Advanced Algorithmic Techniques (COMP523)

Randomised Algorithms 2

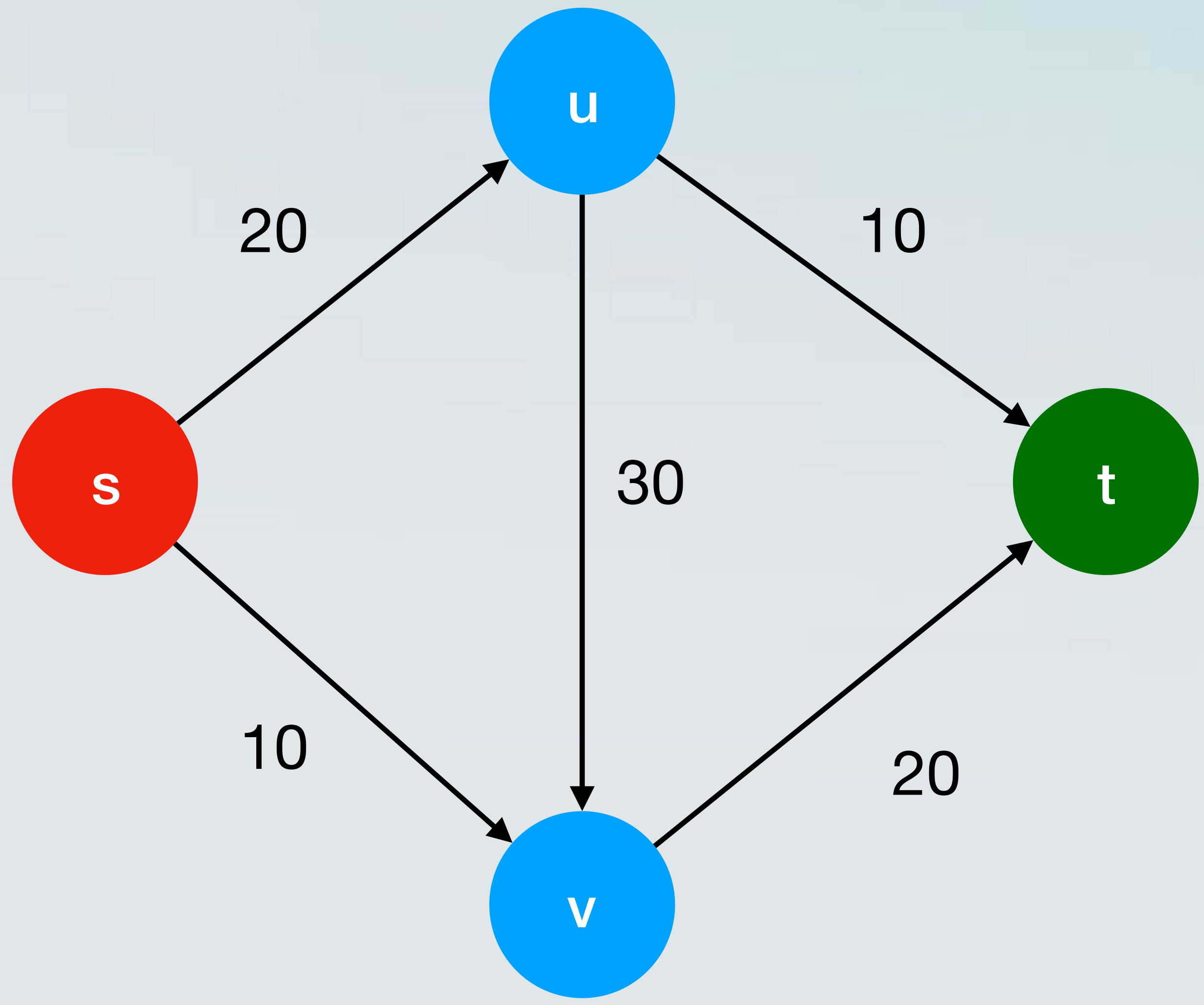
Recap and plan

- **Previous lecture:**
  - Probabilities background.
- **This lecture:**
  - Randomised global cuts in multi-graphs.

Minimum Cut

- A *cut* C is a partition of the nodes of G into two sets S and T, such that s is in S and t is in T.
- The capacity c(S,T) of a cut C is the sum of capacities of all edges "out of S"
  - these are edges (u, v) where u is in S and v is in T.

