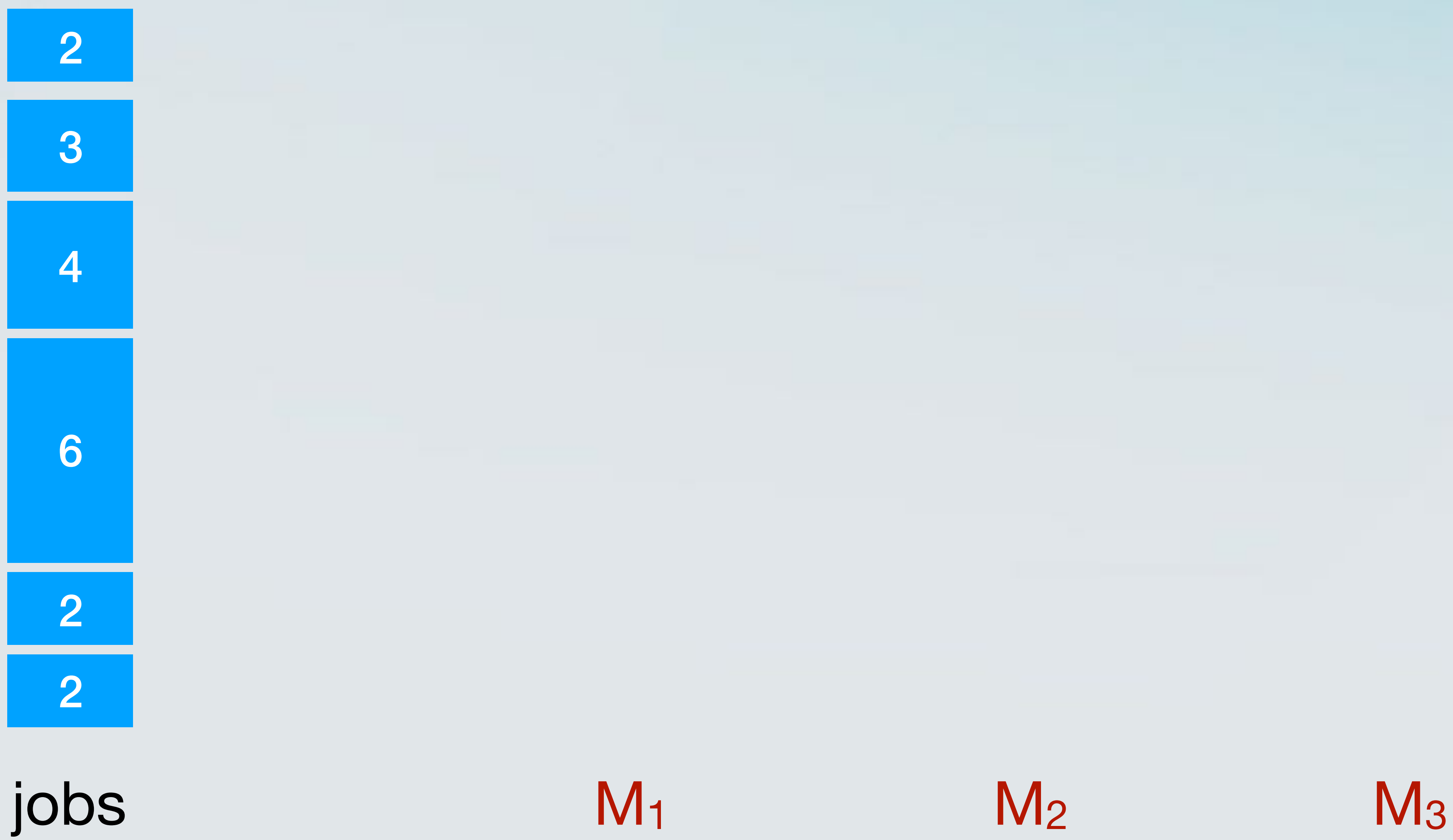
- Greedy algorithms.
- Pricing method (also known as the Primal-Dual method).
- Linear Programming and Rounding.
- Dynamic Programming on rounded inputs.

Application: Load Balancing

- We have a set of m *identical* machines $M_1, \ldots, M_m$
- We have a set of n jobs, with job j having processing time t_j.
- We want to assign every job to some machine.
- Let $A(i)$ be the set of jobs assigned to machine i.
- The load of machine i is $T_i = \sum_{j \in A(i)} t_j$
- The goal is to minimise the makespan, i.e.,

$$T = \max_i T_i$$

Example



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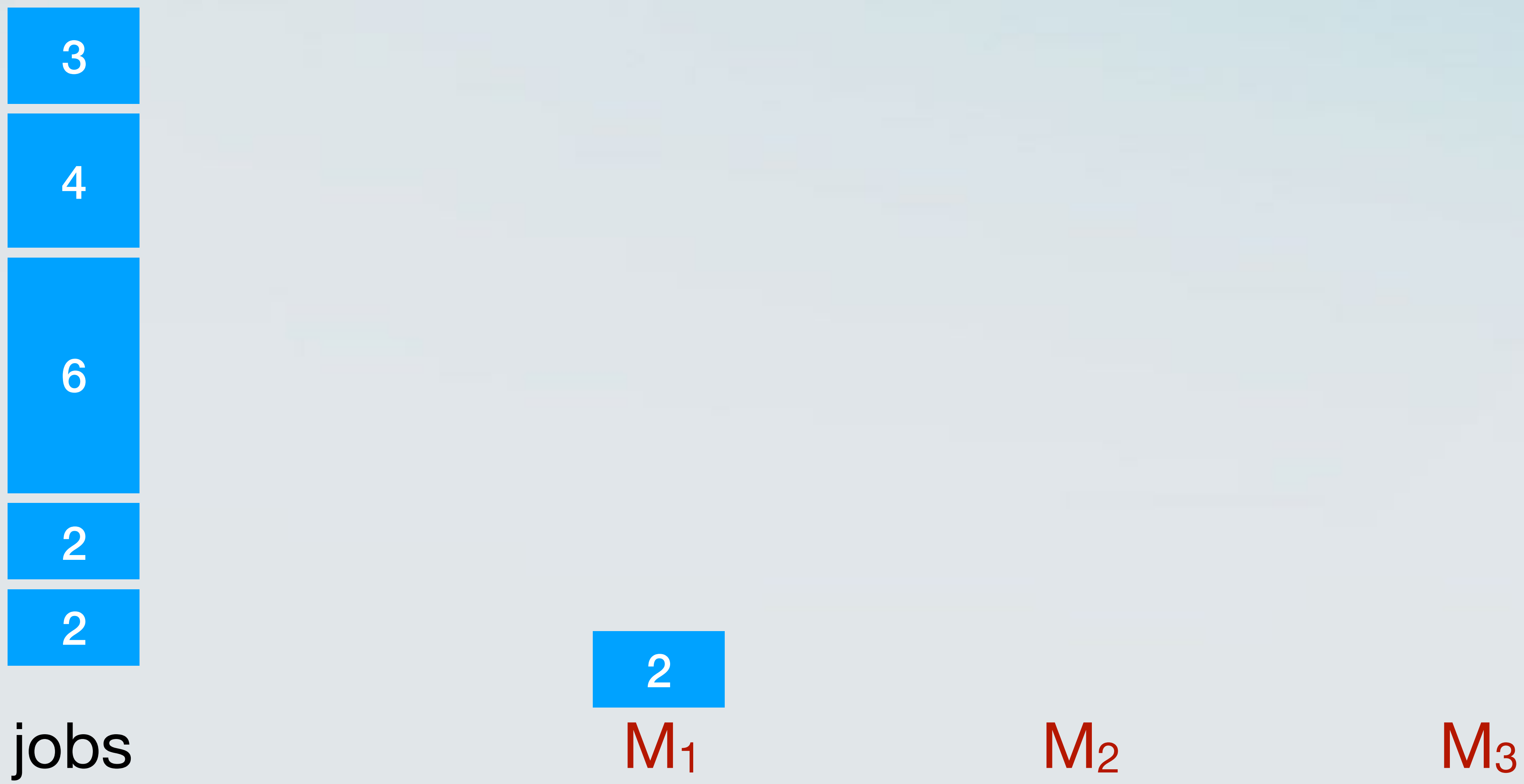

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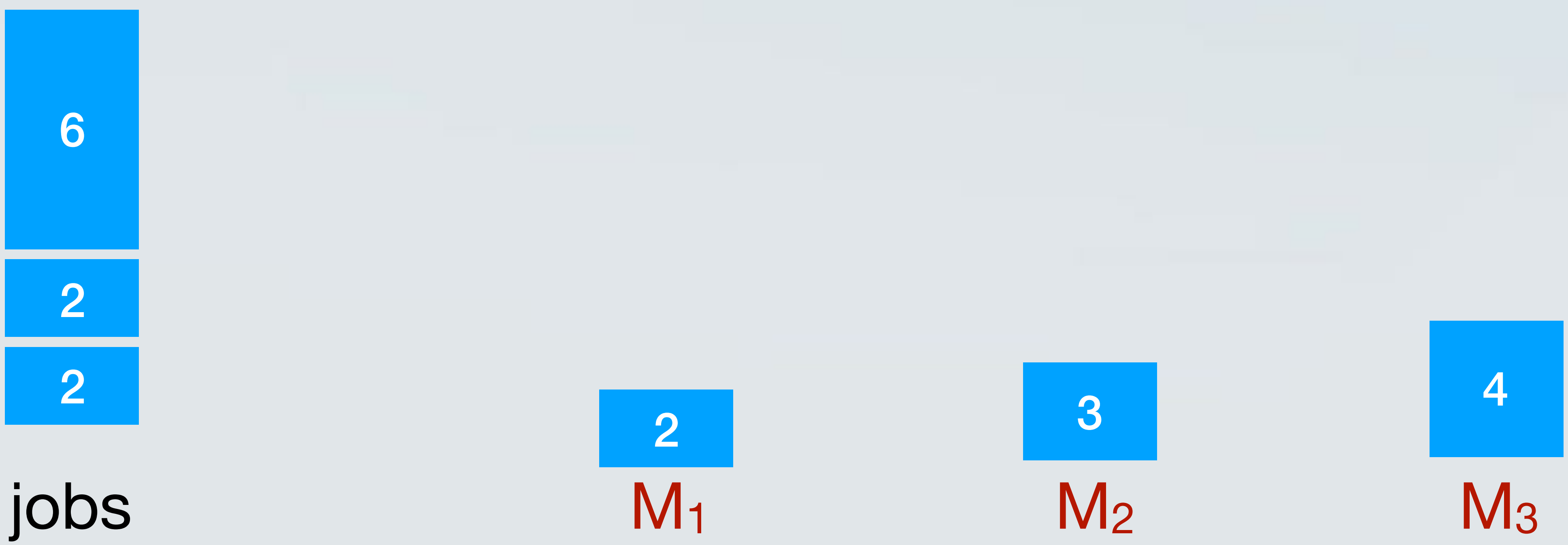

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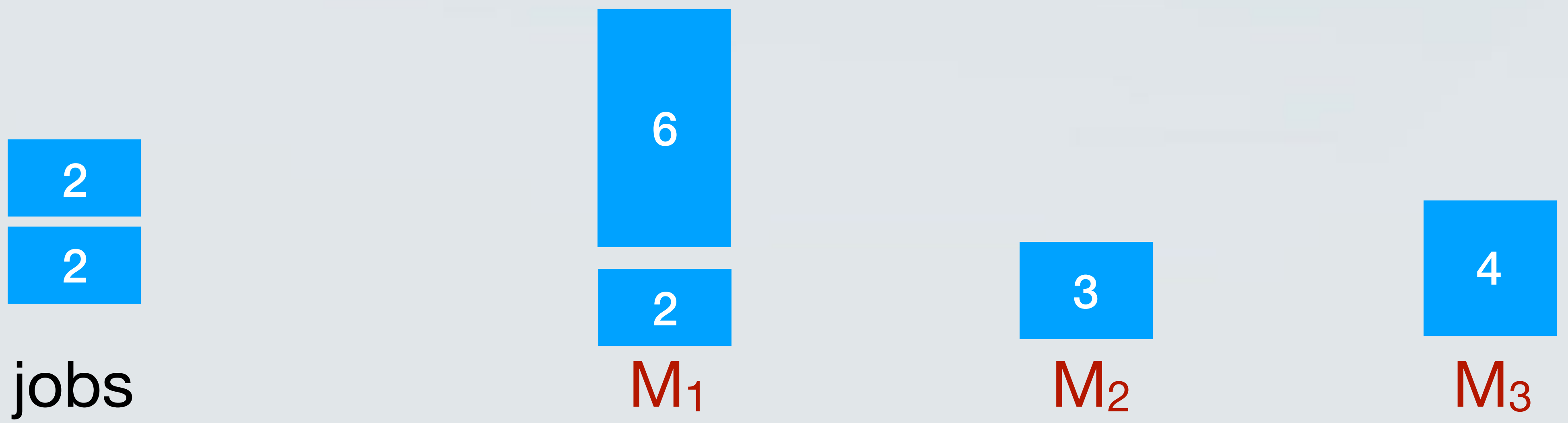

