Advanced Algorithmic Techniques (COMP523)

Approximation Algorithms 3

Recap and plan

- **Previous lecture:**
  - The Pricing Method (Primal-Dual Method).
    - Application: Vertex Cover.
- This lecture:
  - Linear Programming and Rounding.
    - Application: Vertex Cover.
  - Inapproximability of Vertex Cover.
  - Vertex Cover on Bipartite Graphs.

Methods for approximation algorithms

- Greedy algorithms.
- Pricing method (also known as the Primal-Dual method).
- Linear Programming and Rounding.
- Dynamic Programming on rounded inputs.

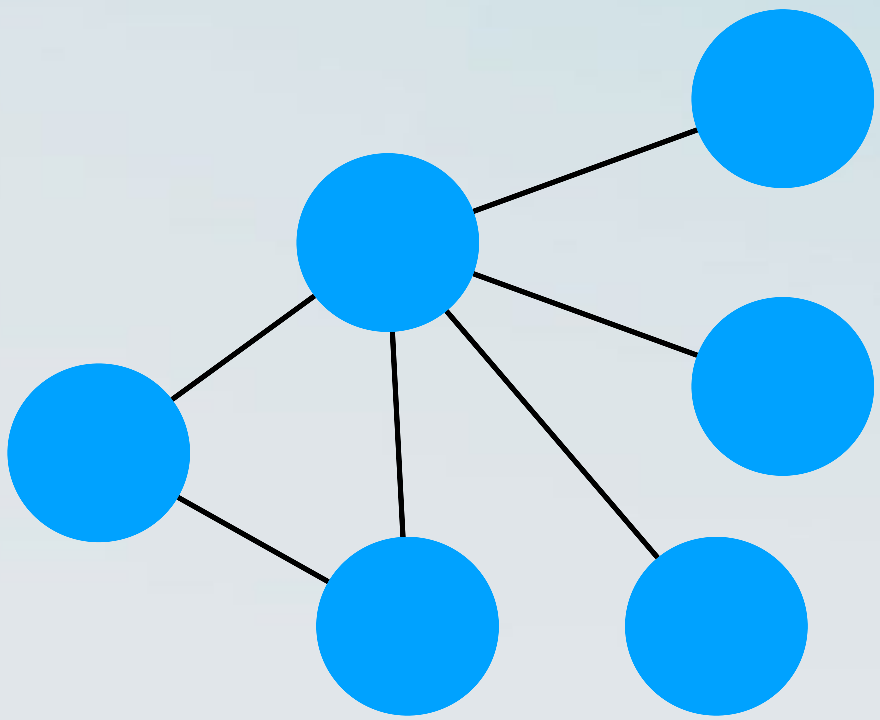
Vertex Cover

- Definition: A vertex cover C of a graph G=(V, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover
  Input: A graph G=(V, E)
  Output: A minimum vertex cover.

Example



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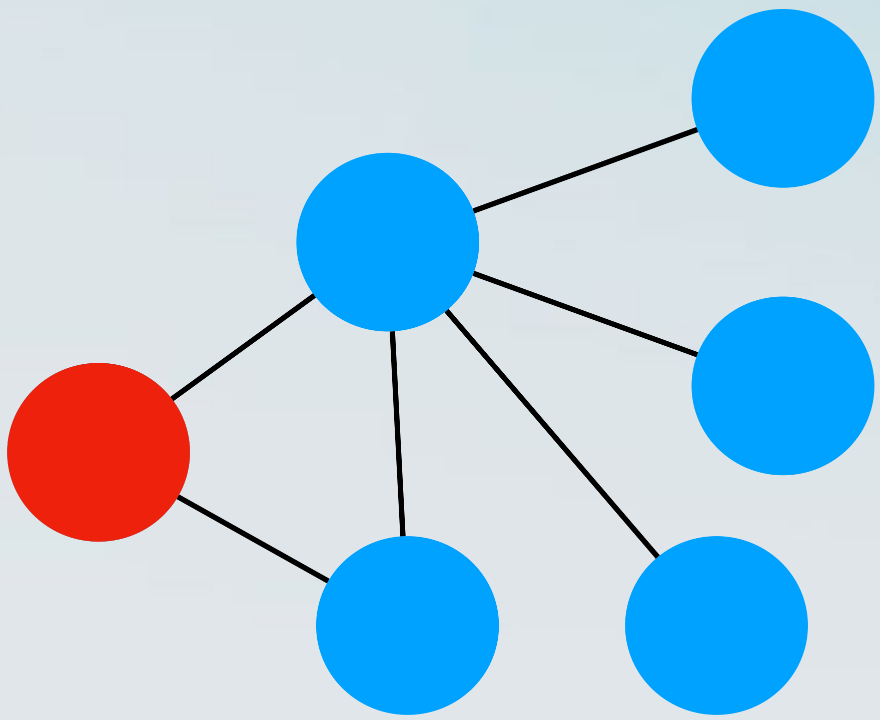

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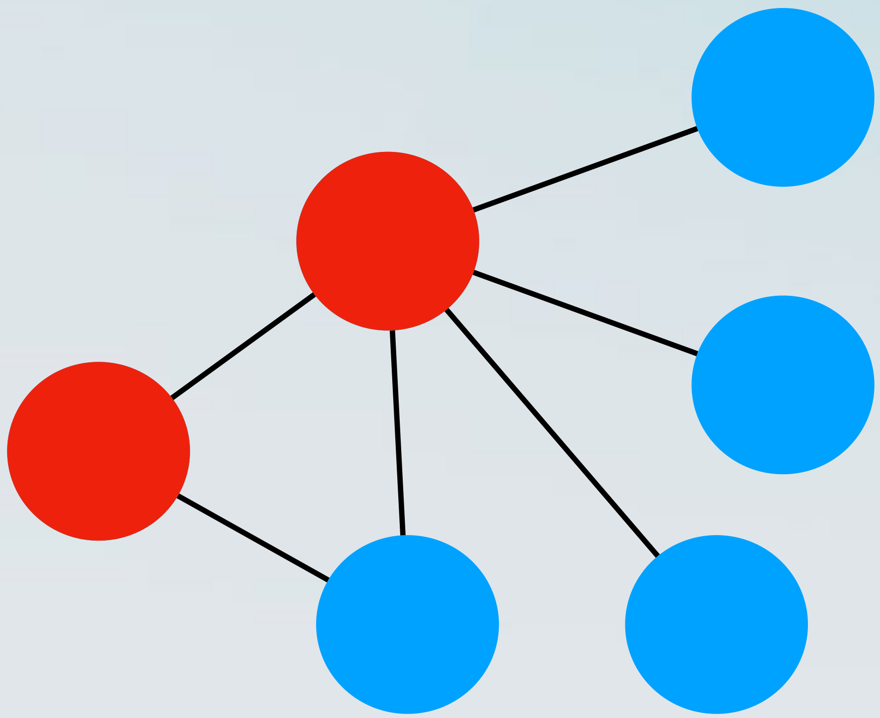

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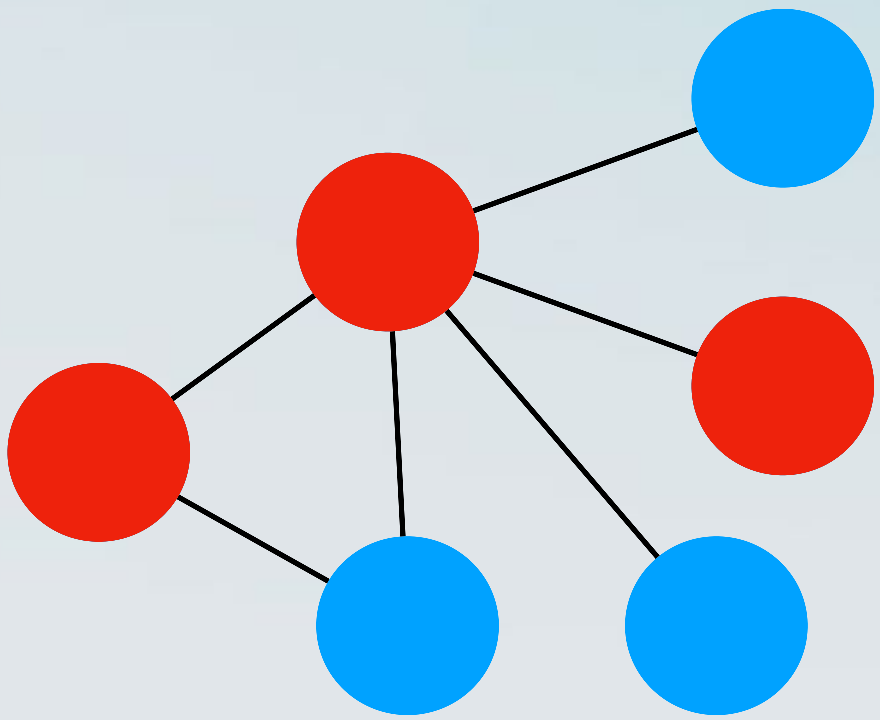

