Advanced Algorithmic Techniques (COMP523)

Approximation Algorithms 4

Recap and plan

- **Previous lecture:**
  - Linear Programming and Rounding.
    - Application: Vertex Cover.
  - Inapproximability of Vertex Cover.
  - Vertex Cover on Bipartite Graphs.
- **This lecture:**
  - Dynamic programming on rounded inputs.
    - Application: Knapsack
  - PTAS and FPTAS

Methods for approximation algorithms

- Greedy algorithms.
- Pricing method (also known as the Primal-Dual method).
- Linear Programming and Rounding.
- Dynamic Programming on rounded inputs.

The 0/1-knapsack problem

- We are given a set of n items {*1*, *2*, … , *n*}.
- Each item *i* has a non-negative weight w_i and a non-negative value v_i.
- We are given a bound W.
- Goal: Select a subset S of the items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i$ is maximised.

7 minute exercise

Design a dynamic programming algorithm for 0/1 knapsack.

Algorithm **SubsetSum**(n,W)

Array M=[0 … n, 0 … W]
Initialise M[0, w] = 0, for each w = 0, 1 , … , W

For i = 1, 2, … , n
  For w = 0 , … , W
    If (w_i > w)
      M[i, w] = M[i-1, w]
    Else
      M[i, w] = **max{**M[i-1, w] , w_i + M[i-1, w-w_i]**}**
    EndIf

Return M[n, W]

0/1-Knapsack in Pseudopolynomial Time

The dynamic programming algorithm for 0/1 knapsack solves knapsack optimally in time polynomial in *n* and W.

Algorithm **Knapsack**(*n*,W)

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Return M[n, W]
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Another pseudopolynomial time algorithm for 0/1-Knapsack

Algorithm **Knapsack**(*n*,W)

Array M=[*0 … n*, 0 … V]
Initialise M[*i*, *0*] = 0, for *i* = *0*, *1* , … , *n*

For i = *1*, *2*, … , *n*
    For V = 1 , … , $\sum_{j=1}^{i} v_j$
      If (V > $\sum_{j=1}^{i-1} v_j$)
          M[*i*, V] = w_i + M[*i-1*, V]
      Else
          M[*i*, V] = **max{**M[*i-1*, V] , w_i + M[*i-1*, max(*0*, V-v_i)]**}**
      EndIf

Return the maximum value V such that M[*n*, V] ≤ W.

Intuition

- We will create subproblems based on the values, not the weights.
- Each subproblem will be defined by an index *i* and target value V.

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For i = $1, 2, \dots, n$
  For V = $1, \dots, \sum_{j=1}^{i} v_j$
    If ($V > \sum_{j=1}^{i-1} v_j$)
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- How many subproblems can we have?
  - At most $O(n^2 v^*)$, where v^* is the maximum value over all the items.
- More details: *Kleinberg and Tardos, Chapter 11, page 648-649.*

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- A polynomial time greedy approximation algorithm with approximation ratio 2.
- Can we get better approximations?

Rounding the values

- We will use a rounding parameter b.
- For each item *i*, let $\tilde{v}_i = \lceil v_i / b \rceil b$
  - It holds that for each item *i*, we have $v_i \leq \tilde{v}_i \leq v_i + b$
  - Let $\hat{v}_i = \tilde{v}_i / b = \lceil v_i / b \rceil$
- Intuition: We divide all the values by some factor b, and then we round up the result to get integer numbers.

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    - It is, when v^* is small (i.e., polynomial in n).

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  - recall: $v_i \leq \tilde{v}_i \leq v_i + b$
  - the optimal values differ by a factor of b.

The algorithm

Knapsack-Approx(ε)

Set $b = (\varepsilon/2n) \max_i v_i$

Run the DP algorithm for knapsack on values $\hat{v}_i$
Return the set S of items found.

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  - We didn't mess up with the weights at all!
  - This is why we could not use the DP algorithm that we knew from previous lectures.

Running Time

- The DP algorithm runs in time O(n^2v*).
- Recall: $v^* = \max_i v_i$
- So here, it runs in time polynomial in n and $\max_i \hat{v}_i$
- It holds that : $\arg\max_i v_i = \arg\max_i \hat{v}_i$
- So we have: $\max_i \hat{v}_i = \hat{v}_j = \lceil v_j / b \rceil = n/\varepsilon$

Running Time

- The overall running time is $O(n^3/\varepsilon)$.
- This is polynomial in the input parameters and $1/\varepsilon$.

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- We know that $\sum_{i \in S} \tilde{v}_i \geq \sum_{i \in S^*} \tilde{v}_i$ (why?)
- We have the following inequalities:

$$\sum_{i \in S^*} v_i \leq \sum_{i \in S^*} \tilde{v}_i \leq \sum_{i \in S} \tilde{v}_i \leq \sum_{i \in S} (v_i + b) \leq nb + \sum_{i \in S} v_i$$

Approximation Ratio

- Recall: $b = (\varepsilon/2n)\max_i v_i$
- Let v_j be the largest value. We have that $v_j = 2nb/\varepsilon$
- We also have that $v_j = \tilde{v}_j$
- Assumption: Each item fits in the knapsack
  - This implies $\sum_{i\in S} \tilde{v}_i \geq \tilde{v}_j = v_j = 2nb/\varepsilon$
- Finally, from the inequalities of the previous slide, we have

$$\sum_{i\in S} v_i \geq \sum_{i\in S} \tilde{v}_i - nb \Rightarrow \sum_{i\in S} v_i \geq (2\epsilon^{-1} - 1)nb$$

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PTAS vs FPTAS

- PTAS (Polynomial Time Approximation Scheme):
  An approximation algorithm which, given an ε, runs in time polynomial in the input parameters and has approximation ratio $1+\varepsilon$.

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- What is the algorithm that we designed for knapsack? A PTAS or an FPTAS?

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  - We can pick ε=1/k, and we have a 1+ε approximation in time $O((1/\varepsilon)n^{1/\varepsilon})$.
    - This is polynomial in n but not in 1/ε.

Inapproximability

- Definition: A problem P is *strongly* NP-hard, when there is a polynomial time reduction from a *strongly* NP-hard to problem to it.

- For a *strongly* NP-hard problem P,
  - There is **no** Fully Polynomial Time Approximation Scheme (FPTAS).
  - There is **no** pseudo-polynomial time algorithm that solves it exactly.

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- Limitations of techniques (e.g., integrality gap).
- Inapproximability
  - How do we prove this?
  - Sometimes easy, sometimes hard, mostly hard!