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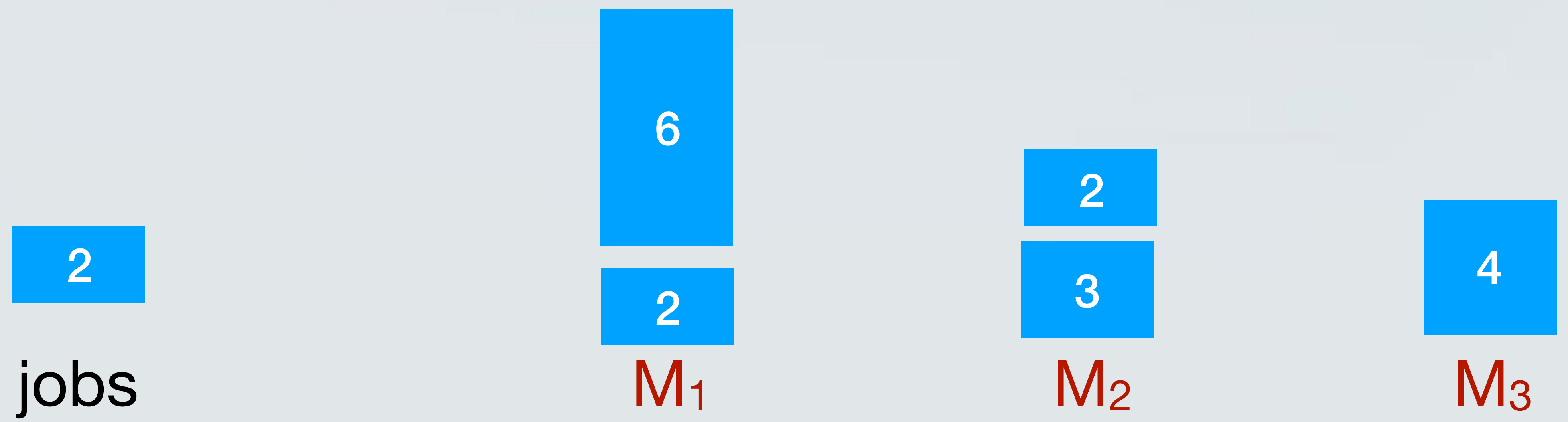

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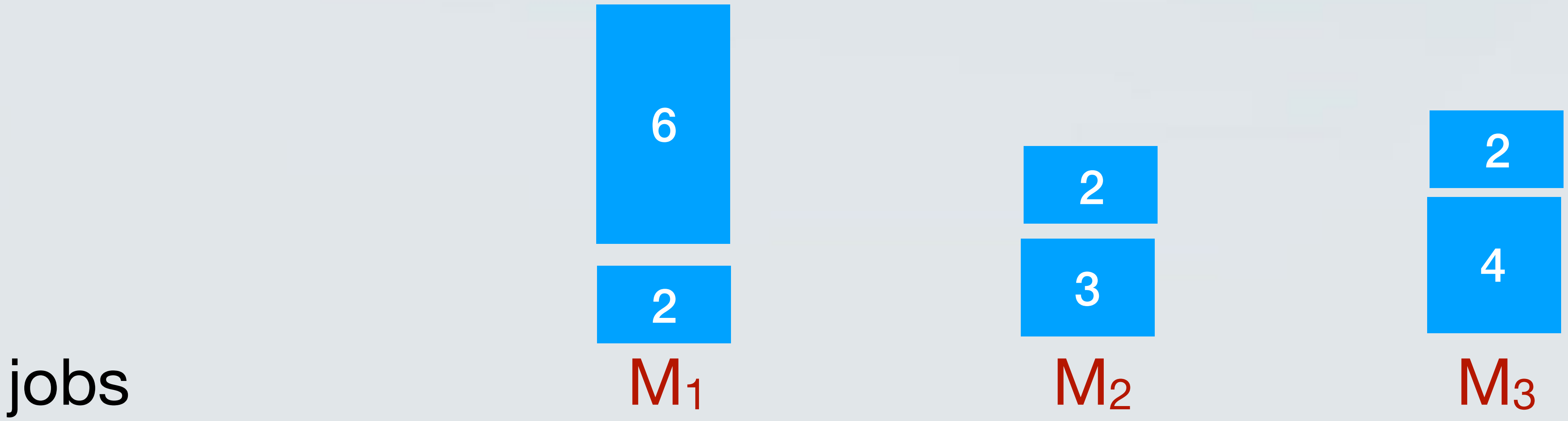

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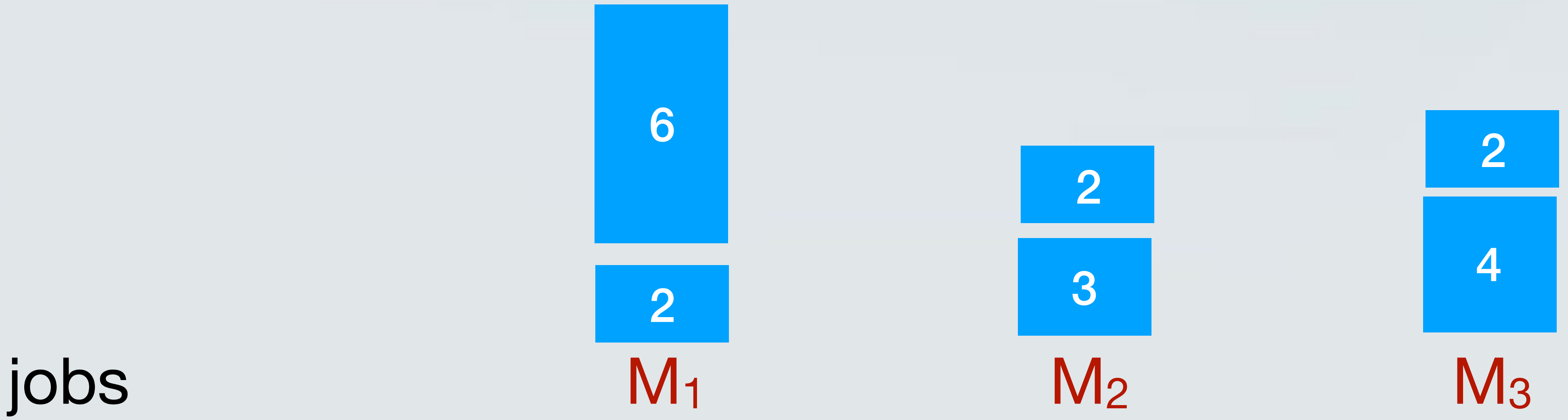

makespan = 8

Load Balancing

- The load balancing problem on identical machines is NP-hard.
- We will design greedy approximation algorithms for it.

Greedy algorithm

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- Assign it to the machine with the smallest load so far.
- Remove it from the pile of jobs.

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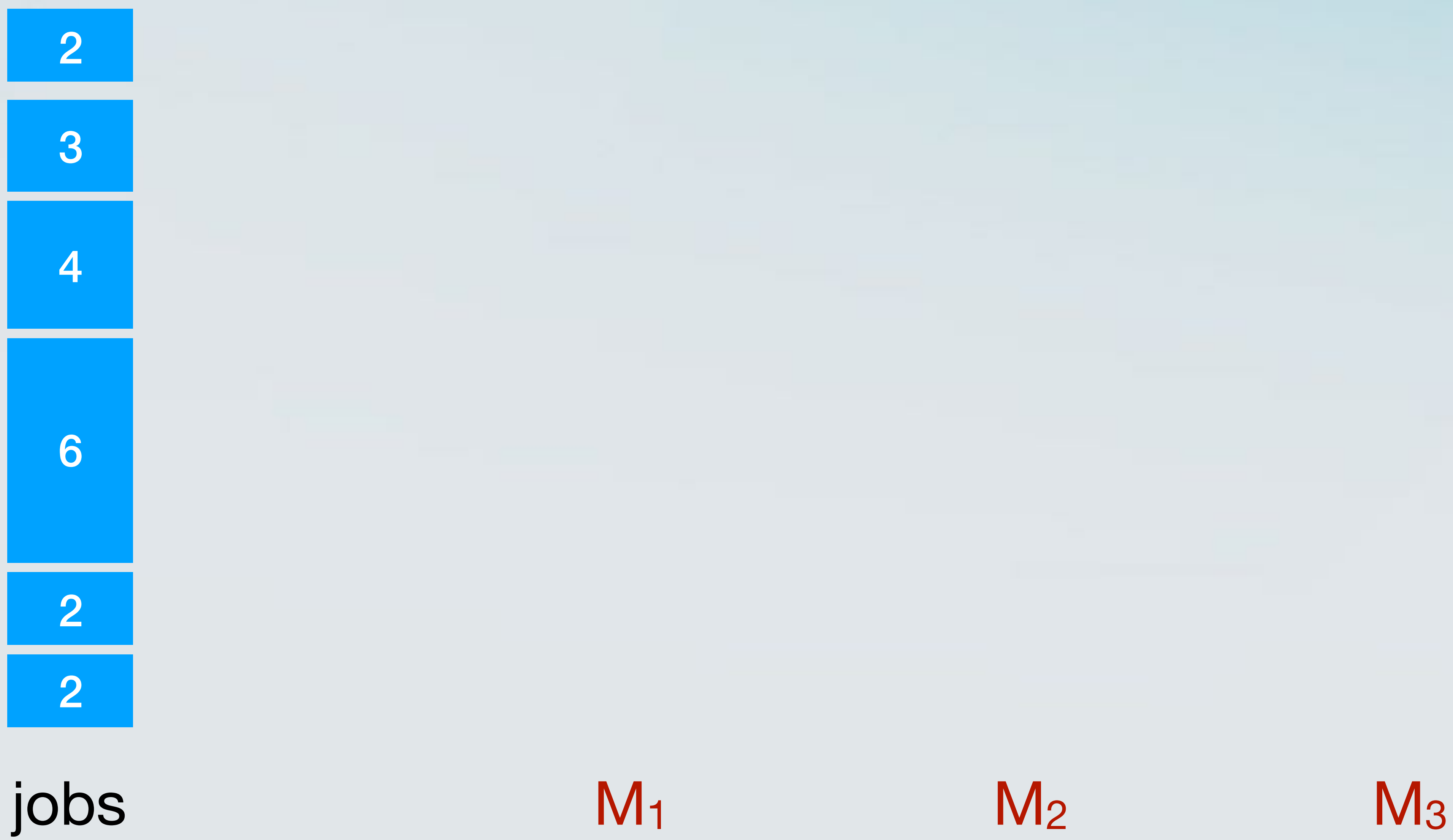
Algorithm **Greedy-Balance**

Start with no jobs assigned
Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i
For $j = 1, \ldots, n$
    Let M_i be the machine that achieves the minimum $\min_k T_k$
    Assign job j to machine M_i
    Set $A(i) = A(i) \cup \{ j \}$
    Set $T_i = T_i + t_j$
EndFor