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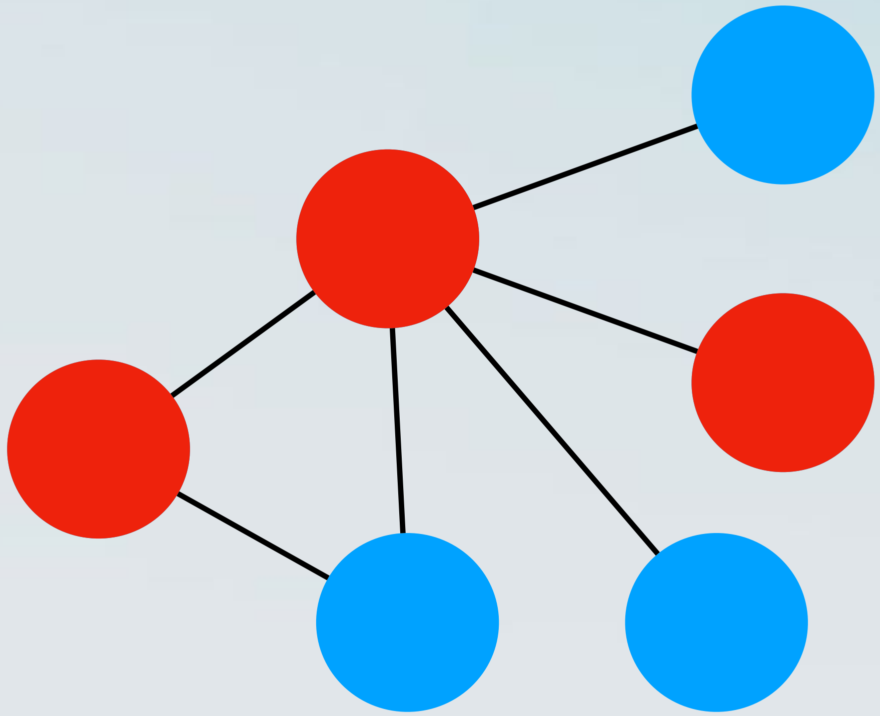
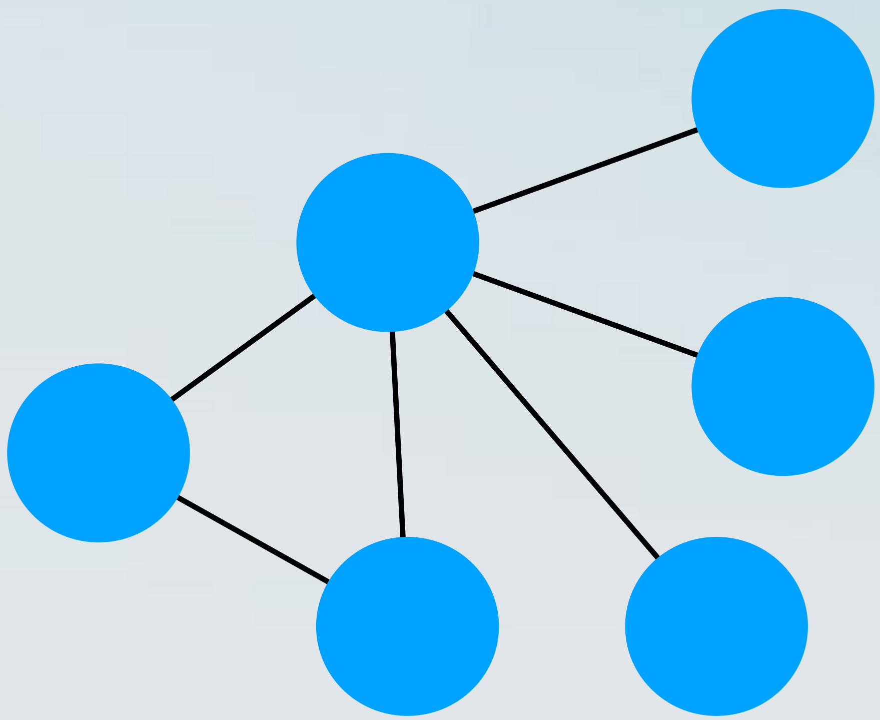


A vertex cover

Example



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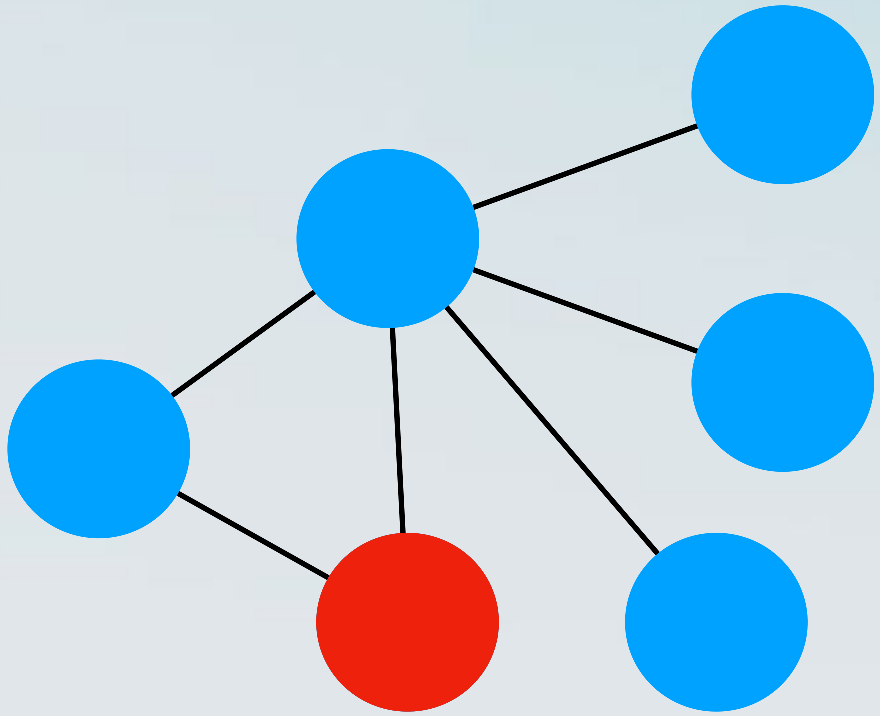

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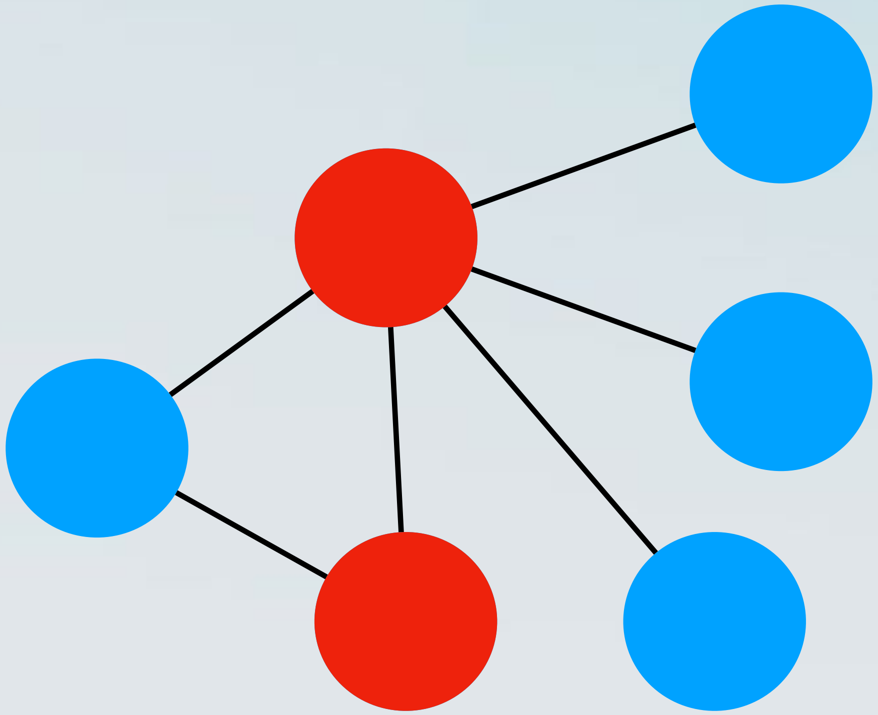

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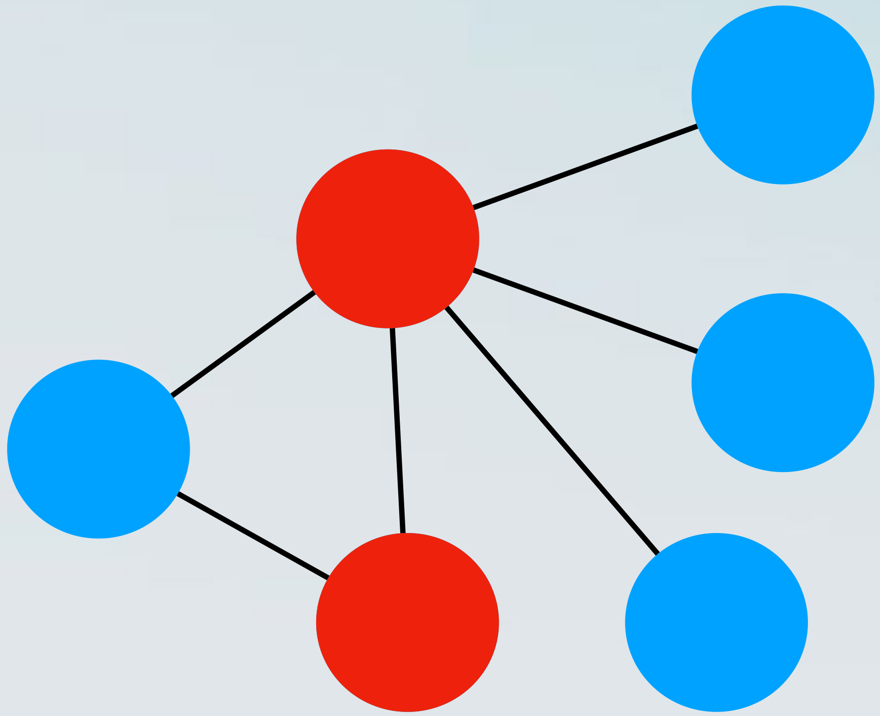

A minimum vertex cover

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- Vertex Cover
  Input: A graph G=(V, E)
  Output: A minimum (weight) vertex cover.

Vertex Cover

- Last time, we designed a polynomial time approximation algorithm for the weighted vertex cover problem.
- We will design another polynomial time approximation algorithm for the weighted vertex cover problem.
    - The second algorithm will be based on a technique called “ILP relaxation and rounding”.

Vertex Cover as an ILP

Minimise $\sum_{i \in V} x_i w_i$

subject to

$x_i + x_j \geq 1,$ **for all** $(i, j) \in E$

$x_i \geq 0,$ **for all** $i \in V$

$x_i \in \{0, 1\},$ **for all** $i \in V$

Vertex Cover as an ILP

Minimise $\sum_{i \in V} x_i w_i$

For each edge, one of the endpoints has to be in the vertex cover.

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A vertex is either included in the vertex cover (value 1) or not (value 0).