Example



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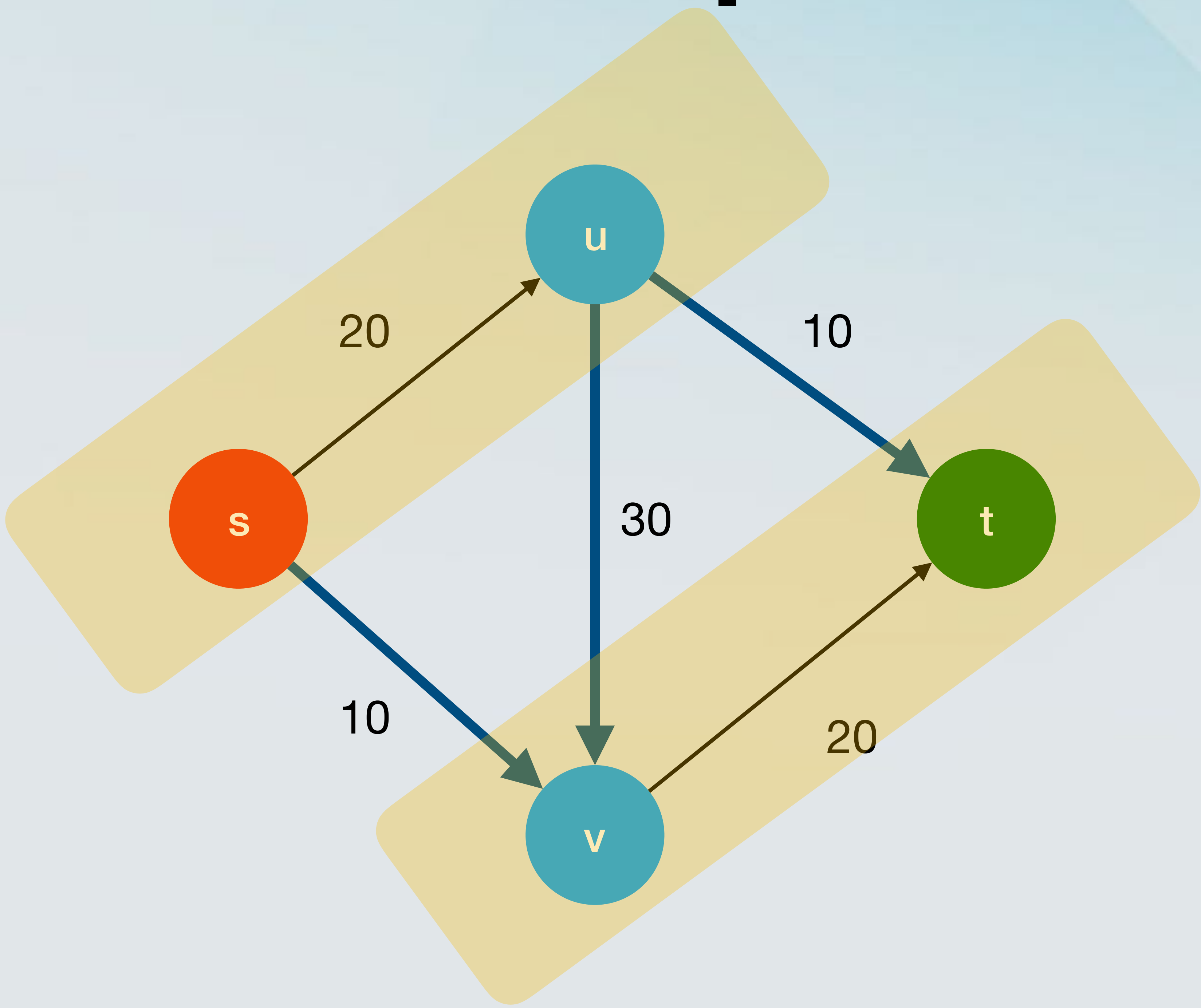

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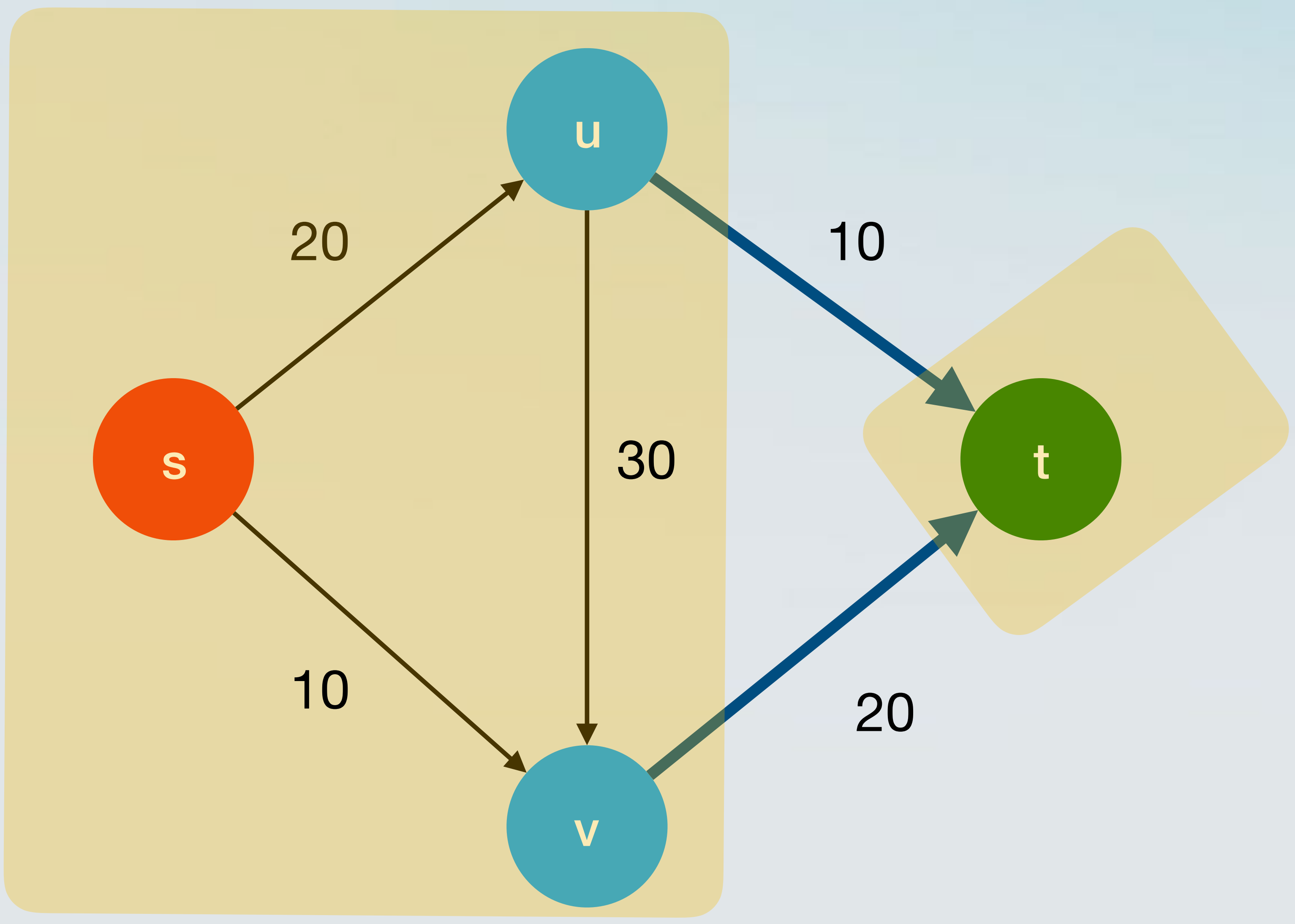

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  - Replace every undirected edge with two directed edges, one in the forward and one in the backward direction. Set the capacity of those edges to be *1*.
  - Pick two arbitrary nodes s, t in V, and find the minimum s-t cut (how?)