Example



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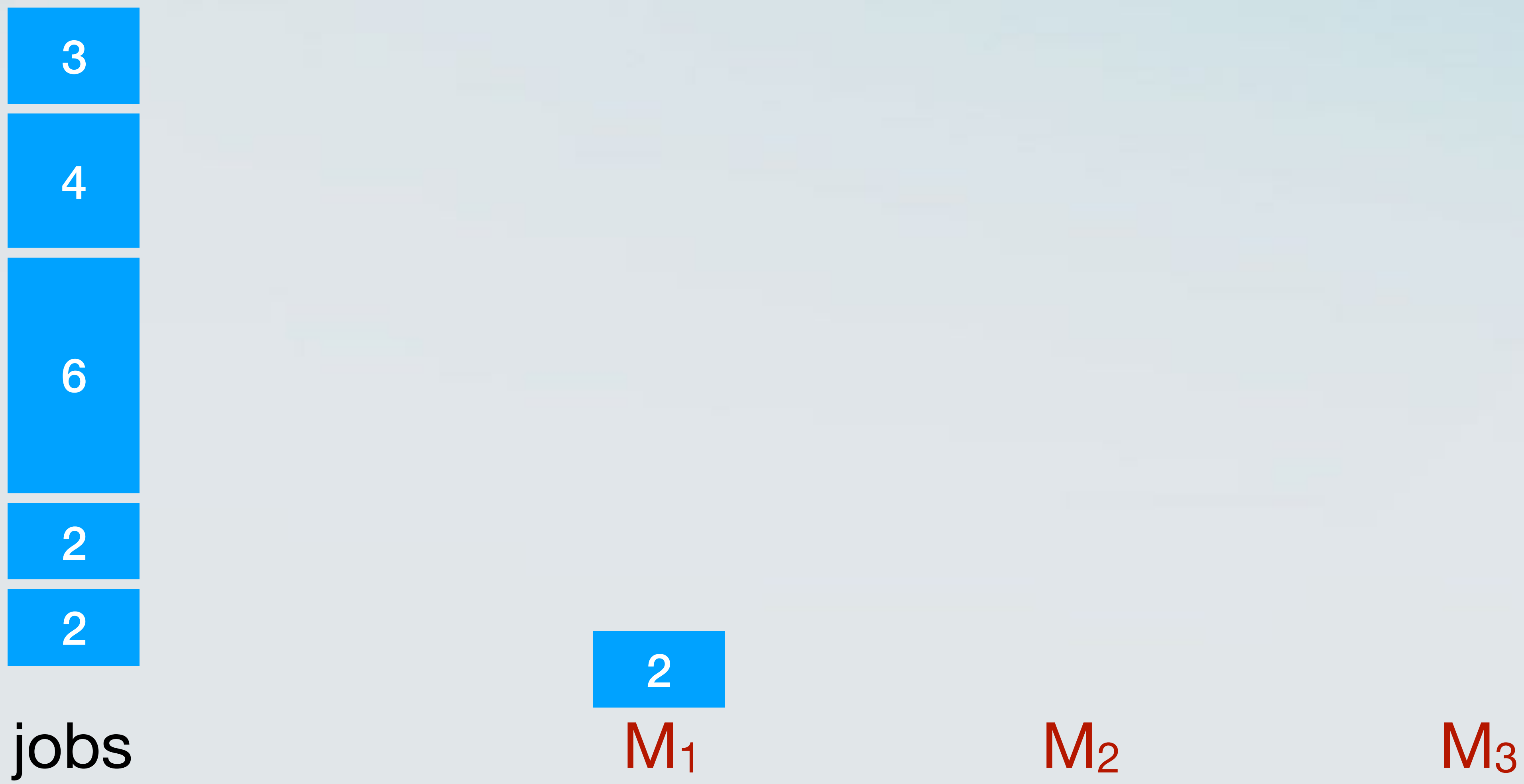

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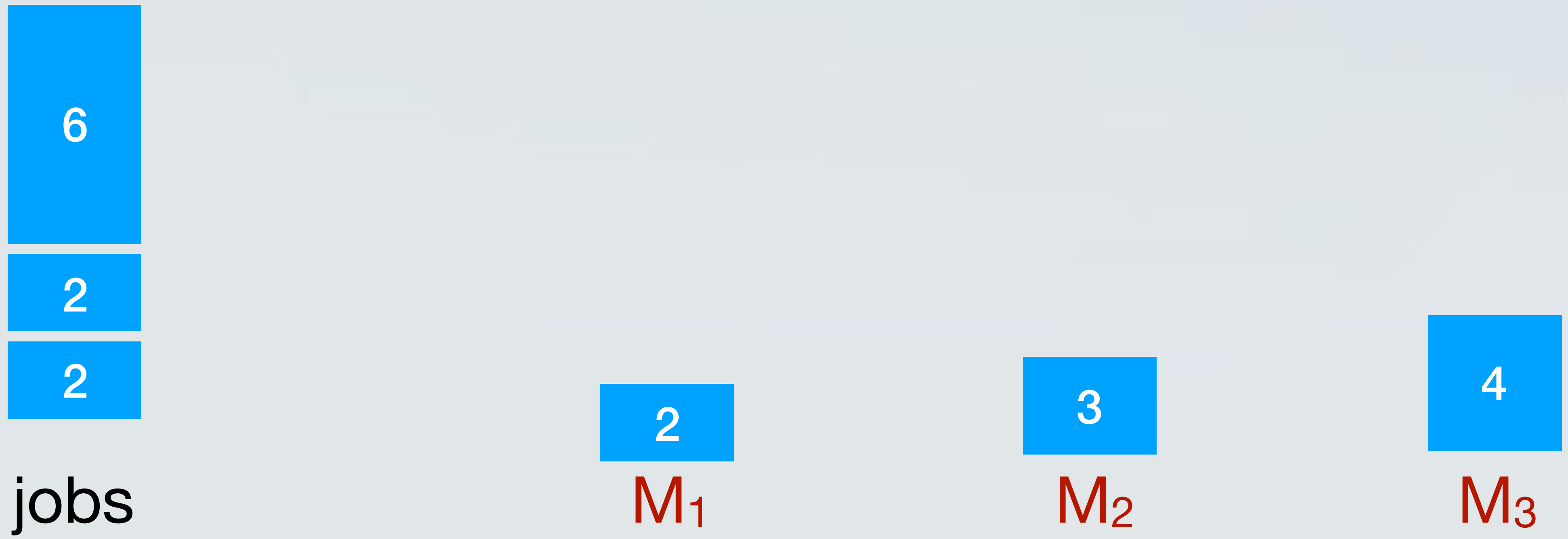

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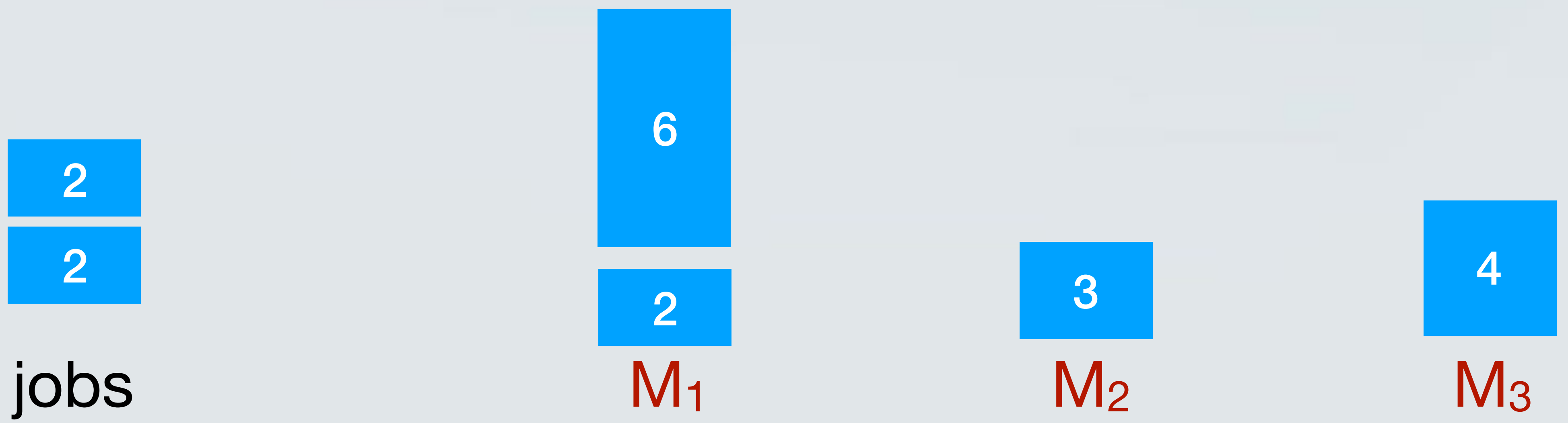

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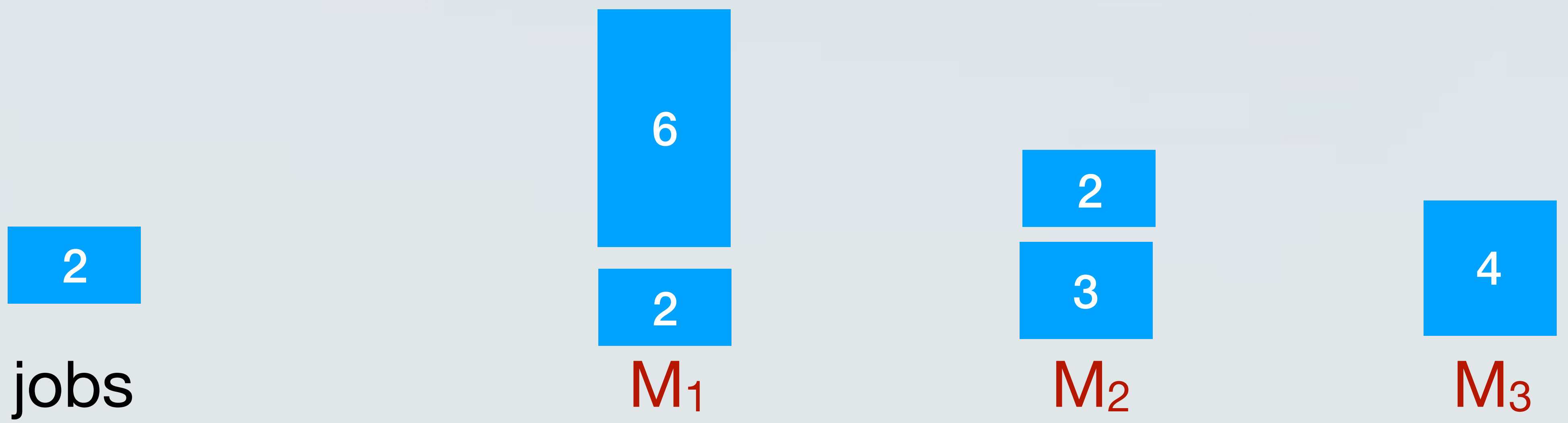

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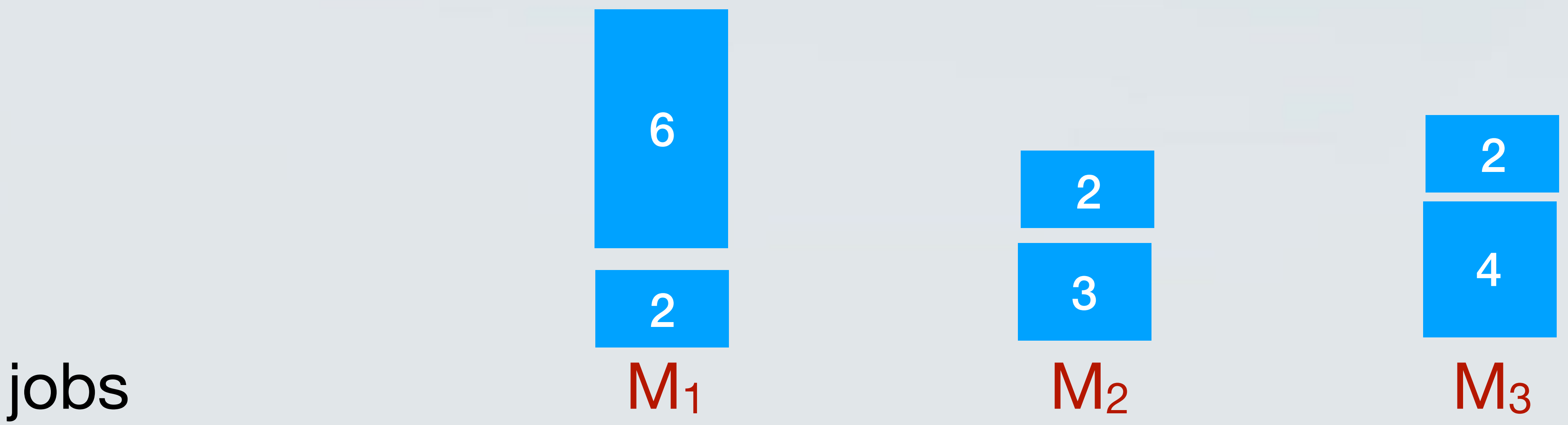