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Possible fractional values, e.g.,
$x_i = 0.3$, $x_j = 0.7$

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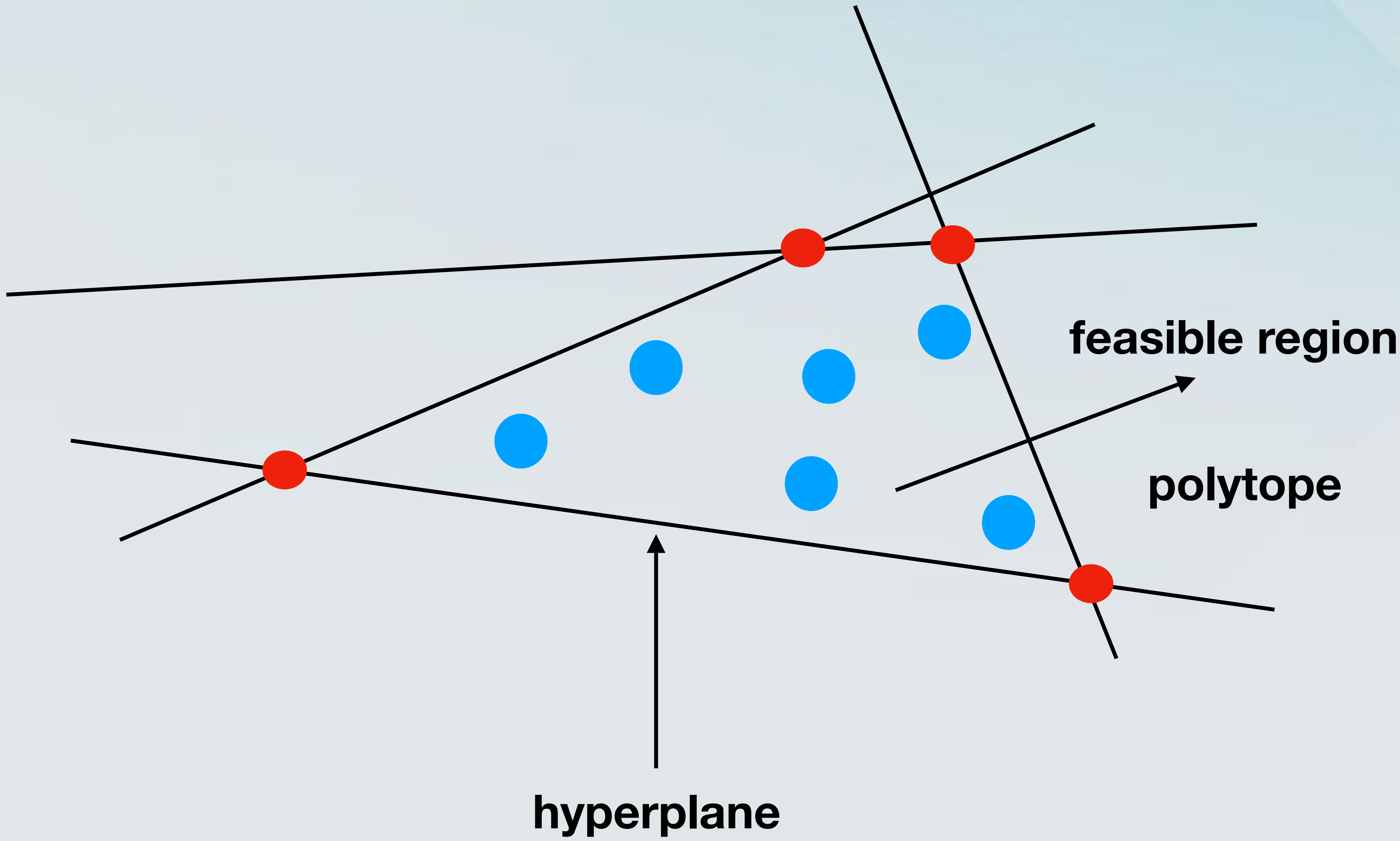
- Is the value of this “fractional” vertex cover, smaller or larger than the value of the minimum weight vertex cover?

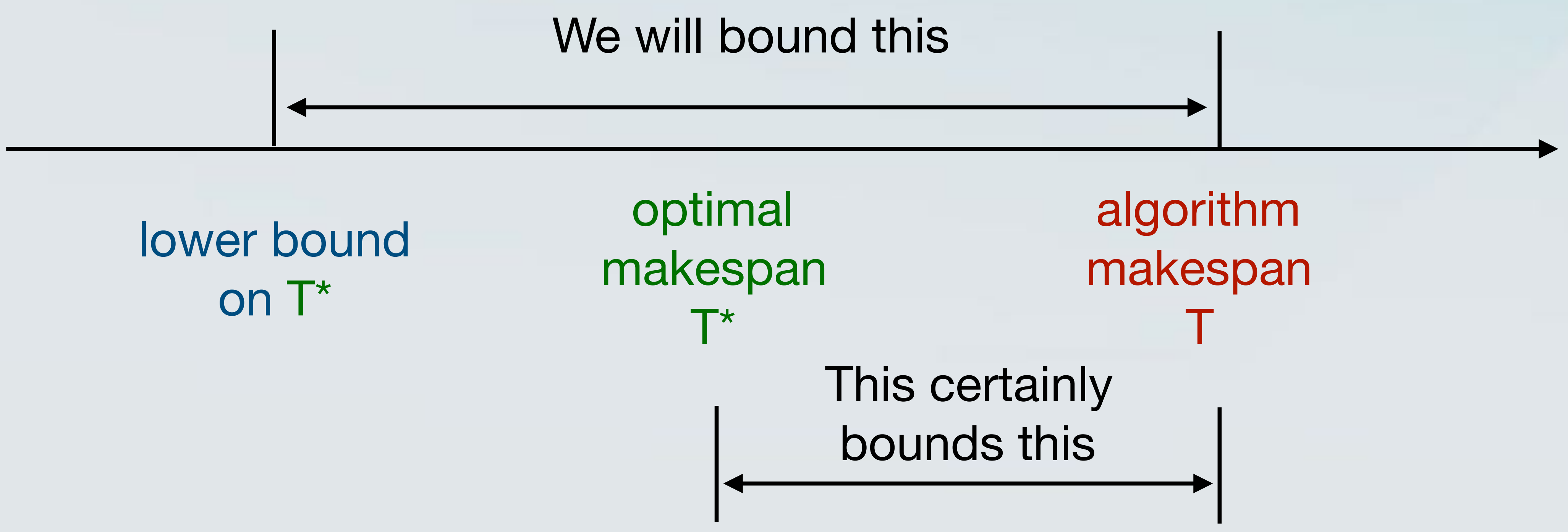
ILP vs LP-relaxation



candidate optimal solution to the relaxation

candidate optimal solution to the ILP

Recall: Load Balancing



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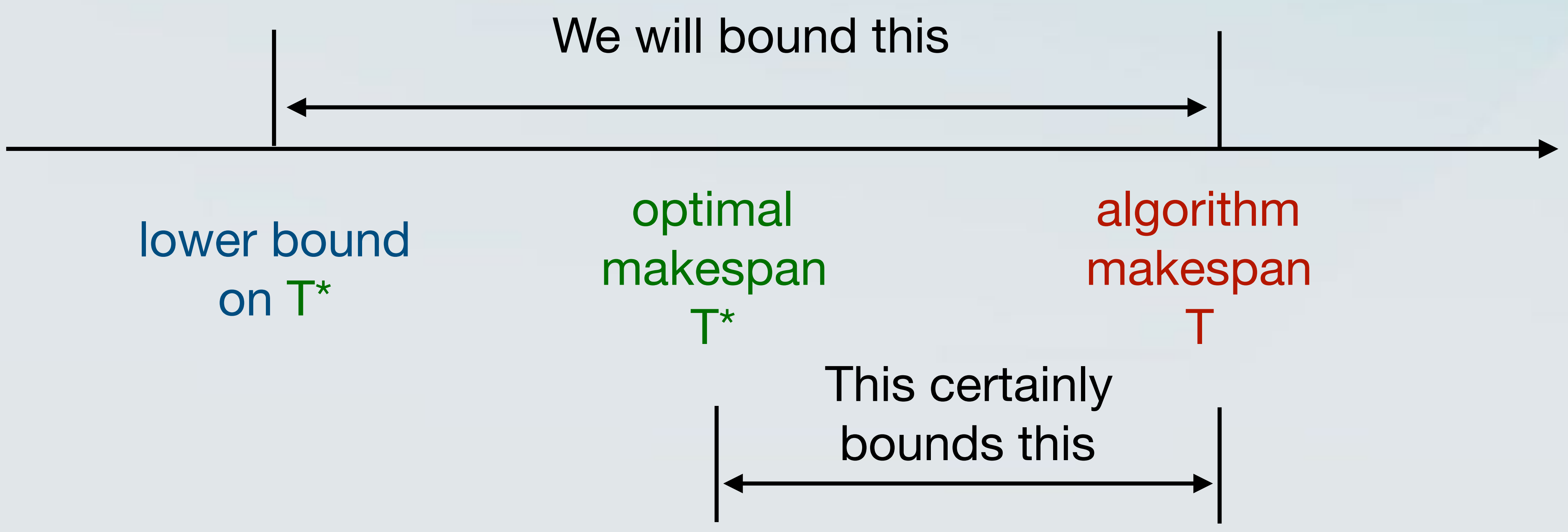

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Back to vertex cover

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  - The weight of the minimum weight "fractional" vertex cover.
  - i.e., the optimal value of the LP-relaxation of the vertex cover ILP.

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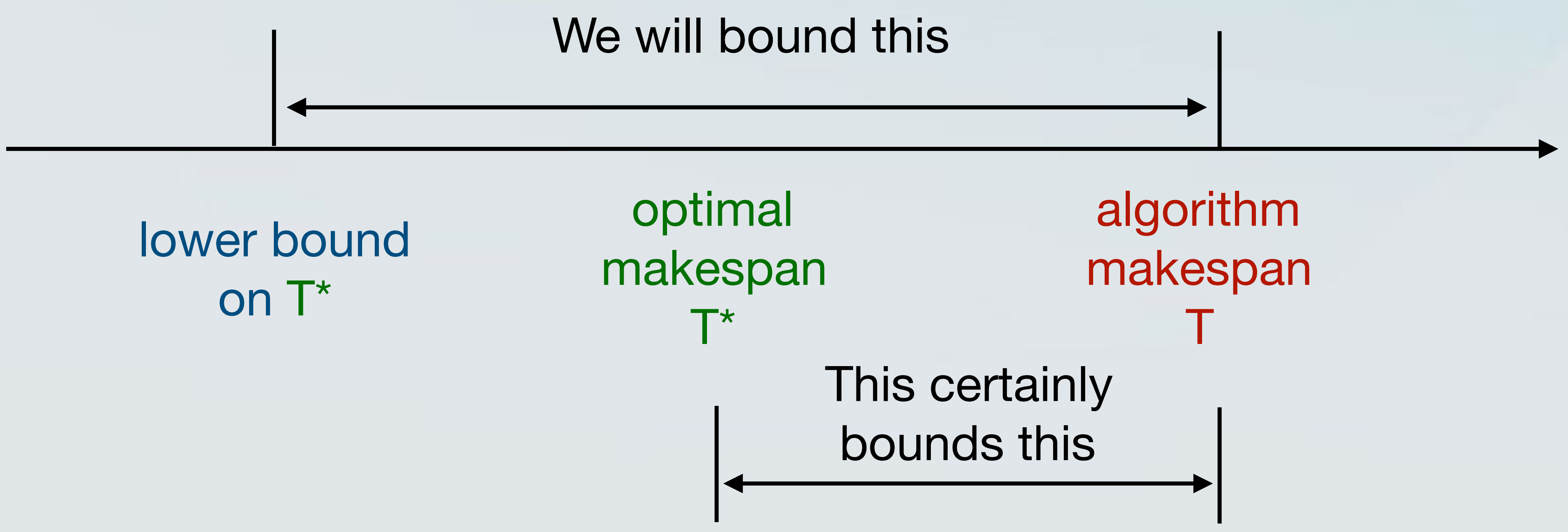