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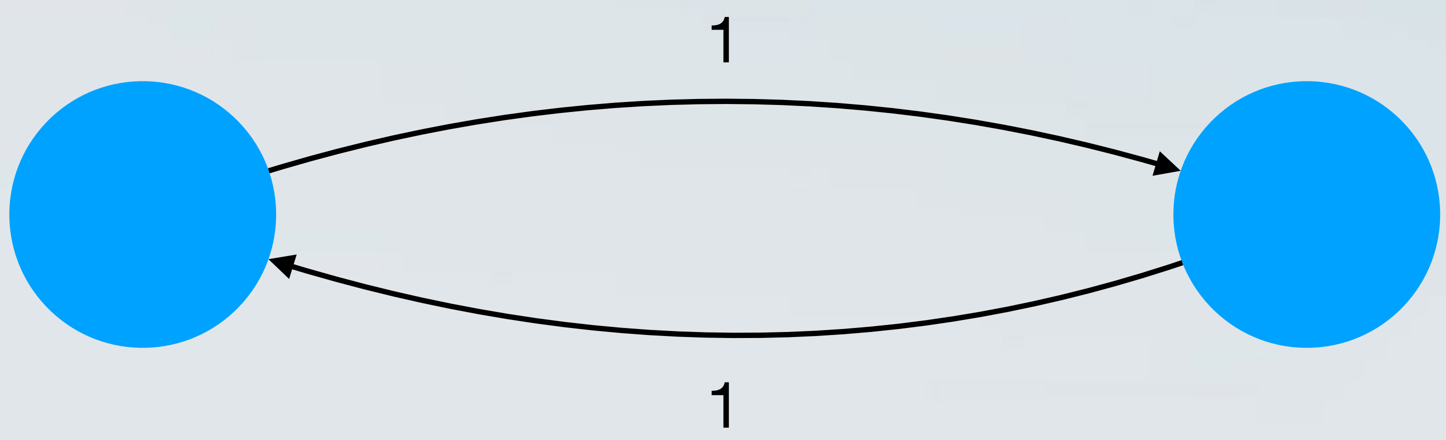

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- This is a polynomial-time algorithm, when the max-flow algorithm is polynomial-time.

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  - There can be multiple "parallel" edges between two nodes.
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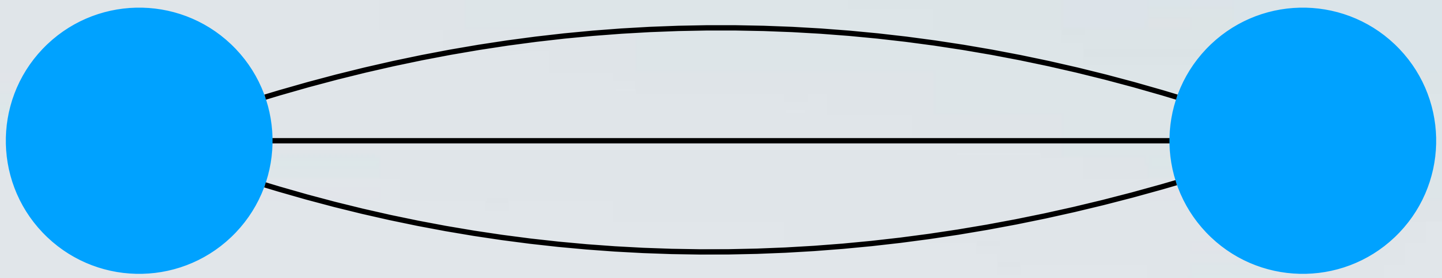

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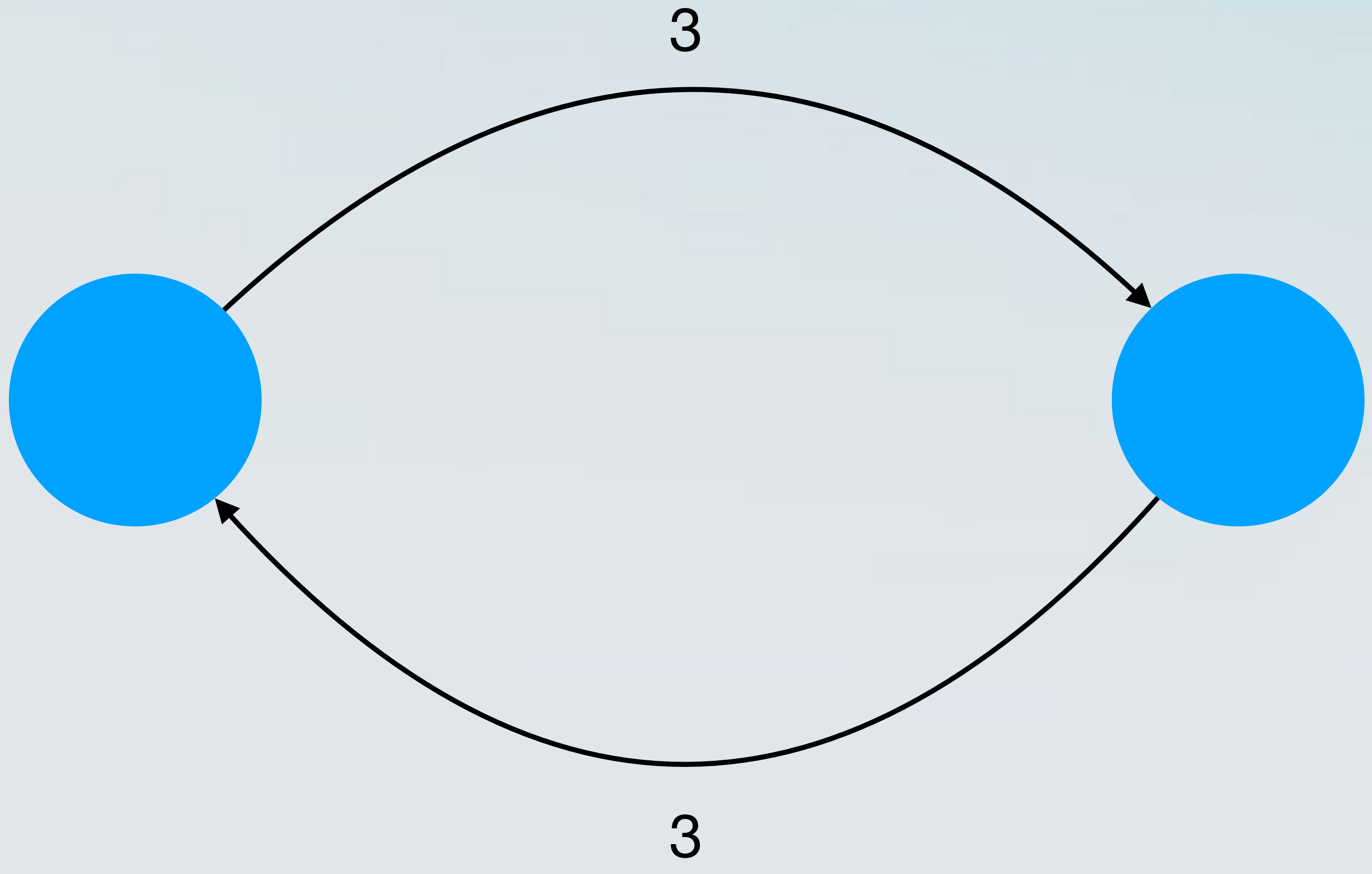