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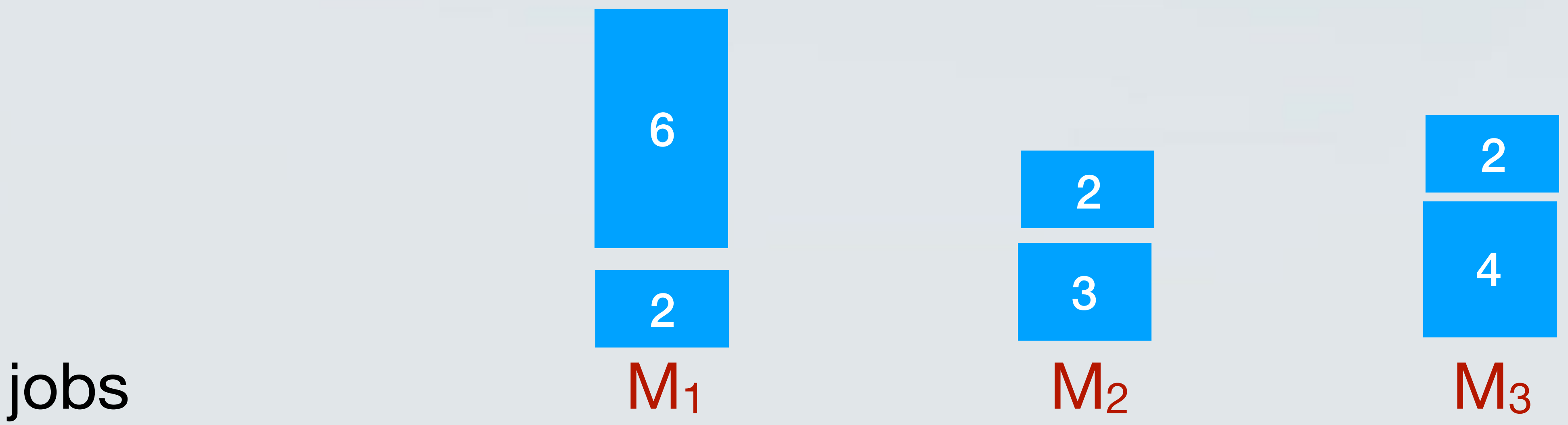

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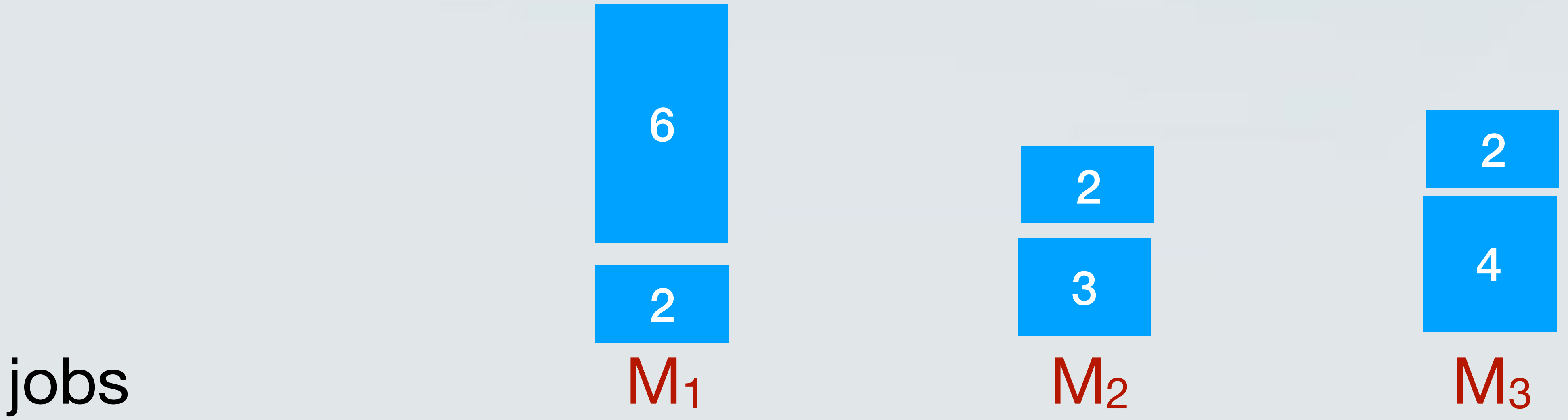

makespan = 8

A makespan of 7 is possible

Notation

- Let T be the makespan achieved by **Greedy-Balance**.
- Let T^* be the optimal makespan.

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    - Bounding the optimal from below (for minimisation problems) and from above (for maximisation problems).

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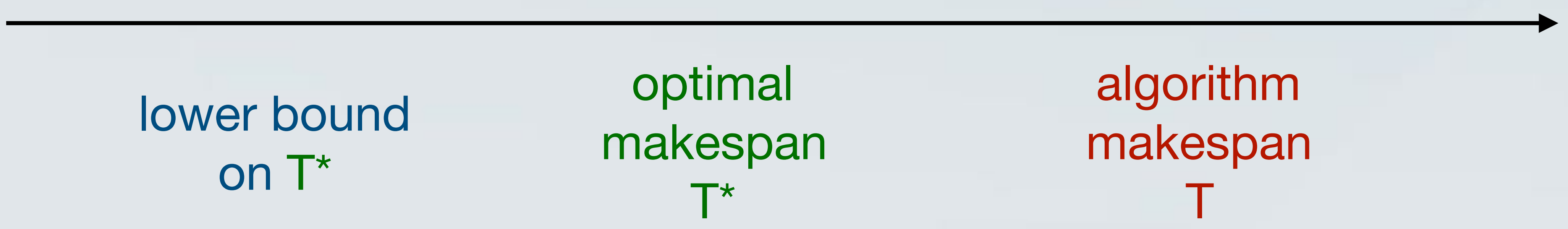

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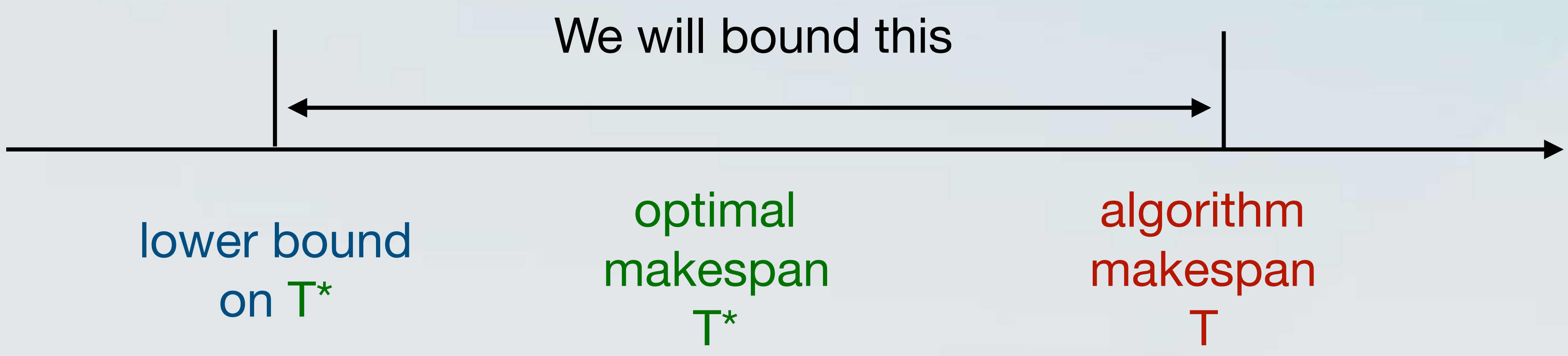

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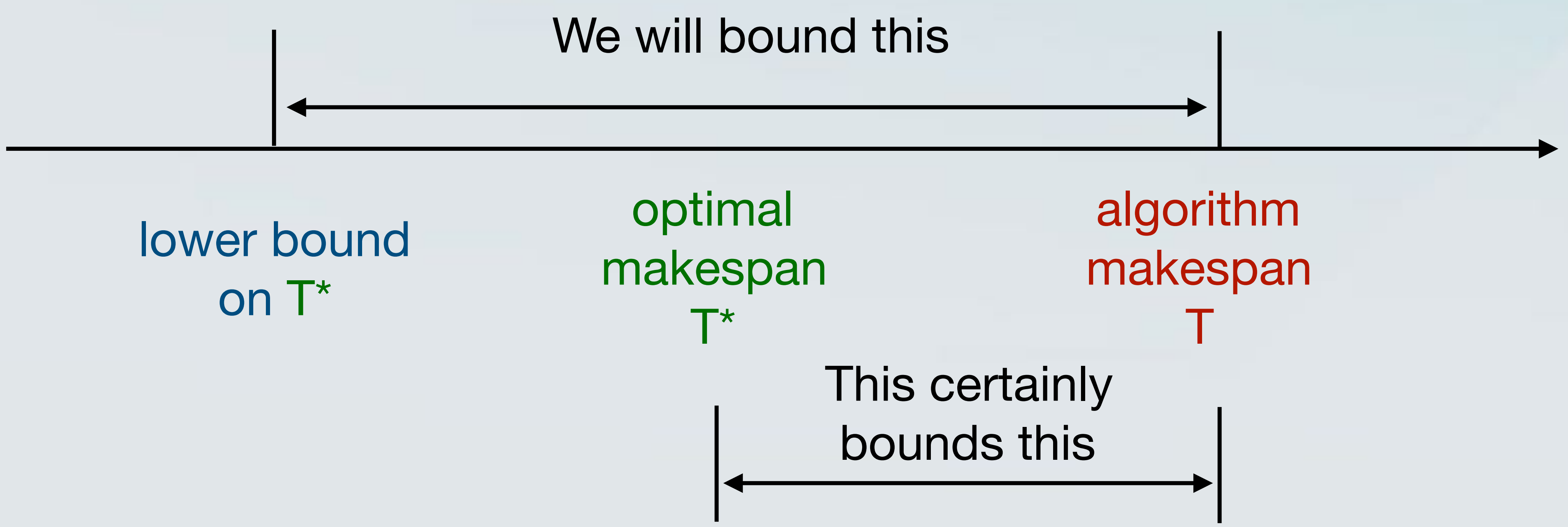

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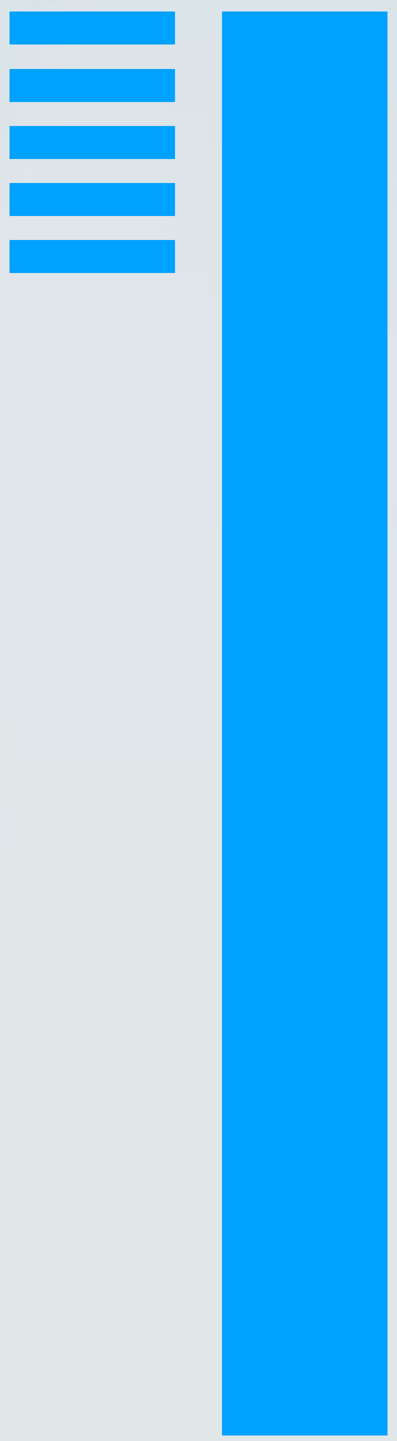
- Consider the *total processing time* of all the jobs (the sum of the processing times t_j).
- One of the m machines must be allocated at least an $1/m$ fraction of the total work.
- We have that:

$$T^* \geq \frac{1}{m} \sum_{j=1}^{n} t_j$$

Is this a good bound?