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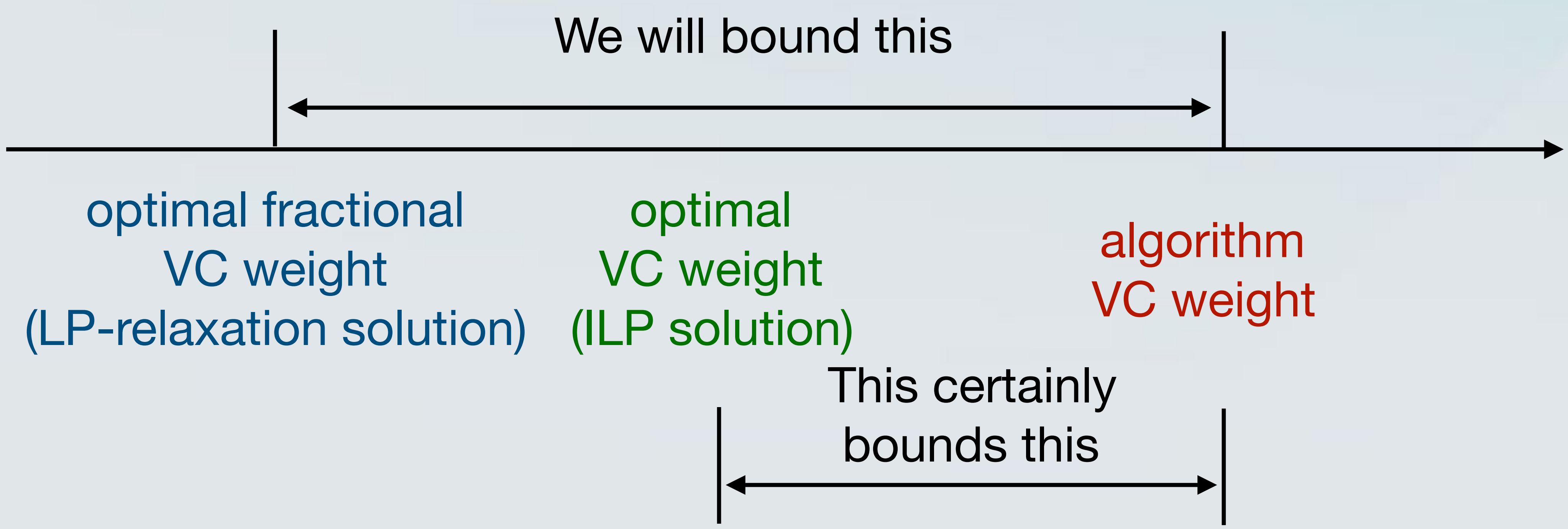
Lower bounding the optimal



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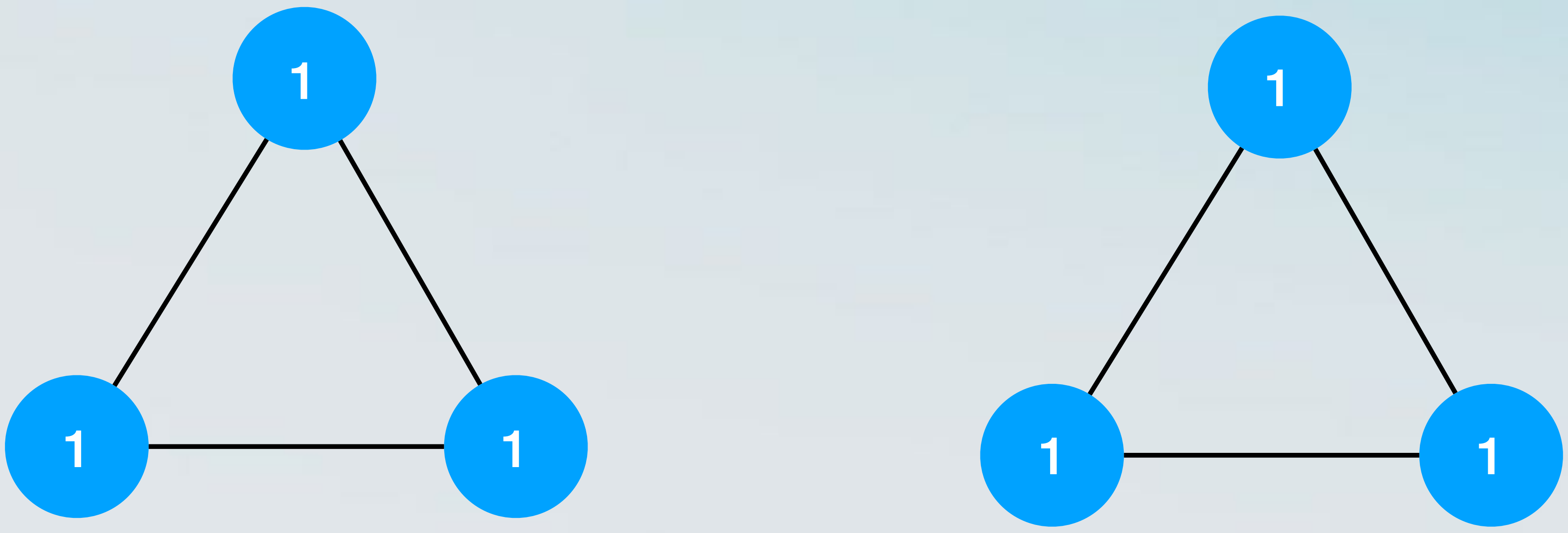

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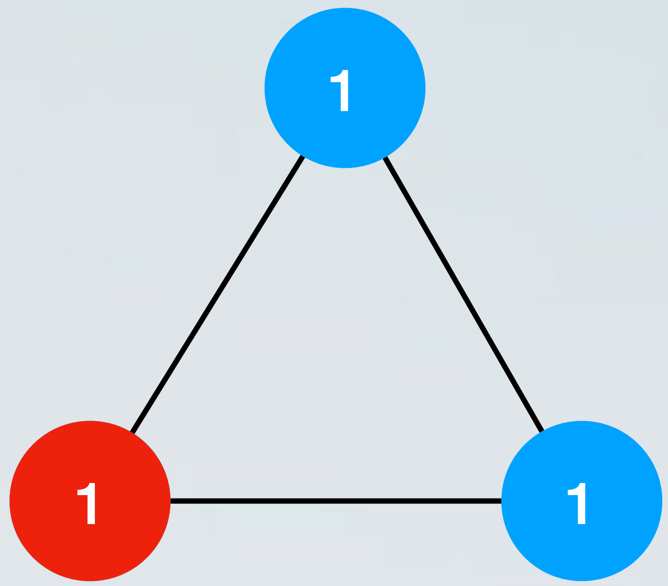

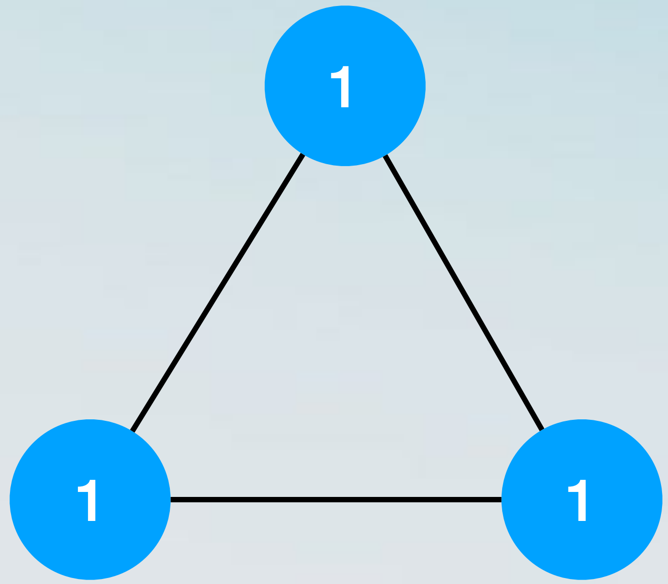

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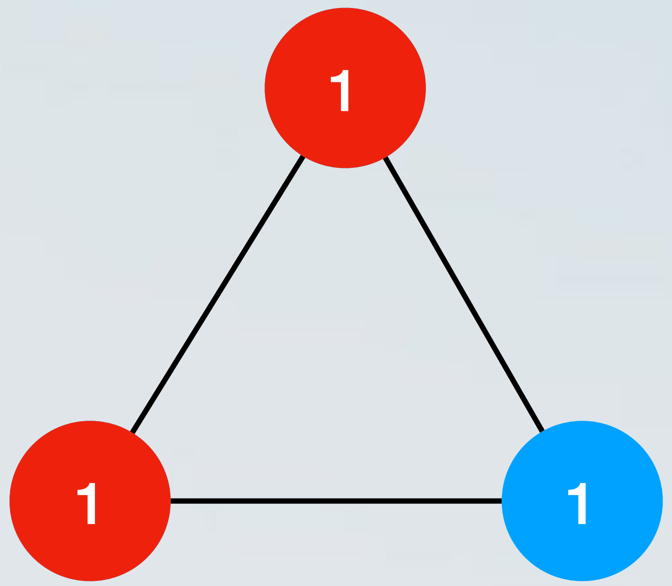