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  - When we are left with two supernodes w_1 and w_2, the corresponding sets of nodes are A and B.

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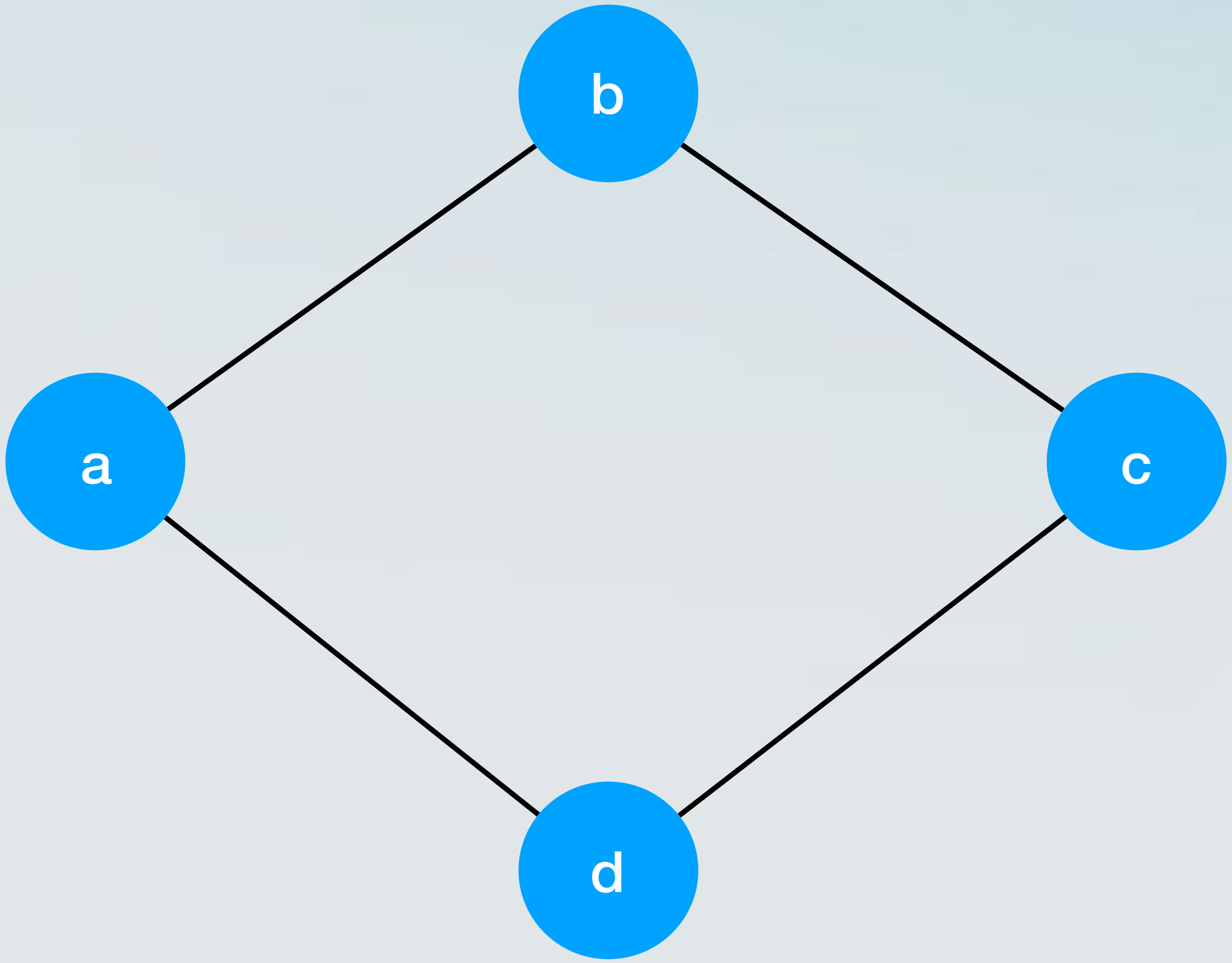