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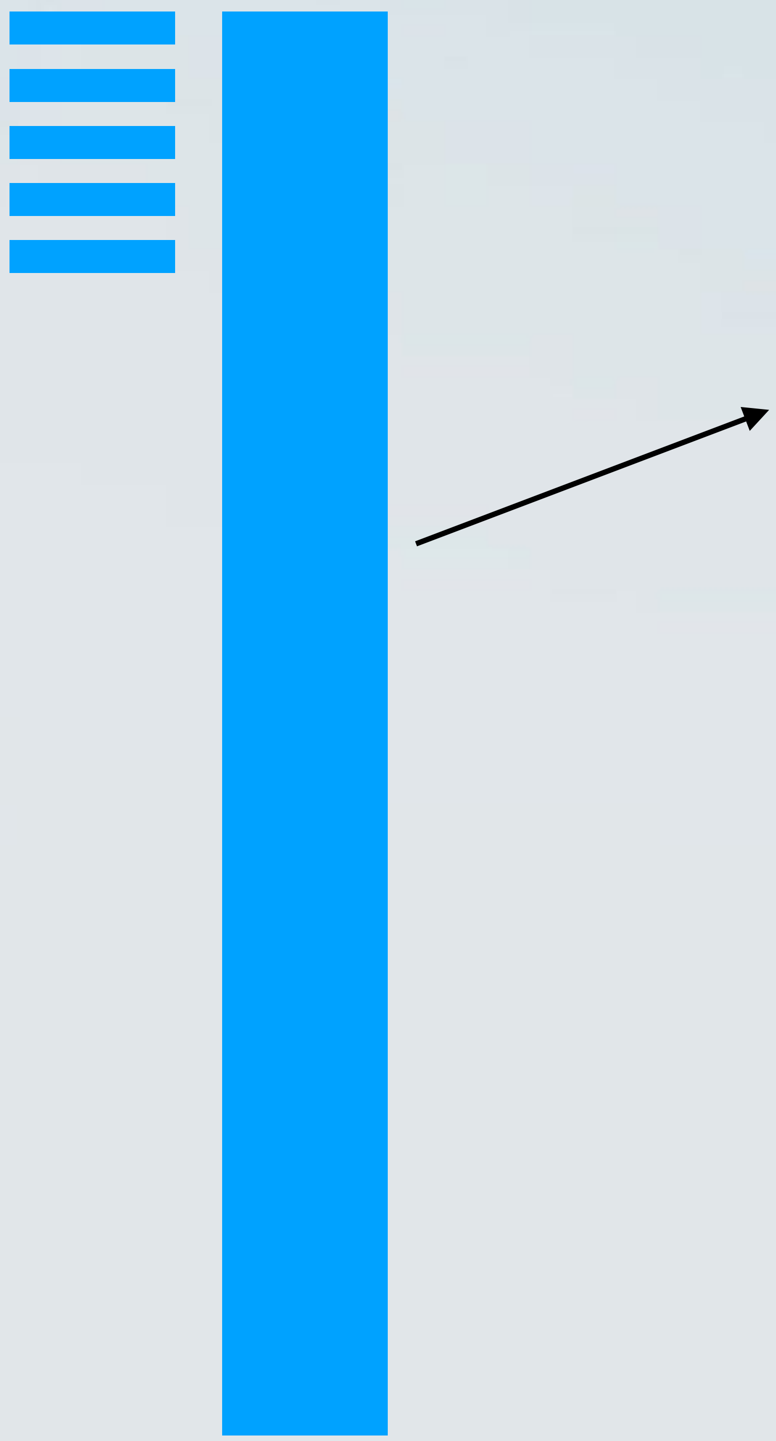

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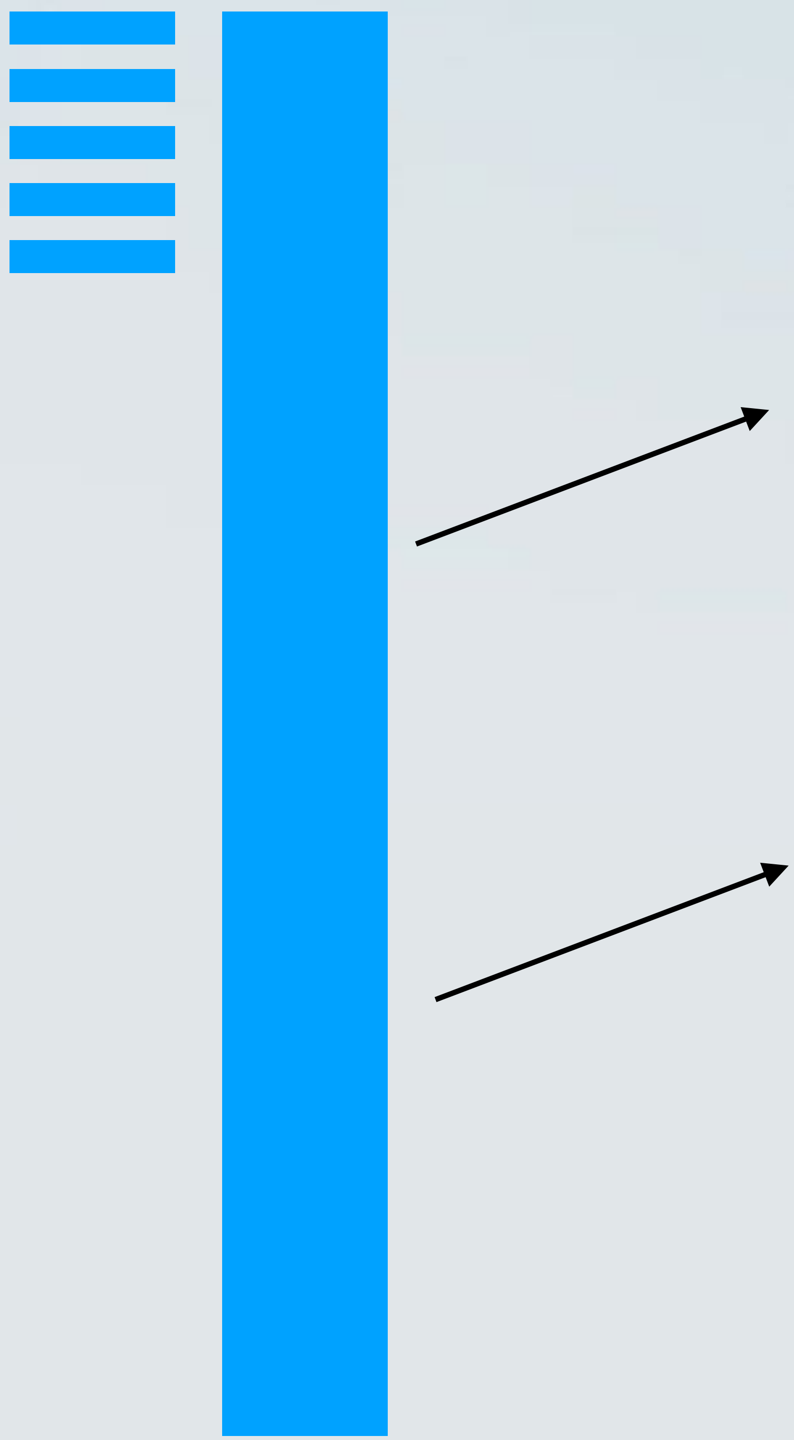

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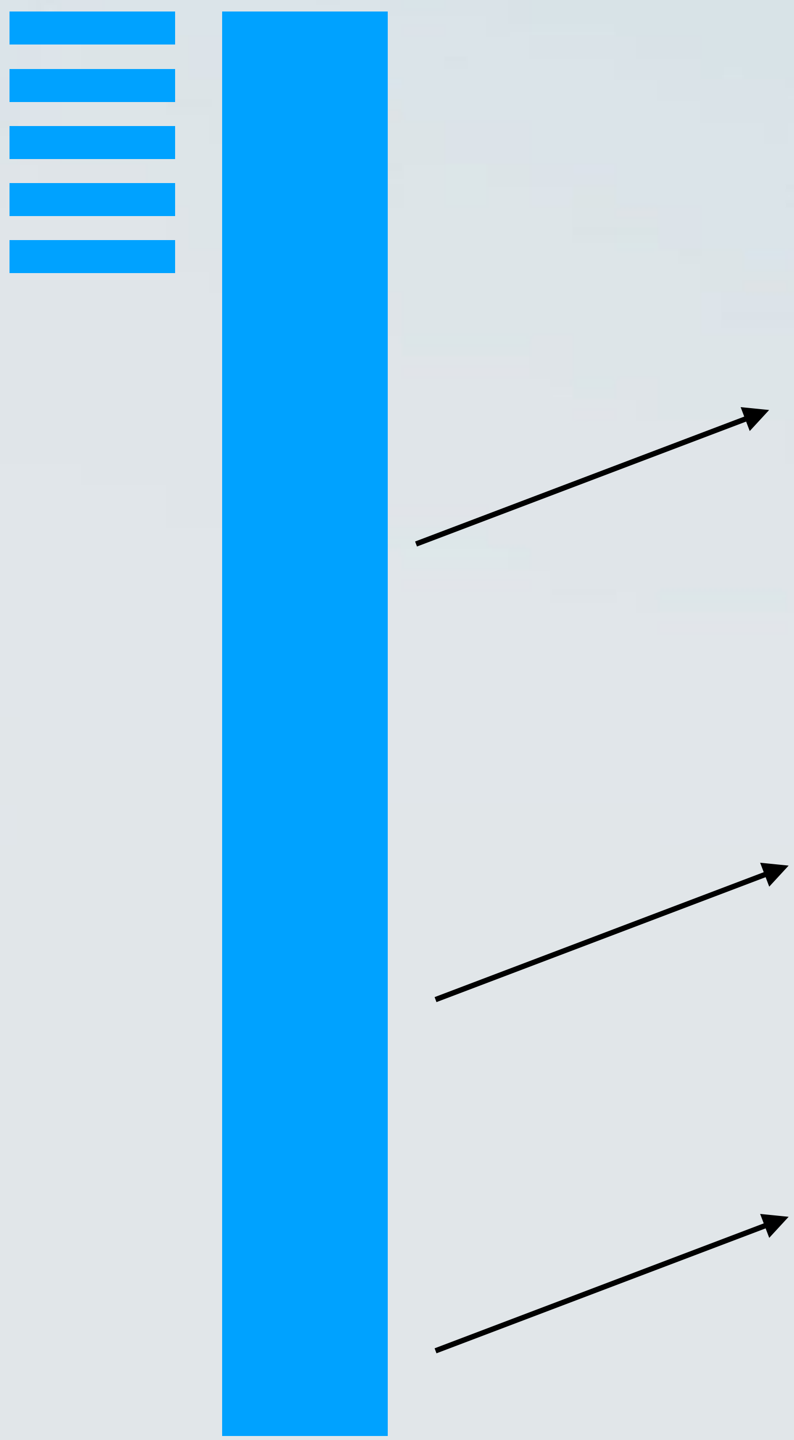