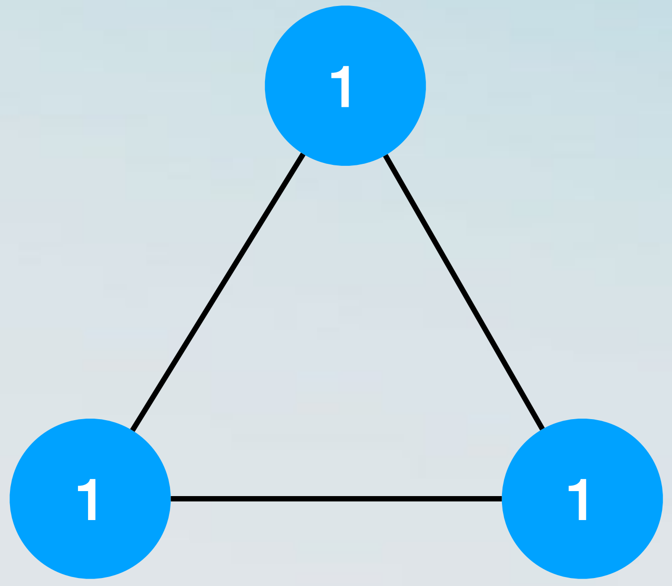

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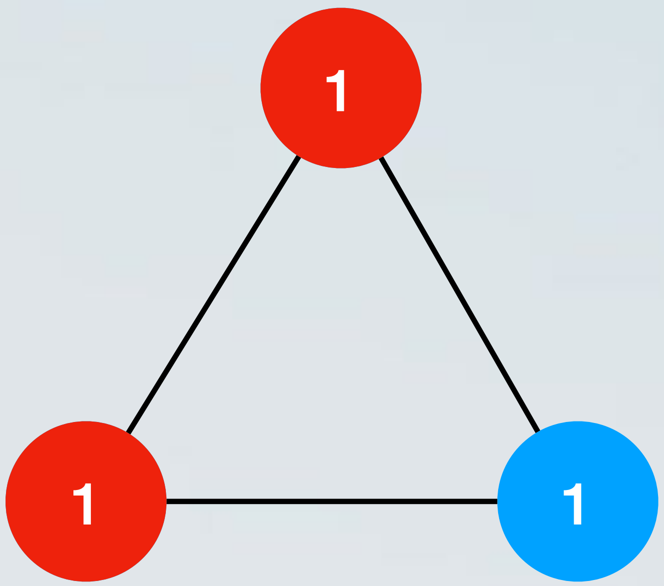
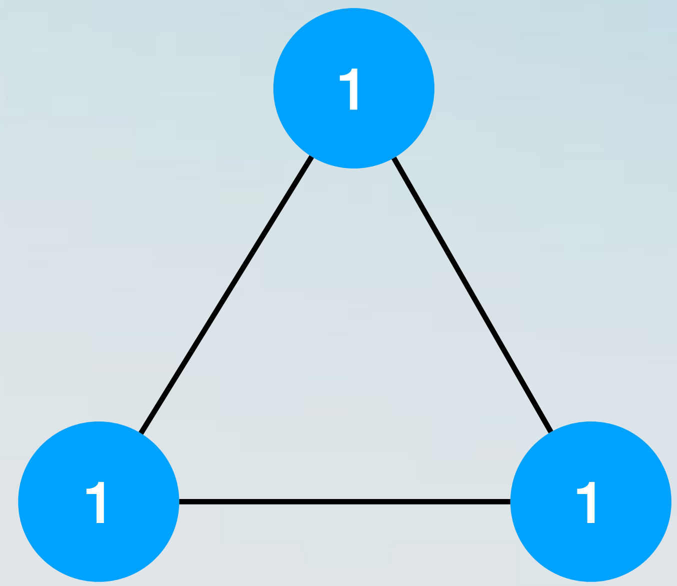


Min weight integral
vertex cover.
weight = 2



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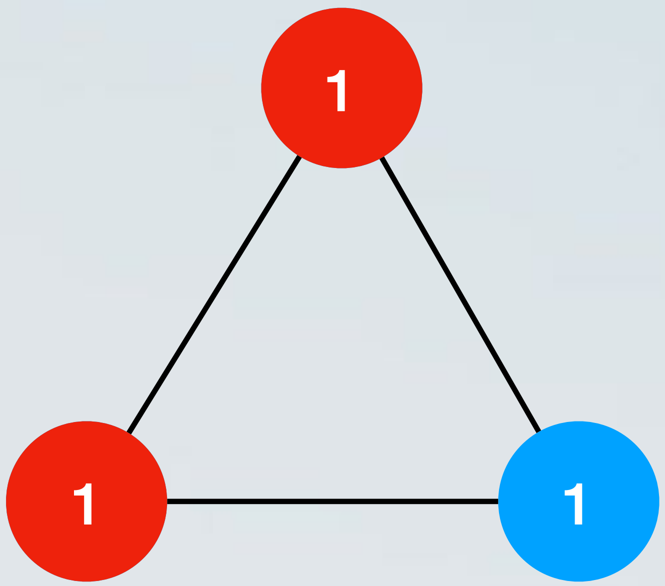

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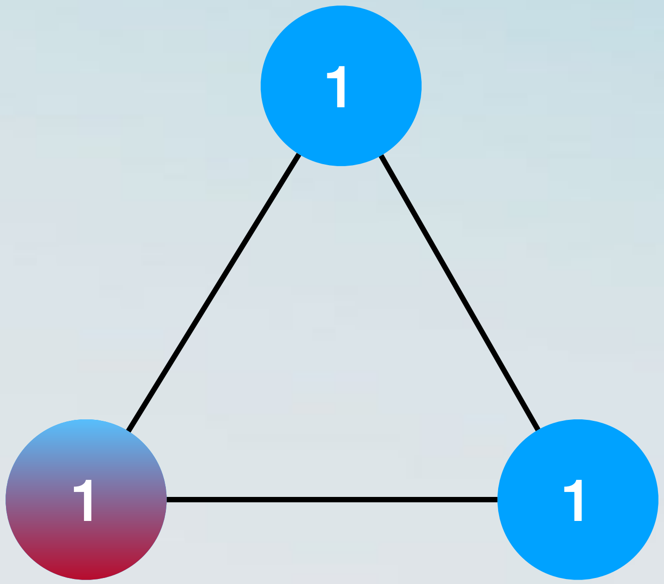

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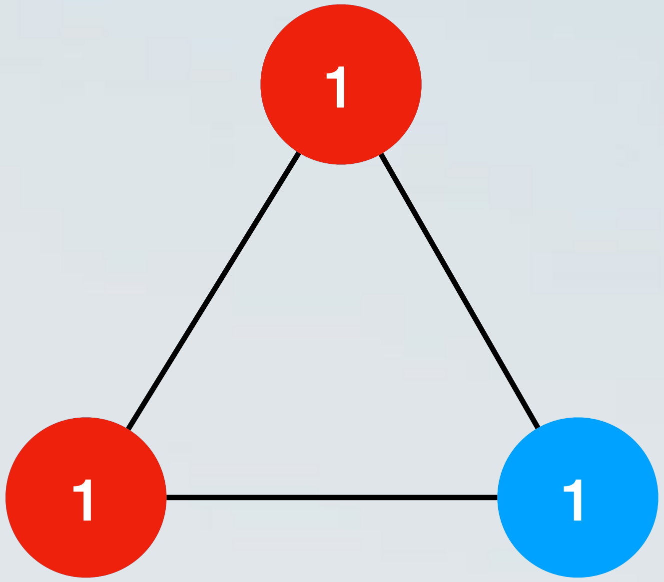

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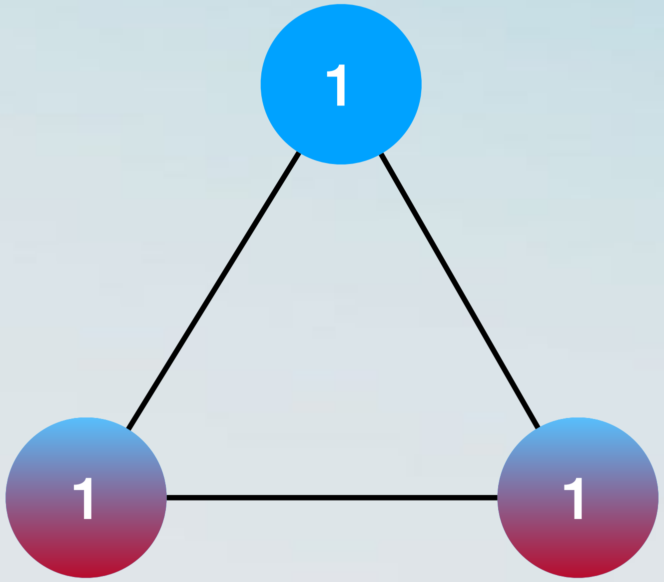

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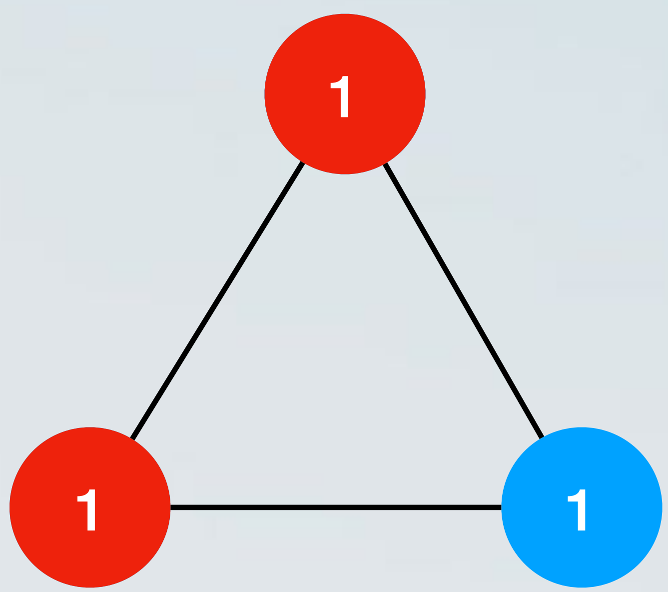

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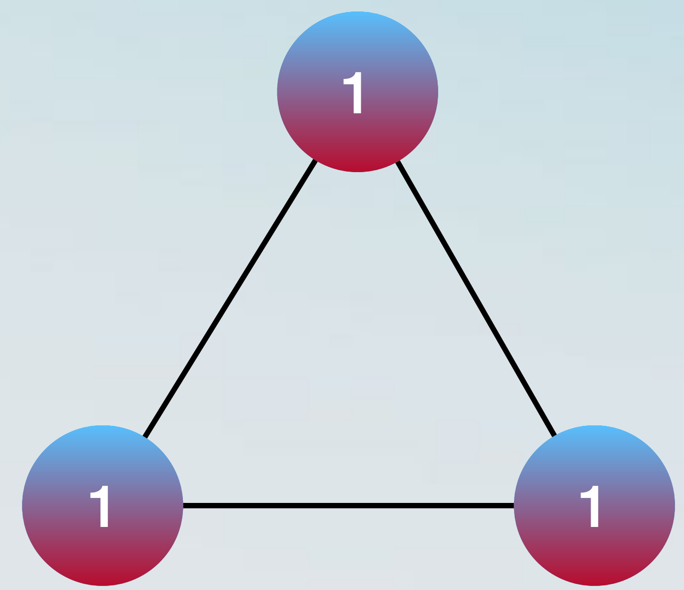