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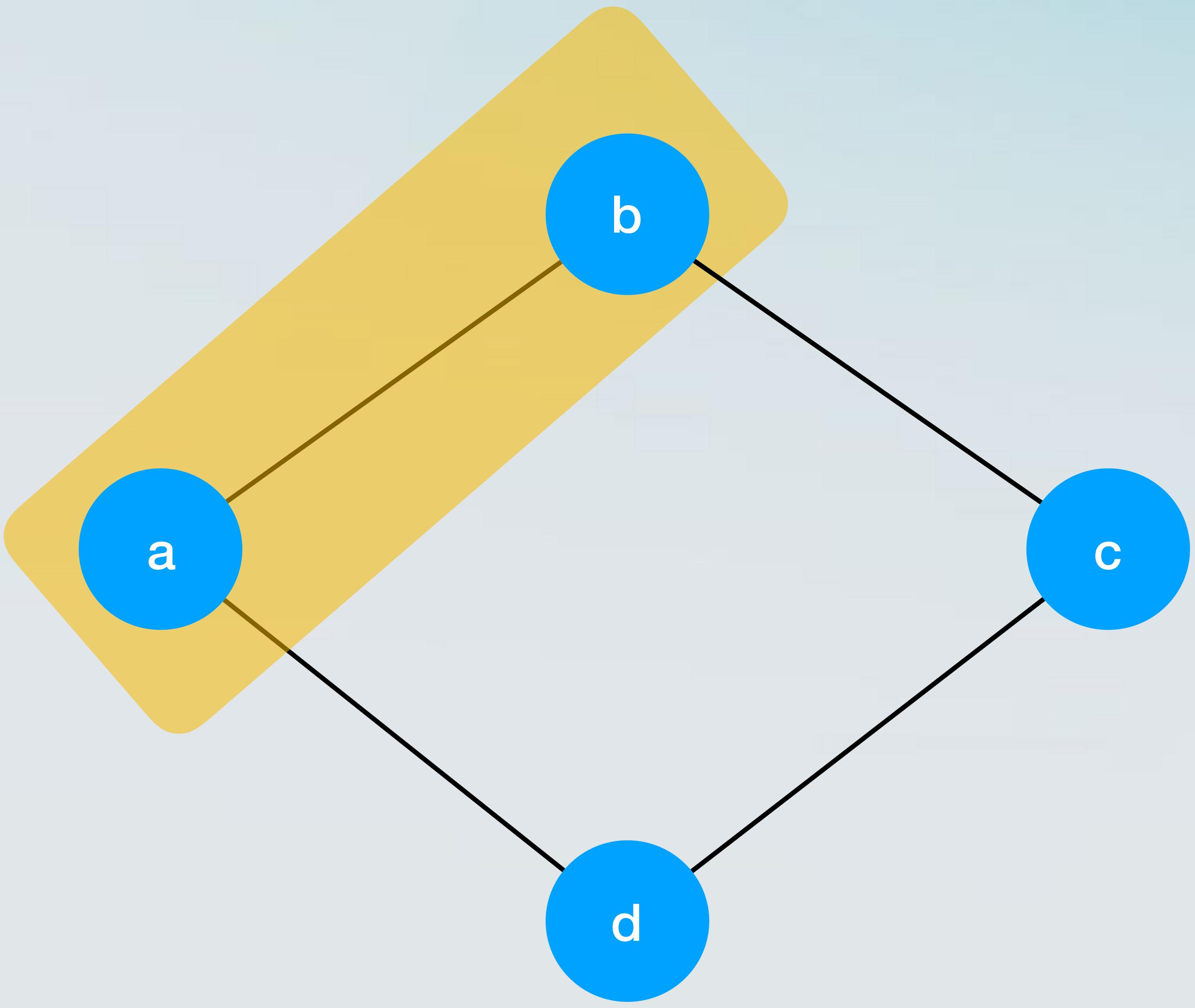

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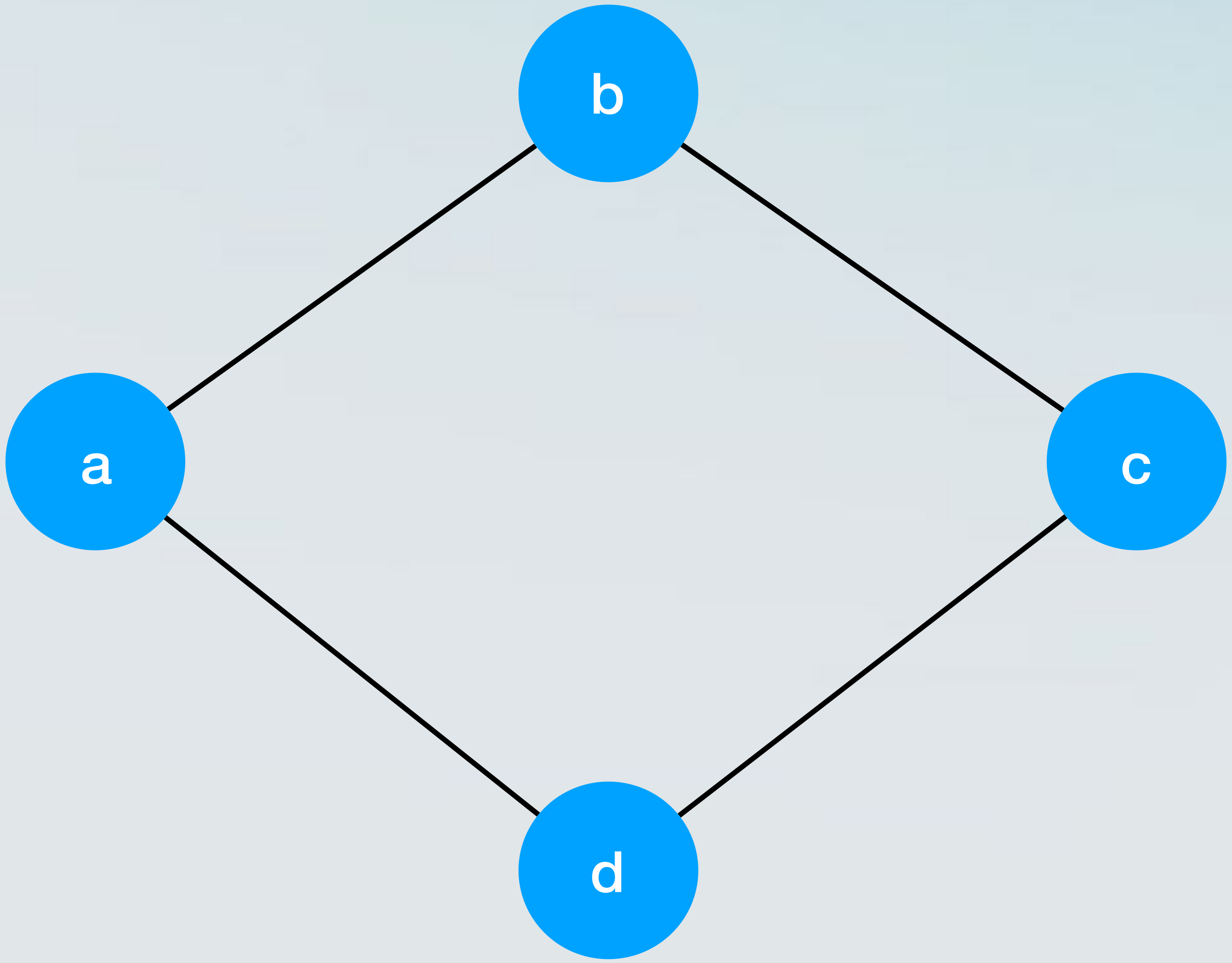