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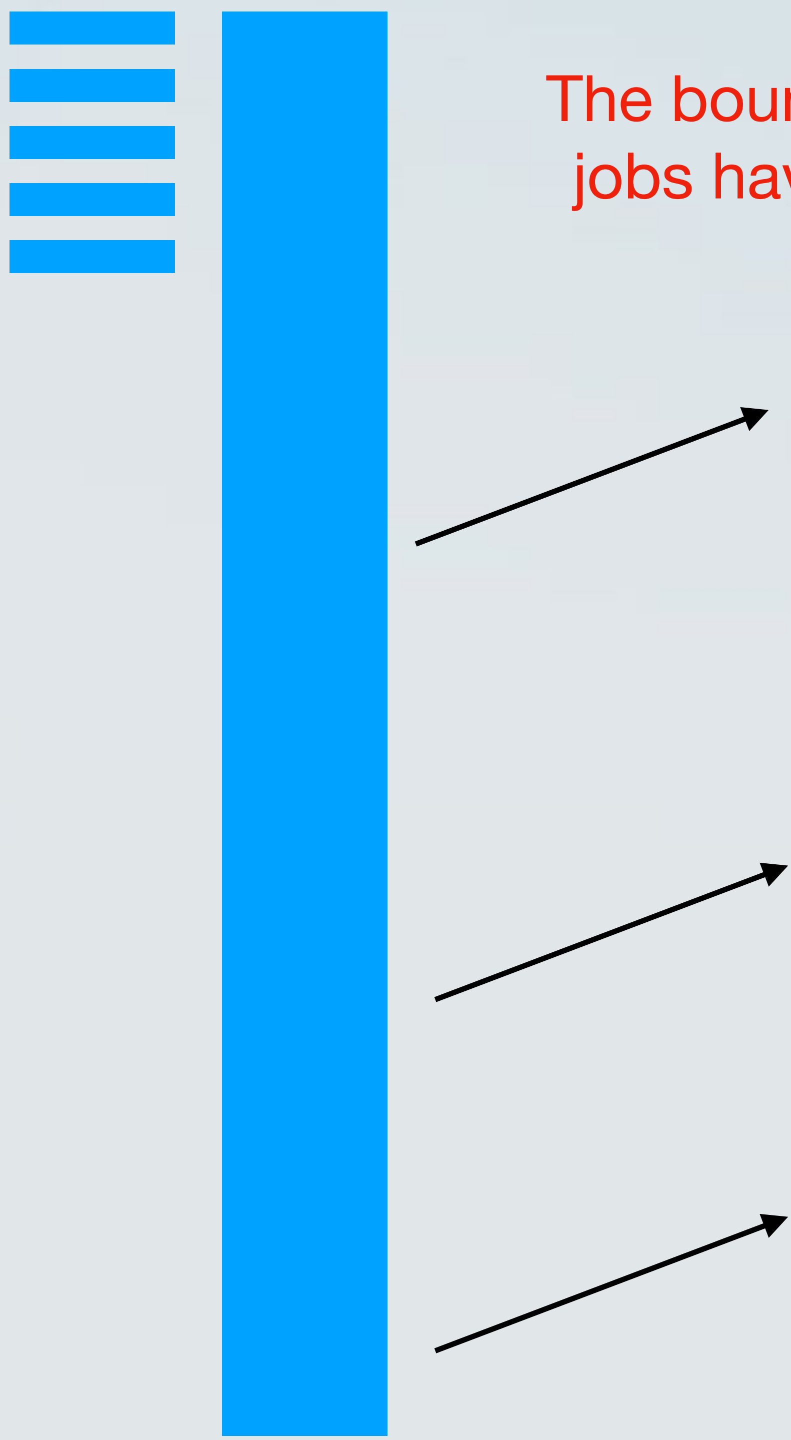

The bound can be good in situation where jobs have fairly similar processing times.

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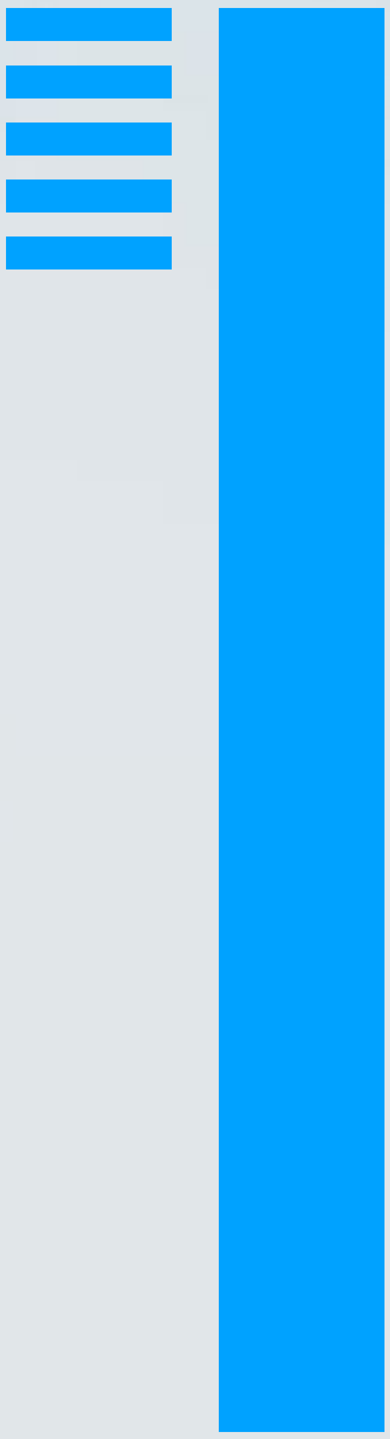

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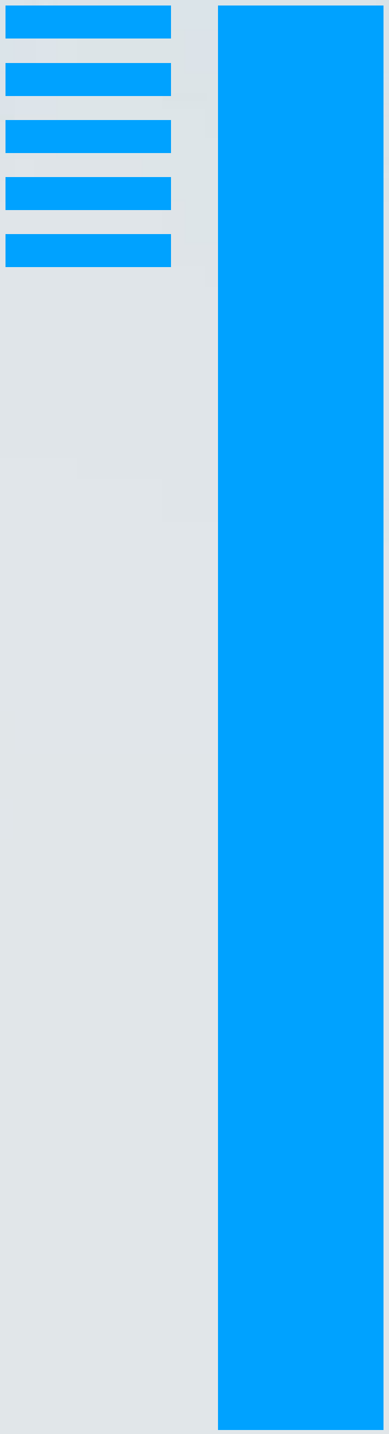

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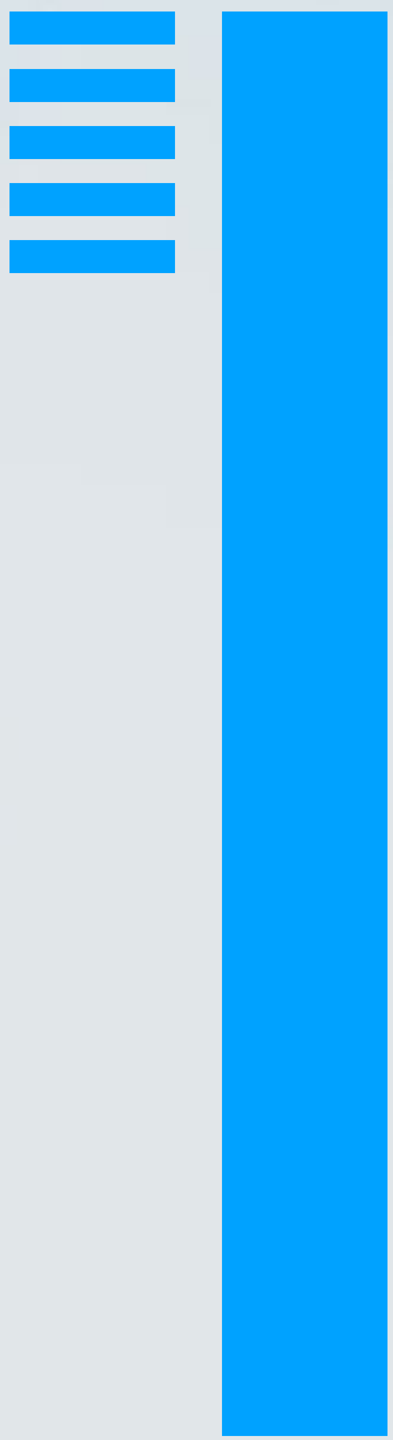

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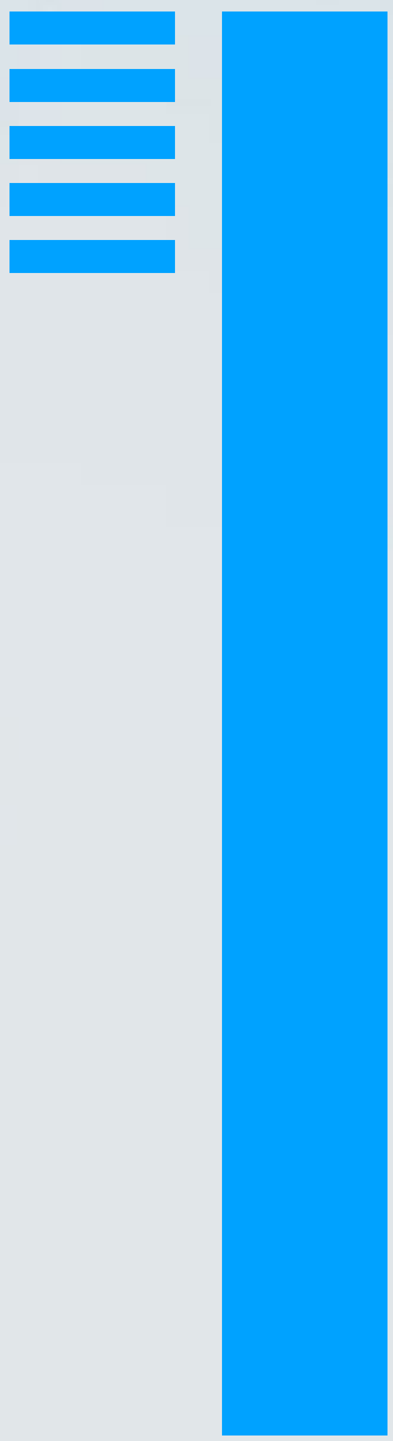

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In other cases, it might not be such a good bound.

But we will actually use both bounds!

Lower bounding the optimal

- Two lower bounds:

$$T^* \geq \frac{1}{m} \sum_{j=1}^{n} t_j$$

$$T^* \geq \max_j t_j$$

The performance of Greedy-Balance

- Theorem: Algorithm Greedy-Balance produces an assignment of jobs to machines with makespan $T \leq 2T^*$.