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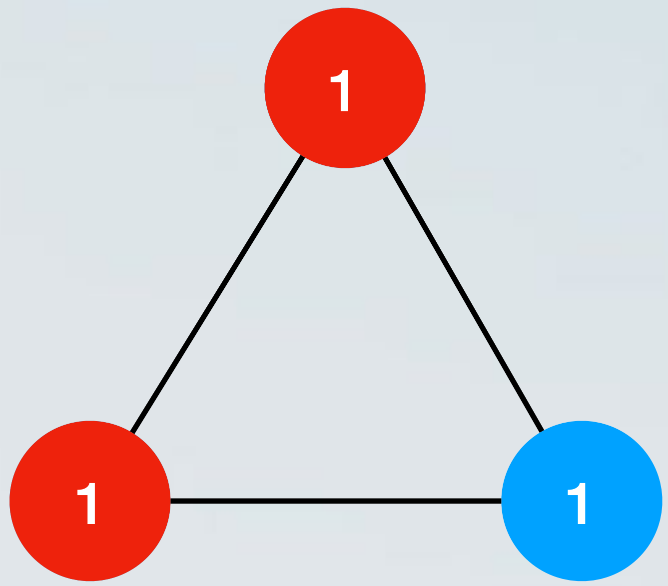

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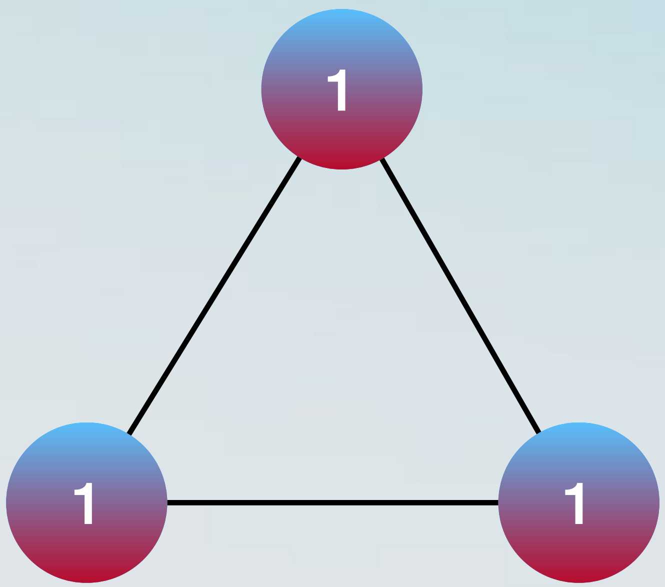

Min weight fractional
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  - We set variable x_i to *1* if $x_i \geq 1/2$ and to *0* otherwise.

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  - But then the constraint $x_i + x_i \geq 1$ would be violated, and this would not be a feasible solution to the LP-relaxation.

Vertex Cover LP-relaxation

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subject to $x_i + x_j \geq 1,$ **for all** $(i, j) \in E$

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weight of the VC that we computed

weight of min-weight fractional VC

Approximation ratio

- The LP-relaxation and round algorithm for vertex cover has an approximation ratio of 2.
- We already knew that 2 was possible, from the Pricing method algorithm (Primal-Dual).
- In this case, the ILP-relaxation and round algorithm seems conceptually simpler.
- In other cases, rounding the solution will not be so straightforward.

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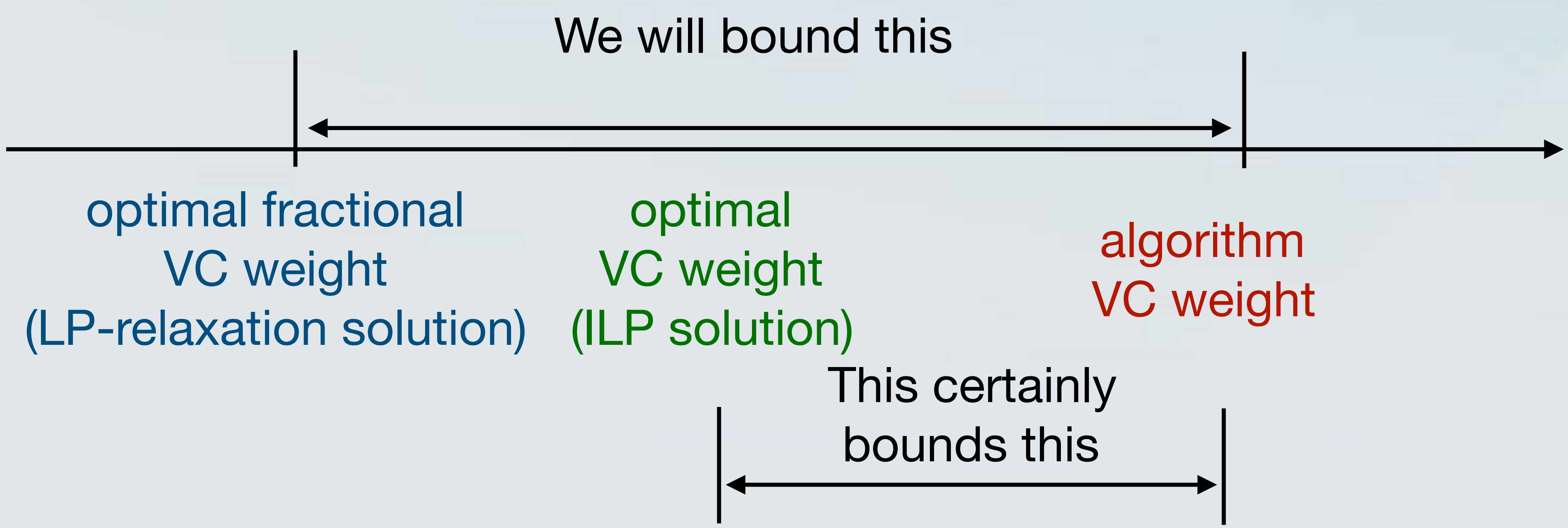
- Maybe we can use the same technique but round more cleverly?

Lower bounding the optimal



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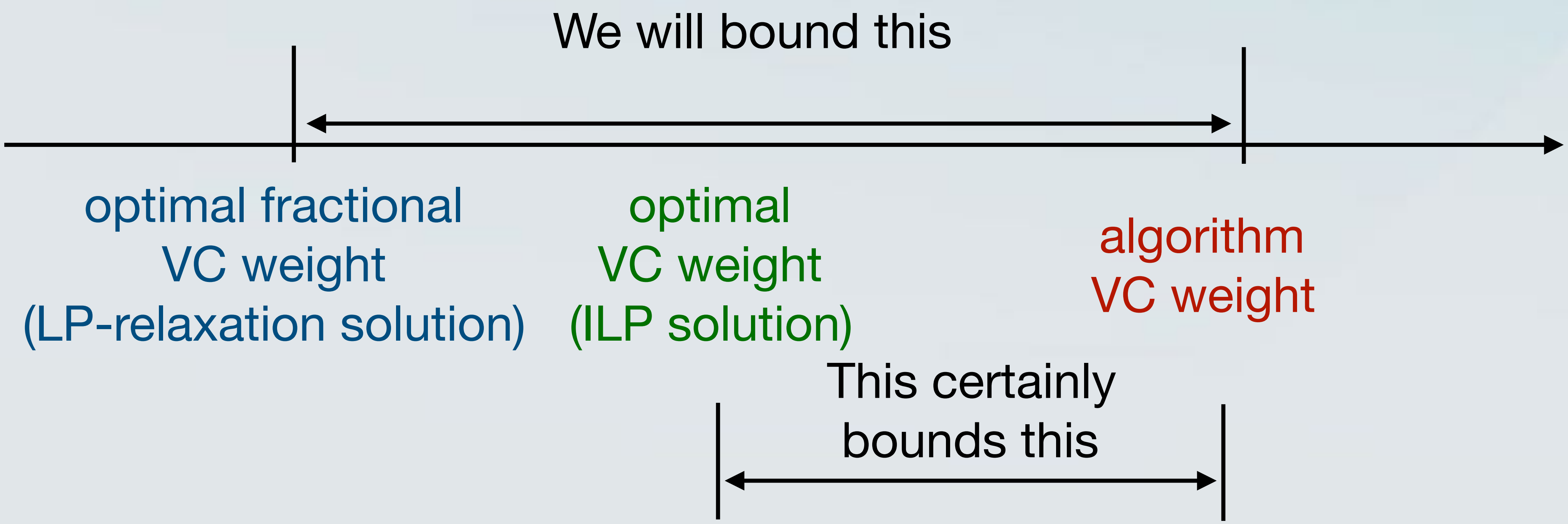

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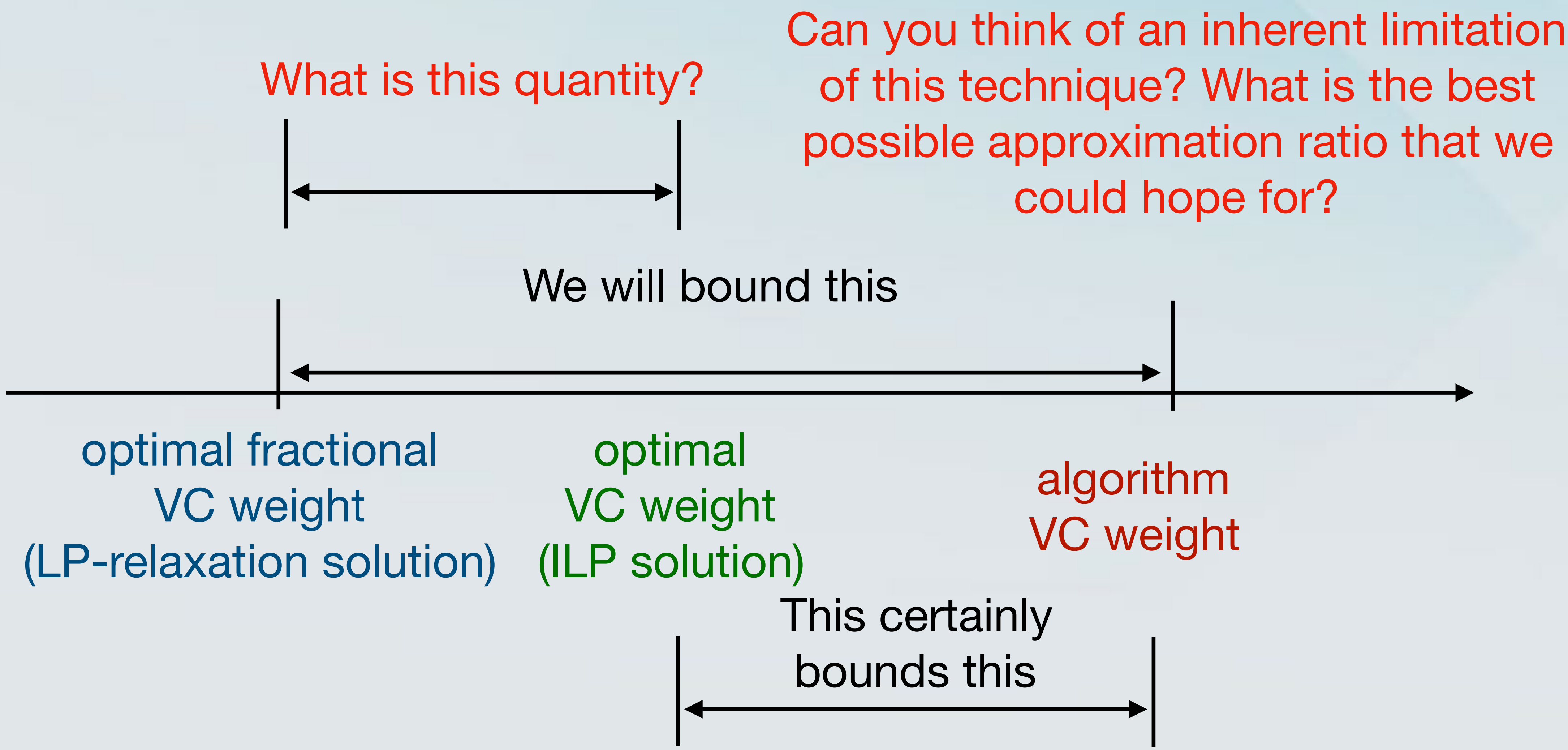