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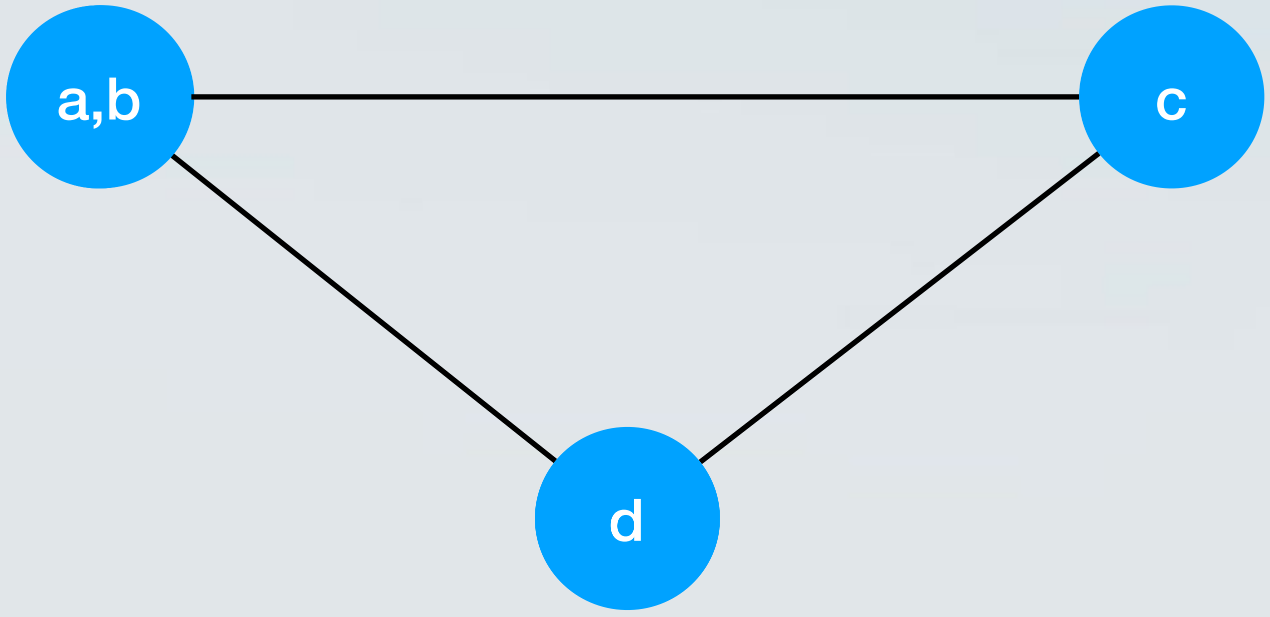

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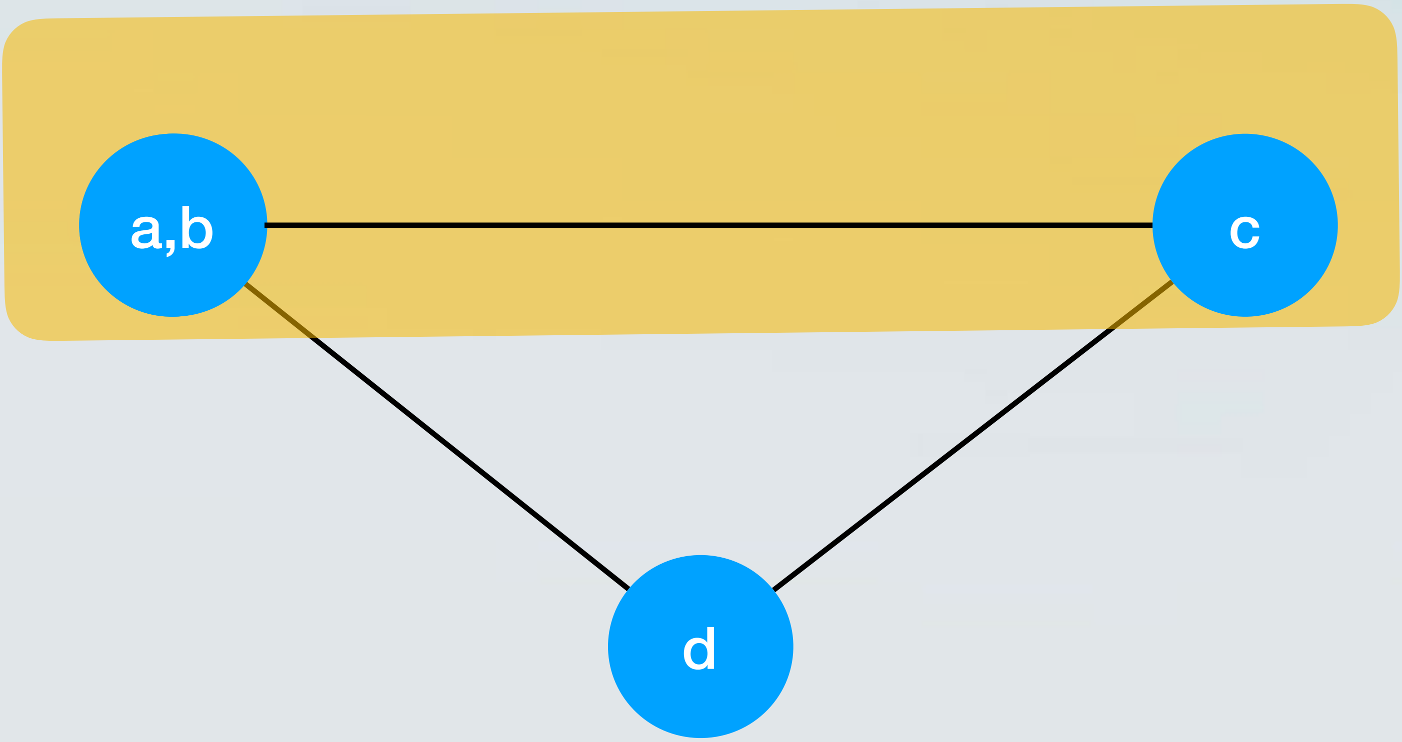