The proof

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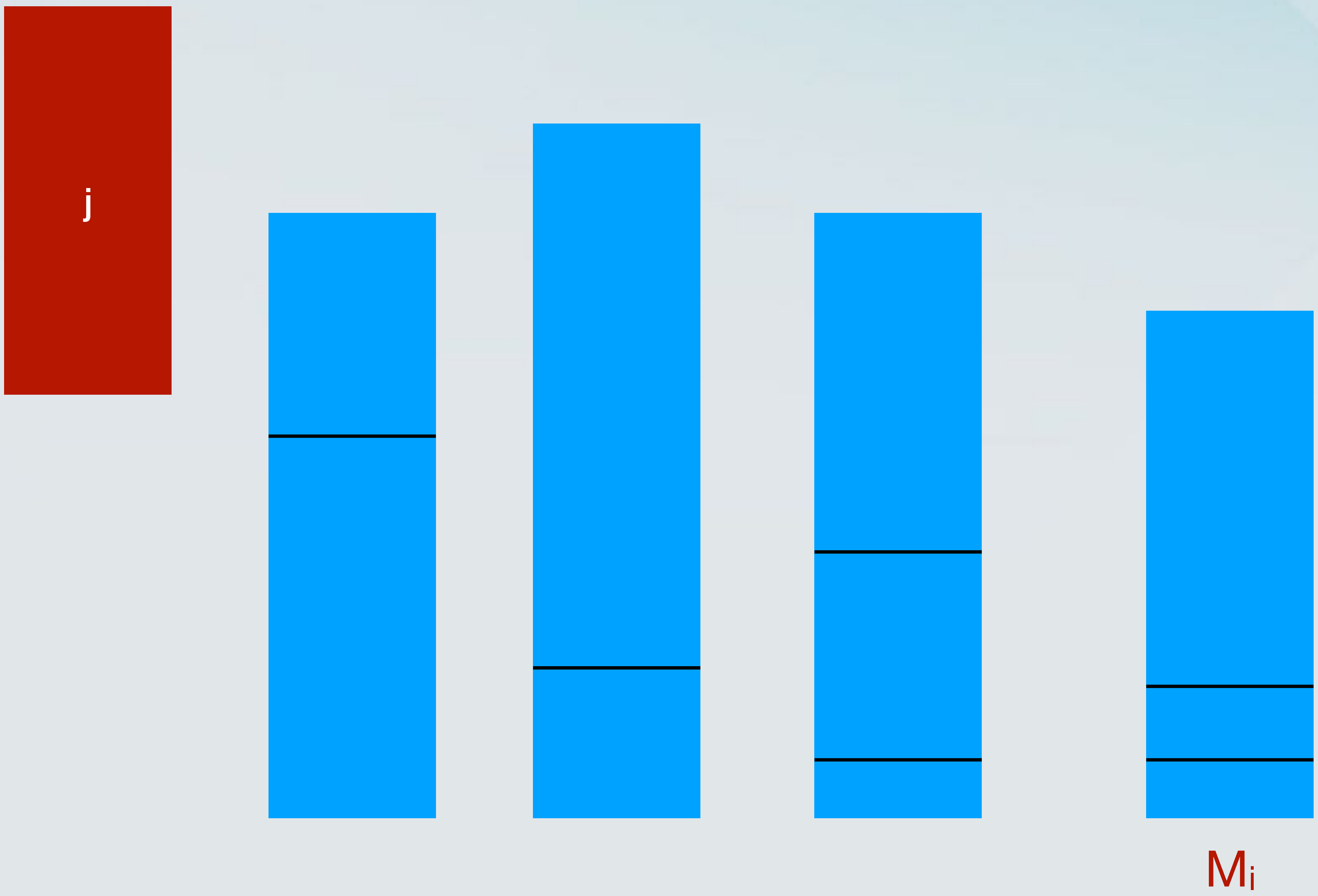

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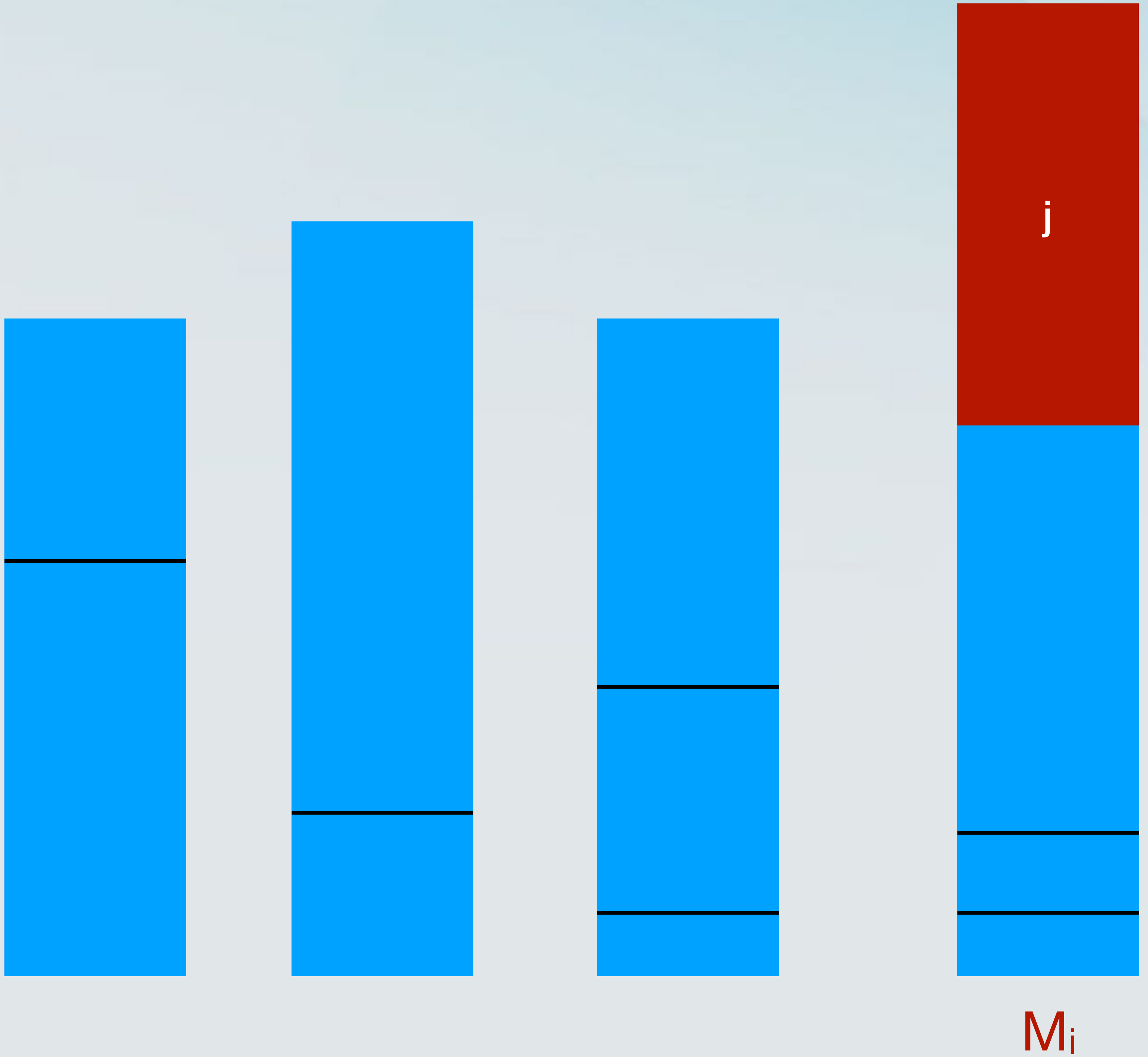

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  - Every other machine has load at least $T_i - t_j$.

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- Summing up over all machines we get:

$$\sum_k T_k \geq m(T_i - t_j) \Rightarrow T_i - t_j \leq \frac{1}{m} \sum_k T_k$$

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$T_i - t_j \leq T^*$ **(first lower bound)**

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- Consider a job j that was assigned to machine M_i by **Greedy-Balance**.

- Consider the time when this assignment took place.
  - The load of machine j was T_i - t_j.
  - This was before we added the job.
  - After we add the job, the load is T_i - t_j + t_j.

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- Consider a job j that was assigned to machine M_i by Greedy-Balance.
- Consider the time when this assignment took place.
  - The load of machine j was $T_i - t_j$.
  - This was before we added the job.
  - After we add the *final* job, the load is $T_i - t_j + t_j$.
  - Obviously $t_j \leq \max_k t_k \leq T^*$

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$$T_i \leq 2T^*$$

$T \leq 2T^*$ **(since j was the final job)**

“*Tight*” examples

- We have shown that the makespan of the solution of **Greedy-Balance** is *at most a 2 factor away* from the optimal makespan.

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  - In other words, is there an example (an *instance*) of the load balancing problem for which the algorithm actually produces a makespan which is *twice as much* as the optimal makespan?

“*Tight*” examples

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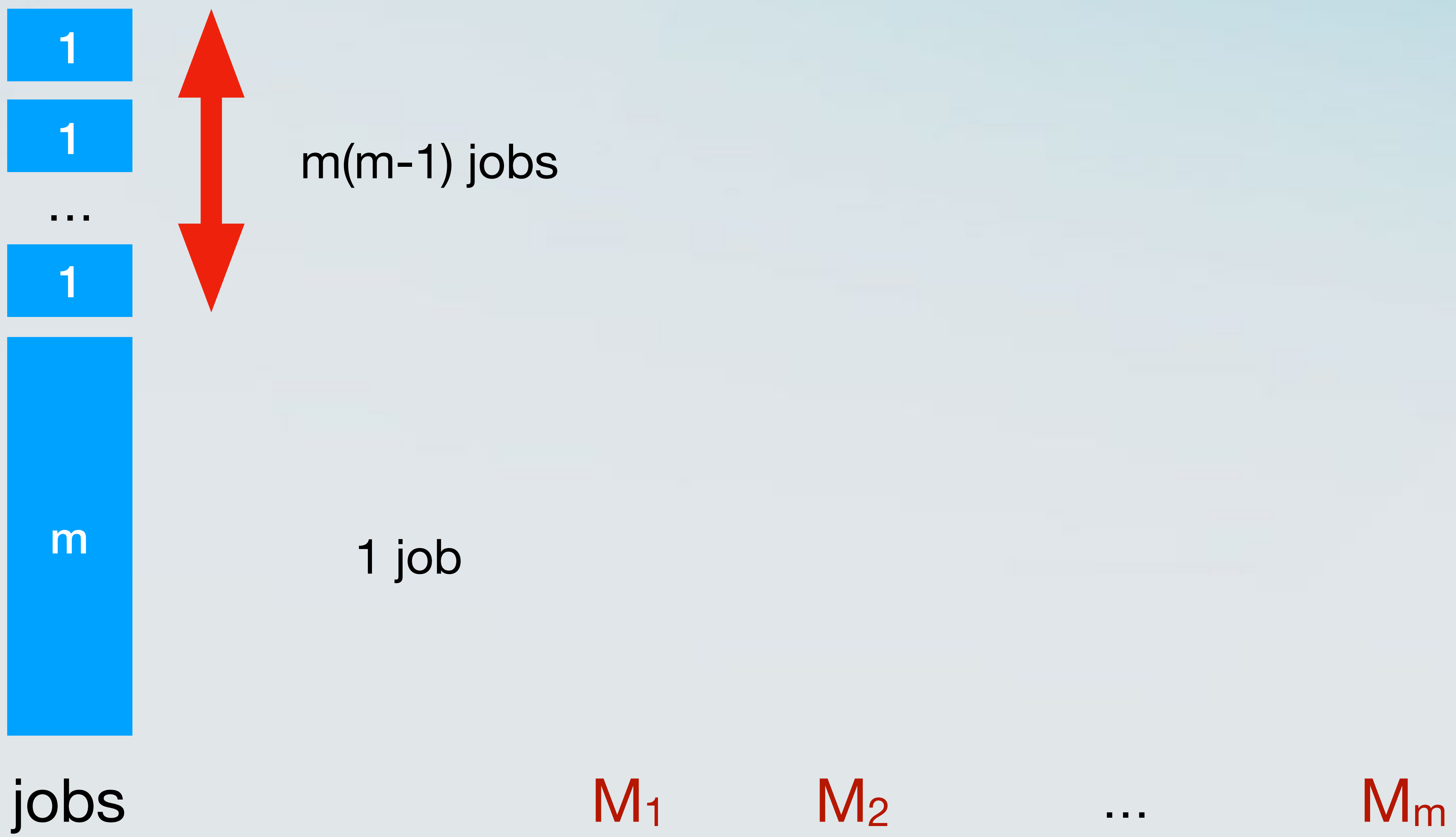
    - In other words, is there an example (an *instance*) of the load balancing problem for which the algorithm actually produces a makespan which is *twice as much* as the optimal makespan?