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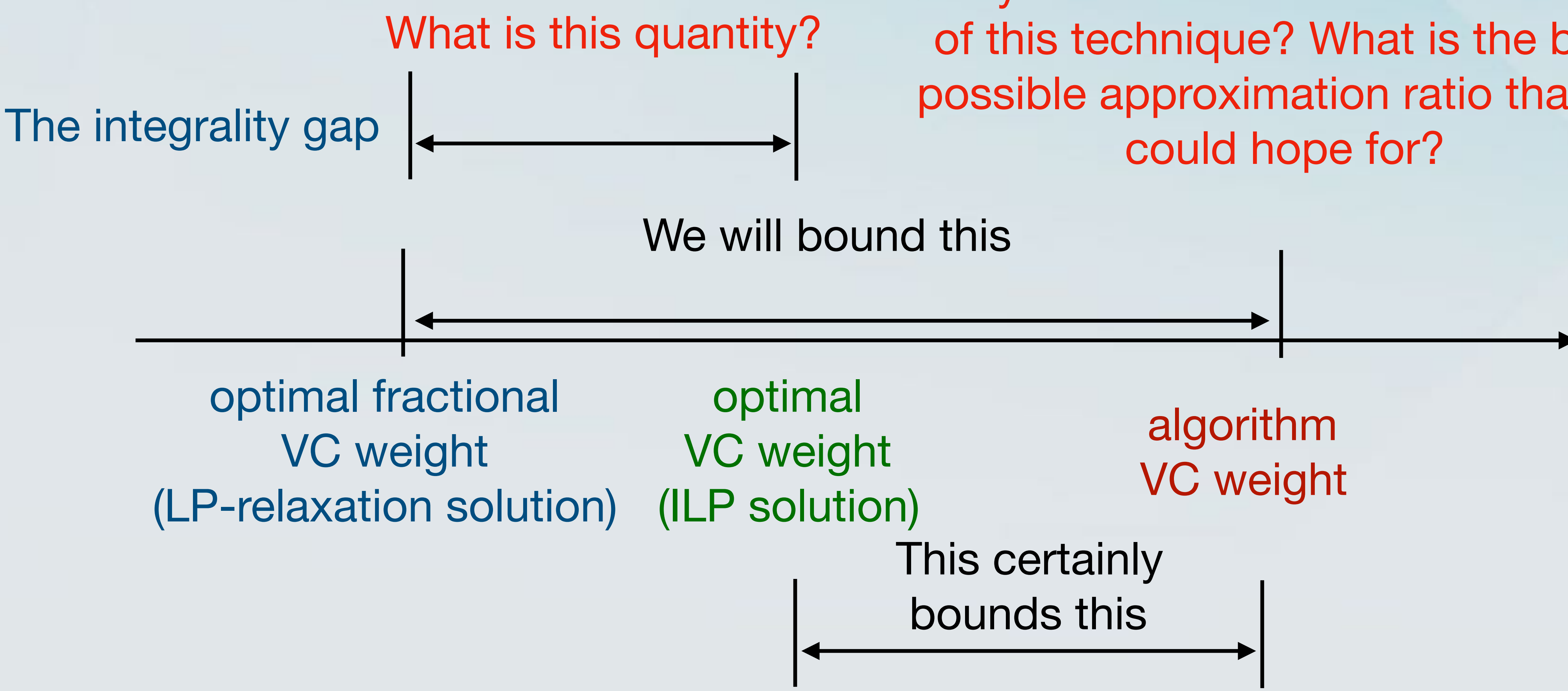

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  - Actually, it does not even mean that LP-relaxation and round technique cannot give you an algorithm with approximation ratio better than α.
  - It means that *with this formulation of the ILP-LP*, α is the best you can hope for.

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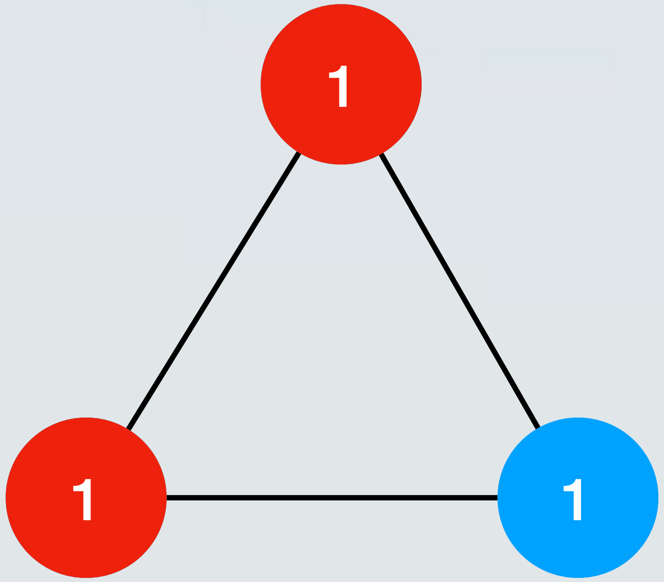
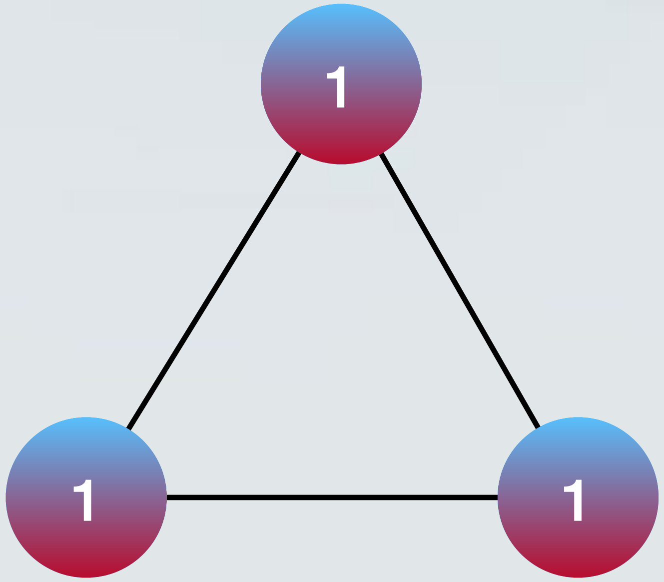


Min weight integral
vertex cover.
weight = 2



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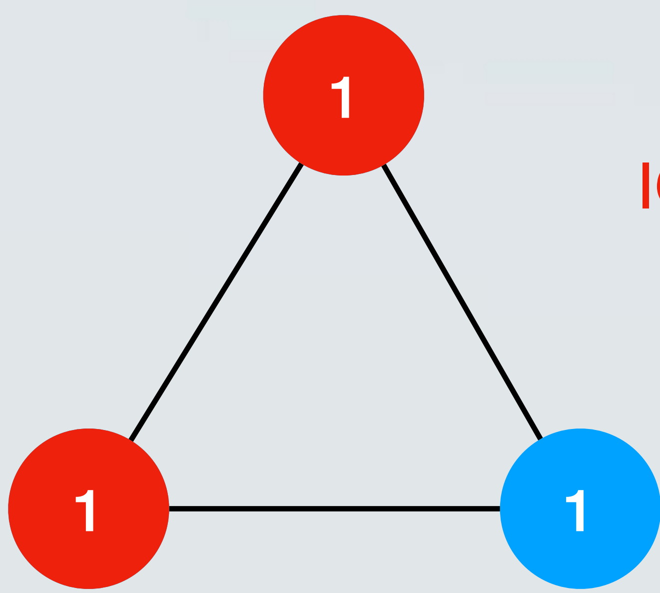

Min weight integral vertex cover.
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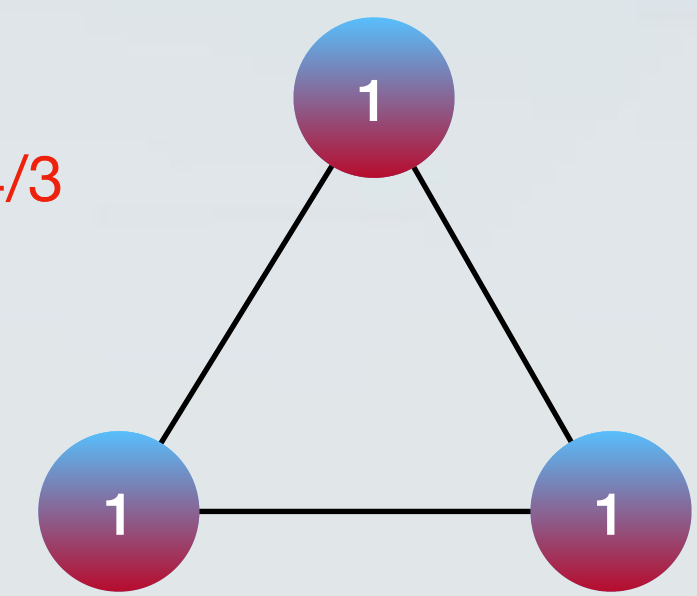
IG ≥ 2/(3/2) = 4/3