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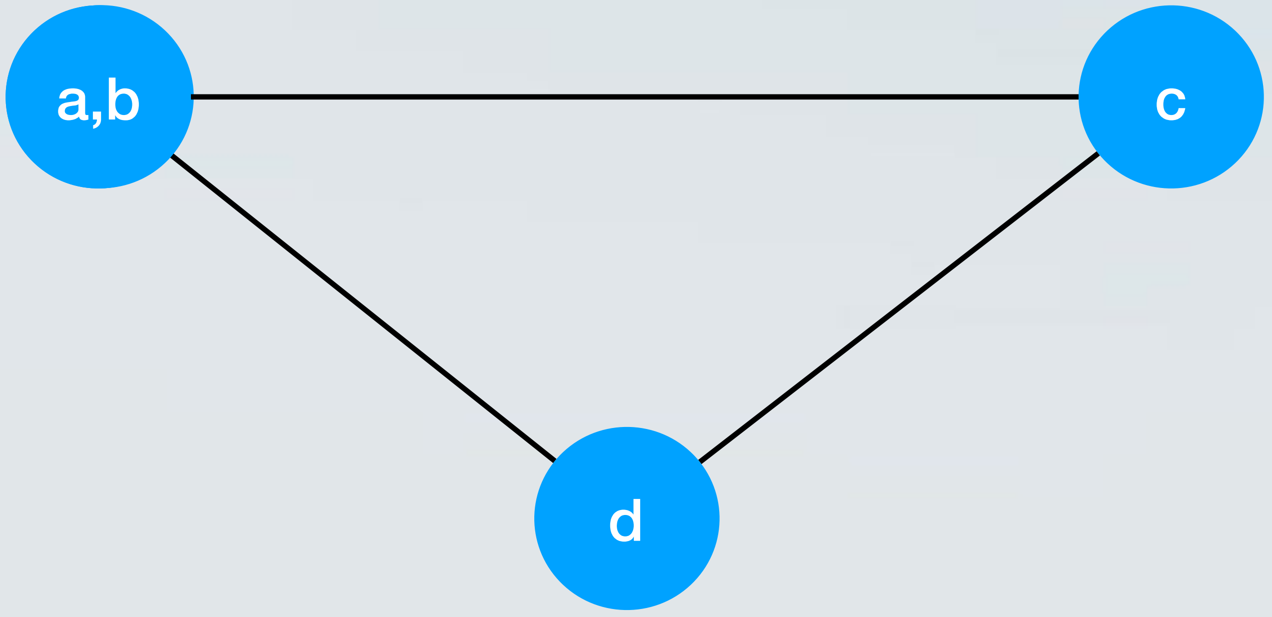

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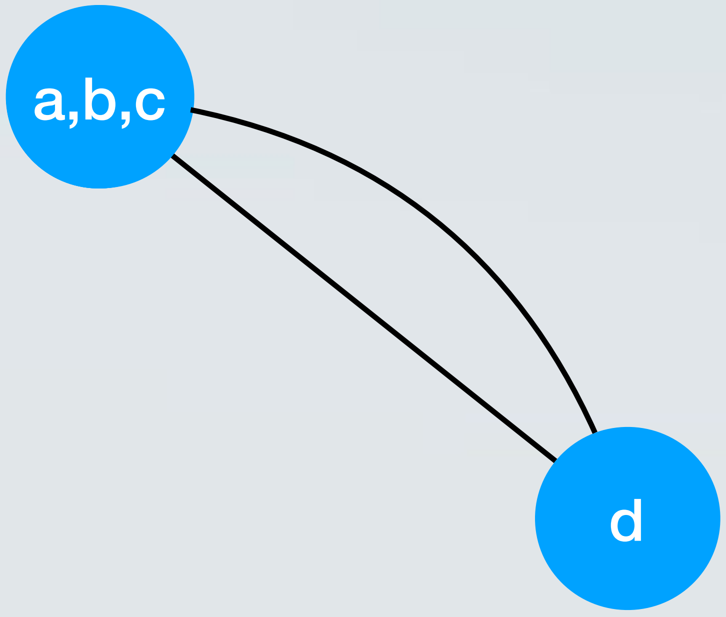

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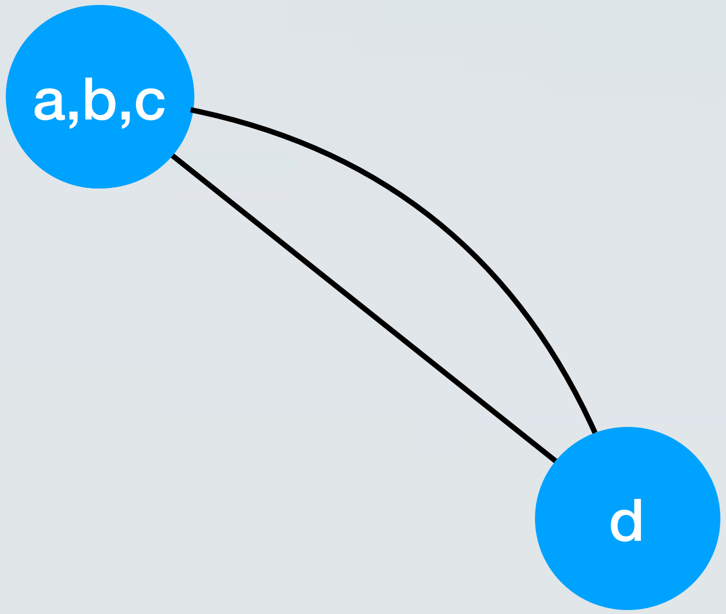