    - In other words, is our analysis of the algorithm *tight*?

Tight example for Greedy-Balance



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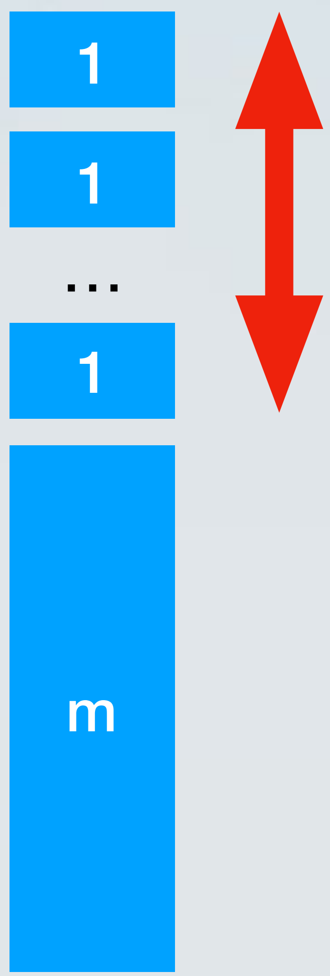


m(m-1) jobs

1 job

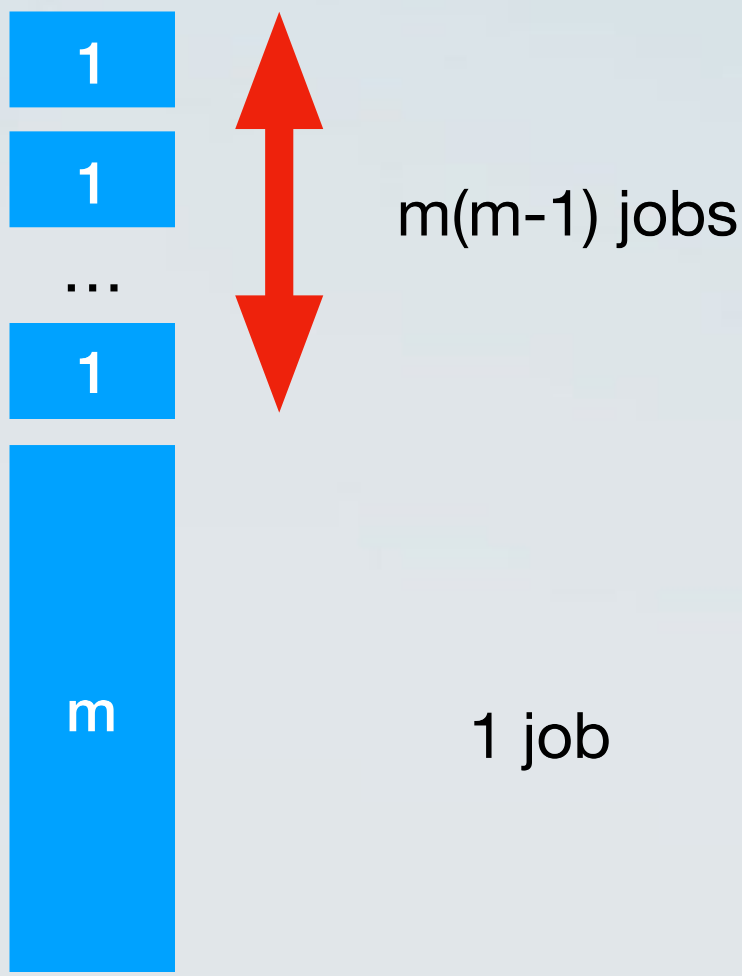
jobs

Greedy-Balance assigns *m-1* “small” jobs to each machine and then finally assigns the “large” job to one machine.

Makespan: 2m-1

M_1 M_2 … M_m

Tight example for Greedy-Balance



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Makespan: 2m-1

The optimal assigns the "large" job to one machine, and evenly spread the "small" jobs over the remaining *m-1* machines.

Makespan: m

jobs M_1 M_2 … M_m

Approximation Ratio

- Consider a minimisation problem P and an objective obj.
  - Here: Load Balancing on identical machines and makespan.
  - Consider an approximation algorithm A.
  - Consider an input x to the problem P.
  - Let obj(A(x)) be the value of the objective from the solution of A on x.
  - Let opt(x) be the minimum possible value of the objective on x.

Approximation ratio

- The approximation ratio of A is defined as

$$\max_x \; obj(A(x)) \,/\, opt(x)$$

  - i.e., the worst case ratio of the objective achieved by the algorithm over the optimal value of the objective, over all possible inputs to the problem.

Approximation ratio

All inputs to the problem P

Algorithm A

Optimal

Approximation ratio

All inputs to the problem P

Input A

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Ratio on Input A

Approximation ratio

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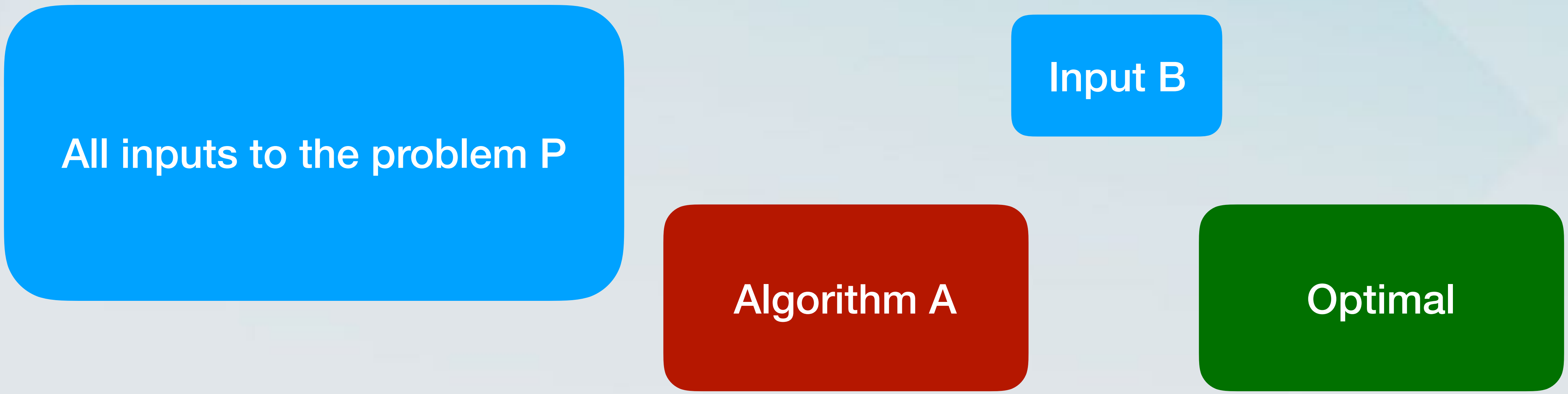

Ratio on Input A

Approximation ratio

All inputs to the problem P

Input B

Algorithm A

Optimal

compute obj on Input B

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Ratio on Input B

Approximation ratio

All inputs to the problem P

Input ...

Algorithm A

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Take the maximum

Approximation Ratio

- That means that:
  - In order to prove an upper bound on the approximation ratio, we have to somehow argue about *all* inputs to the problem.
  - In order to prove a lower bound on the approximation ratio, we have to argue about *one* input to the problem.

Approximation ratio

- For maximisation problems, we define

$$\max_x \text{opt}(x) \,/\, \text{obj}(A(x))$$

  - i.e., the worst case ratio of the optimal value of the objective over the value of the objective achieved by the algorithm, over all possible inputs to the problem.

  - Convention, to have approximation ratios always be ≥ 1.

Challenges

- What does “*close*” to the optimal mean? How do we measure that?
- How do we make such an argument, if we cannot really find the optimal?
- How do we know if our algorithm is the best possible? Can we get “*closer*” to the optimal?

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A better greedy algorithm for load balancing

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    - Pick any job.
    - Assign it to the machine with the smallest load so far.
    - Remove it from the pile of jobs.

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  - Pick any job.
  - Assign it to the machine with the smallest load so far.
  - Remove it from the pile of jobs.

We did not really take into account the order in which we consider the jobs.

A better greedy algorithm for load balancing

- Sorted-Balance:
  - Sort the jobs in non-increasing order of processing times.
  - Pick a job according to this order.
  - Assign it to the machine with the smallest load so far.
  - Remove it from the pile of jobs.

Intuition

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- Assume that we have at most m jobs.
- What is the approximation ratio of Sorted-Balance?
  - Each job goes to a different machine.
  - Sorted-Balance produces an optimal allocation.
  - The same was actually true for Greedy-Balance.

A third lower bound for opt

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  - Consider the first $m+1$ jobs in sorted order.
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  - Since there are m machines, there must be one machine that receives at least two of these jobs.

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  - Consider the first $m+1$ jobs in sorted order.
  - Each one of them takes at least t_{m+1} time.
  - Since there are m machines, there must be one machine that receives at least two of these jobs.
  - The load on this machine will be at least $2t_{m+1}$.

Lower bounding the optimal

- Two lower bounds:

$$T^* \geq \frac{1}{m}\sum_{j=1}^{n} t_j$$

$$T^* \geq \max_j t_j$$

Lower bounding the optimal

- Three lower bounds:

$$T^* \geq \frac{1}{m}\sum_{j=1}^{n} t_j$$

$$T^* \geq \max_j t_j$$

$$T^* \geq 2t_{m+1}$$

The performance of Sorted-Balance

- Theorem: Algorithm Sorted-Balance produces an assignment of jobs to machines with makespan $T \leq (3/2)T^*$.

The proof

- Let M_i be the machine with the maximum load according to the assignment of **Sorted-Balance**.

- If M_i is assigned a single job, the outcome is optimal.

- Assume M_i that is assigned at least two jobs and let j be the last job assigned to the machine.

  - Note that $j \geq m+1$

  - Therefore, $t_j \leq t_{m+1} \leq (1/2)T^*$

The proof (still the same argument)

- Every other machine has load at least T_i - t_j.
- Summing up over all machines we get:

$$\sum_k T_k \geq m(T_i - t_j) \Rightarrow T_i - t_j \leq \frac{1}{m} \sum_k T_k$$

$T_i - t_j \leq T^*$ **(first lower bound)**

The proof (previous argument)

- Consider a job j that was assigned to machine M_i by **Greedy-Balance**.

- Consider the time when this assignment took place.

  - The load of machine j was $T_i - t_j$.

  - This was before we added the job.

  - After we add the *final* job, the load is $T_i - t_j + t_j$.

  - Obviously $t_j \leq \max_k t_k \leq T^*$

The proof (new argument)

- Consider a job j that was assigned to machine M_i by **Greedy-Balance**.

- Consider the time when this assignment took place.

  - The load of machine j was $T_i - t_j$.

  - This was before we added the job.

  - After we add the *final* job, the load is $T_i - t_j + t_j$.

  - We established that $t_j \leq t_{m+1} \leq (1/2)T^*$

The proof

$T_i - t_j \leq T^*$ **(first lower bound)**

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$$T_i \leq \frac{3}{2}T^*$$

$$T \leq \frac{3}{2}T^*$$

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- For the load balancing problem on identical machines, there is a Polynomial Time Approximation Scheme (PTAS).

    - An algorithm which, given an input and a constant parameter ε, runs in polynomial time and produces an outcome which is $(1+\varepsilon)$ far from the optimal.

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- Generally:
  - A PTAS (or an FPTAS, more about that later) is the best approximation we can hope for, for an NP-hard problem.
  - Sometimes it is impossible to get that close.
  - Inapproximability **α** of problem P:
    - There is no polynomial time algorithm that achieves an approximation ratio better than **α**.