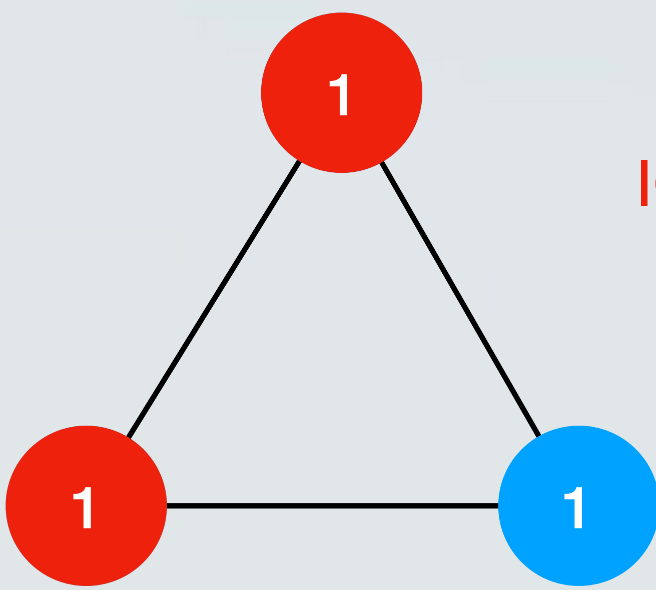
Min weight fractional vertex cover.
weight = 3/2

Integrality gap of VC

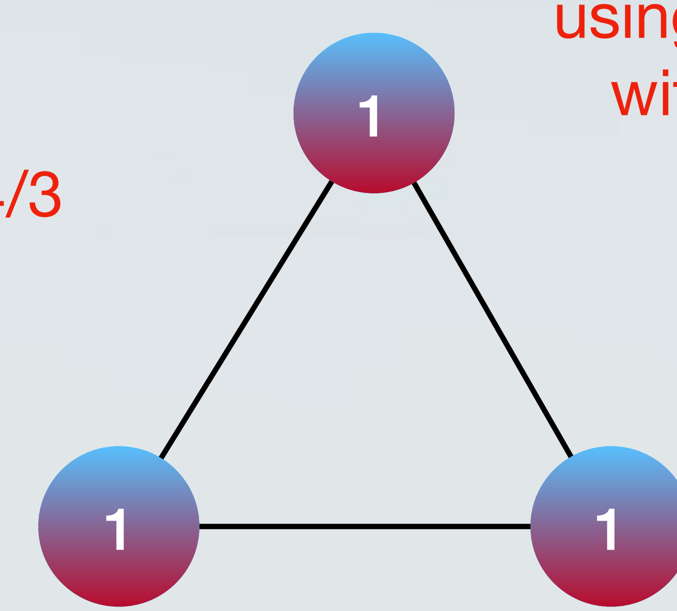
- Do we know *any lower bound* on the integrality gap of vertex cover?
  - i.e., "the IG is at least this much".

We cannot hope to design an algorithm using this formulation with ration better than 4/3.

IG ≥ 2/(3/2) = 4/3



Min weight integral vertex cover.
weight = 2



Min weight fractional vertex cover.
weight = 3/2

Integrality gap of VC

- Can we get any better lower bounds?
- The integrality gap of VC approaches 2 as the number of vertices goes to infinity.
- 5-min exercise: Try to prove this statement.

Limitations of algorithms

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- How do we know if our algorithm is the best possible? Can we get "*closer*" to the optimal?

Limitations of algorithms

- How do we know if our algorithm is the best possible? Can we get "*closer*" to the optimal?

- Maybe we can use the same technique but round more cleverly? No.

Inapproximability of VC

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- Known fact 1: It is impossible (unless P=NP) to design an algorithm with approximation ratio better than 1.363.

- Known fact 2: It is impossible (unless Unique Games is an NP-hard problem) to design an algorithm with approximation ratio better than 2.

Inapproximability of VC

- We saw two different algorithms, both provided an approximation ratio of 2.
- Could we hope to design a better algorithm for the problem?
- The answer to this question depends on our definition of “hope”.
- Known fact 1: It is impossible (unless P=NP) to design an algorithm with approximation ratio better than 1.363.
- Known fact 2: It is impossible (unless Unique Games is an NP-hard problem) to design an algorithm with approximation ratio better than 2.
- Both facts are quite involved to prove.

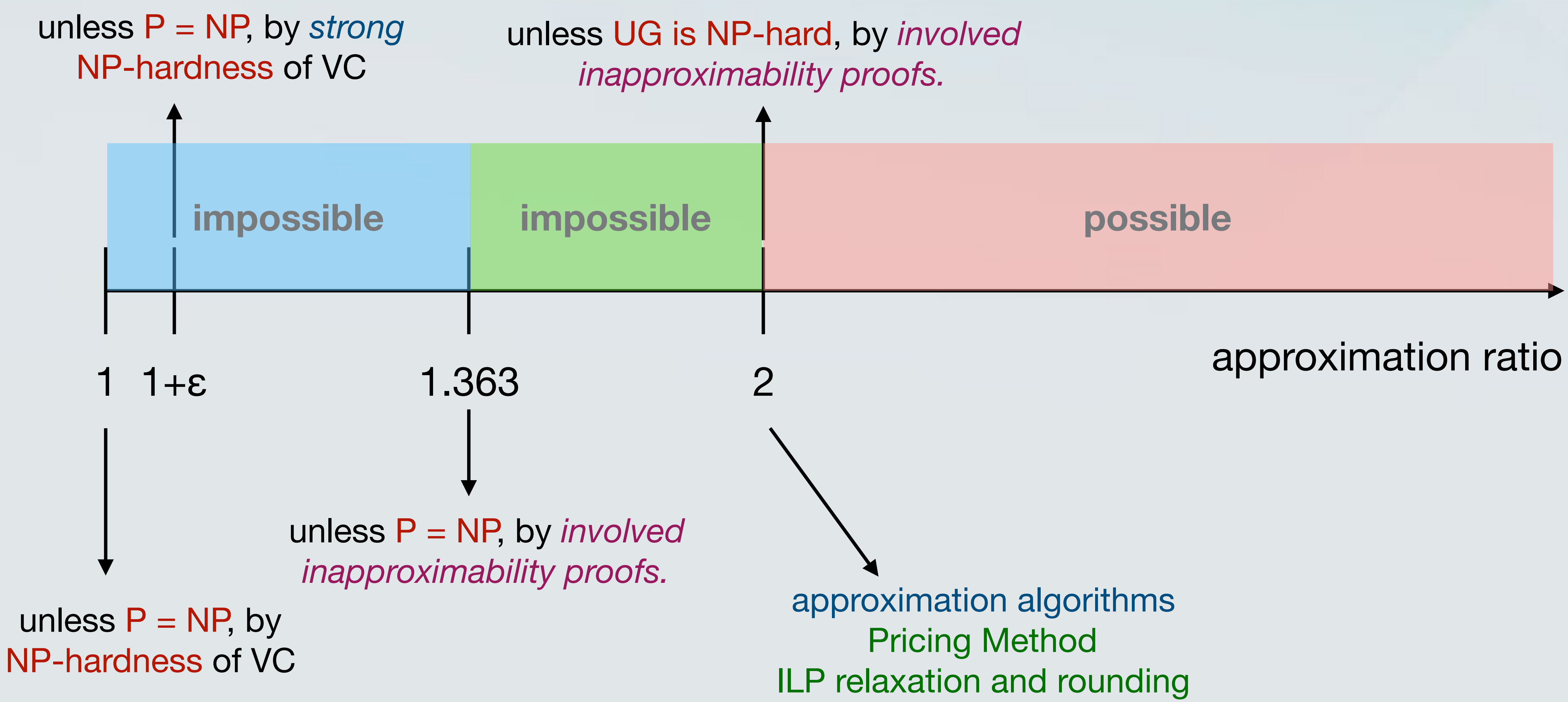
Easier inapproximability

- Definition: A problem P is *strongly* NP-hard, when there is a polynomial time reduction from a *strongly* NP-hard to problem to it.

- For a *strongly* NP-hard problem P,
  - There is **no** Fully Polynomial Time Approximation Scheme (FPTAS - *next lecture*).
  - There is **no** pseudopolynomial time algorithm that solves it exactly.

The approximation landscape for Vertex Cover

- Vertex Cover is *strongly* NP-hard.



Vertex Cover on bipartite graphs

- Definition: A vertex cover C of a bipartite graph G=(A ∪ B, E) is a subset of the nodes such that every edge e in the graph has at least one endpoint in C.

- Definition: A minimum vertex cover is a vertex cover of the smallest possible size.

- Vertex Cover on bipartite graphs
  Input: A bipartite graph G=(A ∪ B, E).
  Output: A minimum vertex cover.

Vertex Cover on bipartite graphs

- We will establish via a series of arguments that VC on bipartite graphs can be solved in polynomial time.

Vertex Cover as an ILP

Minimise $\sum_{i \in V} x_i$

subject to $x_i + x_j \geq 1,$ **for all** $(i, j) \in E$

$x_i \geq 0,$ **for all** $i \in V$

$x_i \in \{0, 1\},$ **for all** $i \in V$

Vertex Cover LP-relaxation

Minimise $\sum_{i \in V} x_i$

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The dual

Maximise $$\sum_{j \in E} y_j$$

subject to $$\sum_{j \text{ is incident to vertex } i} y_j \leq 1, \quad \text{for all } i \in V$$

$$y_j \leq 1, \quad \text{for all } j \in E$$

The ILP corresponding to the dual

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Include as many edges as possible …

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maximum matching!

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König's Theorem

- In a bipartite graph, the *size of the maximum matching* is equal to the *size of the minimum vertex cover*.
    - König's proof is constructive: If starts from a maximum matching and produces a vertex cover, proving that it is minimum.
    - Alternative proof based on *total unimodularity*.

This is Maximum Matching

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This is the LP-relaxation

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Fact: The incidence matrix of a bipartite graph is totally unimodular.

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Fact: The incidence matrix of a bipartite graph is totally unimodular.

This means that size of maximum matching = size of maximum fractional matching.

This is Vertex Cover

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Fact: The constraint matrix is also totally unimodular. It is just the transpose of the constraint matrix of the maximum bipartite matching problem.

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This means that size of maximum matching = size of maximum fractional matching.

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But size of maximum fractional matching = size of minimum fractional VC (why?).

This means that size of maximum matching = size of minimum VC.

How do we find the size of the minimum VC on a bipartite graph?

- Solve the maximum matching on the same bipartite graph.
- The size of this matching is the size of the minimum vertex cover.

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- Solve the maximum matching on the same bipartite graph.
- The size of this matching is the size of the minimum vertex cover.
- Find the minimum vertex cover using the size of the minimum vertex cover.
  - How?

From previous lecture...

- Pick a vertex v in the graph.
  - Remove it (and the incident edges) to get graph G - {v}.
  - Property: If v was in any minimum vertex cover, G - {v} has a minimum vertex cover of size k*-1.
  - Check if the graph G - {v} has a vertex cover of size at most k*-1.
    - Yes: Include v in the vertex cover.
    - No: Do not include v in the vertex cover.
    - Then move to the next vertex.

Summing up

- Vertex Cover is strongly NP-hard in general.
  - In fact, hard to approximate better than 1.363.
  - There exist 2-approximate polynomial time algorithms for the problem.
- On bipartite graphs, the problem is solvable in polynomial time.