A = {a, b, c}
B = {d}

The Contraction Algorithm

Contraction(G)

For each node v, record
    the set S(v) of nodes that have been contracted into v.
  Initially, S(v) = {v} for each v. */* no contractions so far */*

If G has two nodes v_1 and v_2, then return the cut {S(v_1), S(v_1)}.

Else, choose an edge e = (u, v) of G *uniformly at random*.
   Let G' be the graph resulting from contracting e, with a
   new node z_{uv} replacing u and v.

   Define S(z_{uv}) = S(u) ∪ S(v)
   Contraction(G')

EndIf

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  - In other words, there is a set F of edges with one endpoint in A and one endpoint in B, such that |F|=k.
  - We will prove that the contraction algorithm outputs the cut (A, B) with *high probability*.

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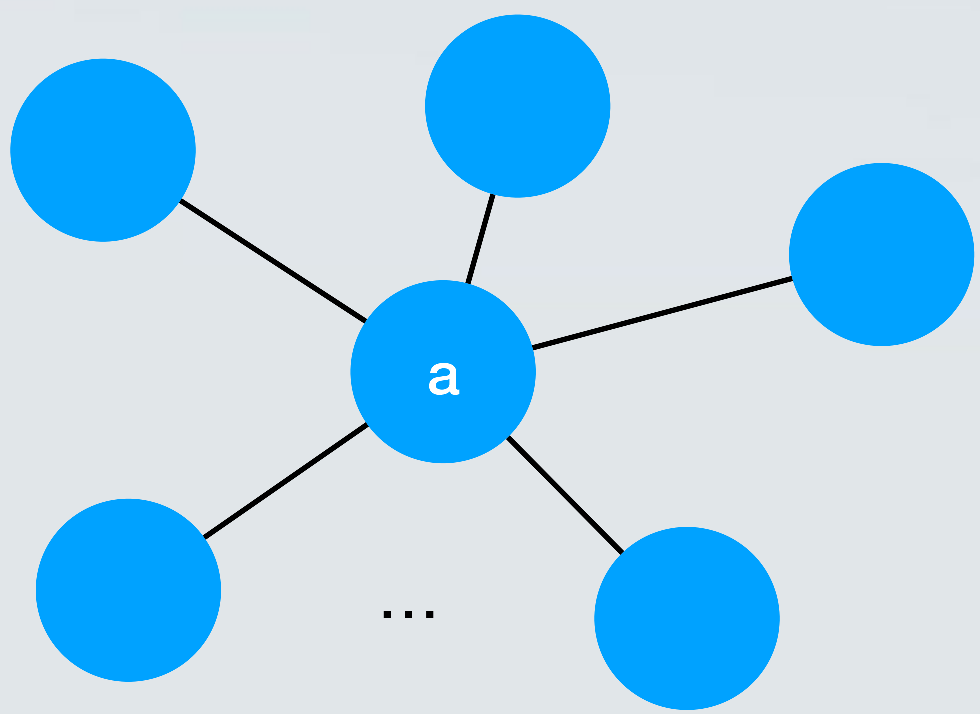

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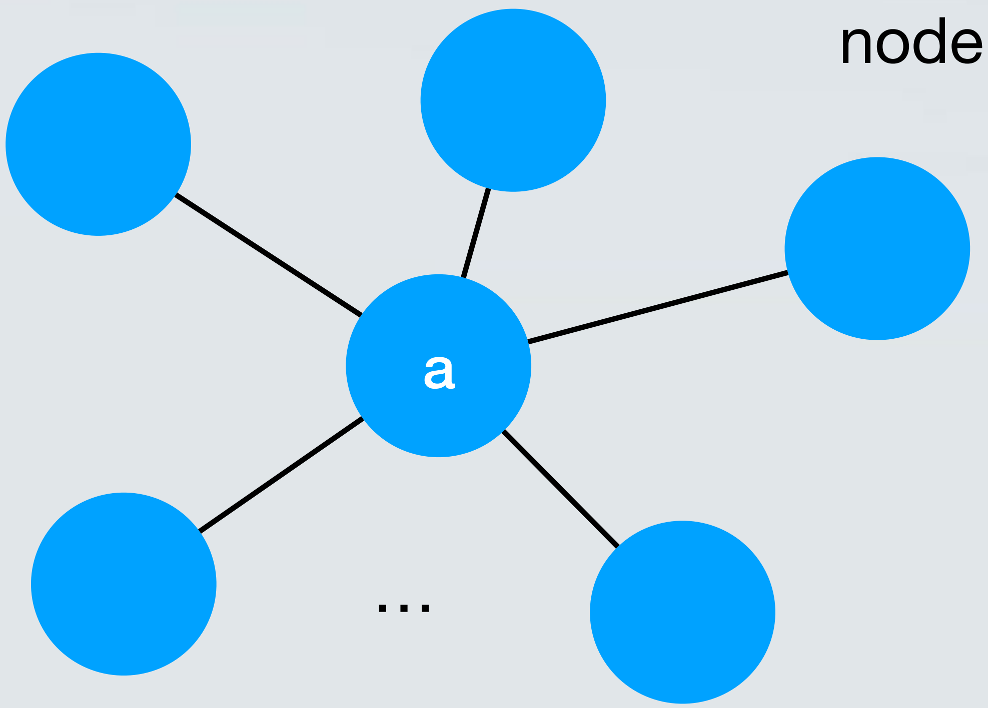

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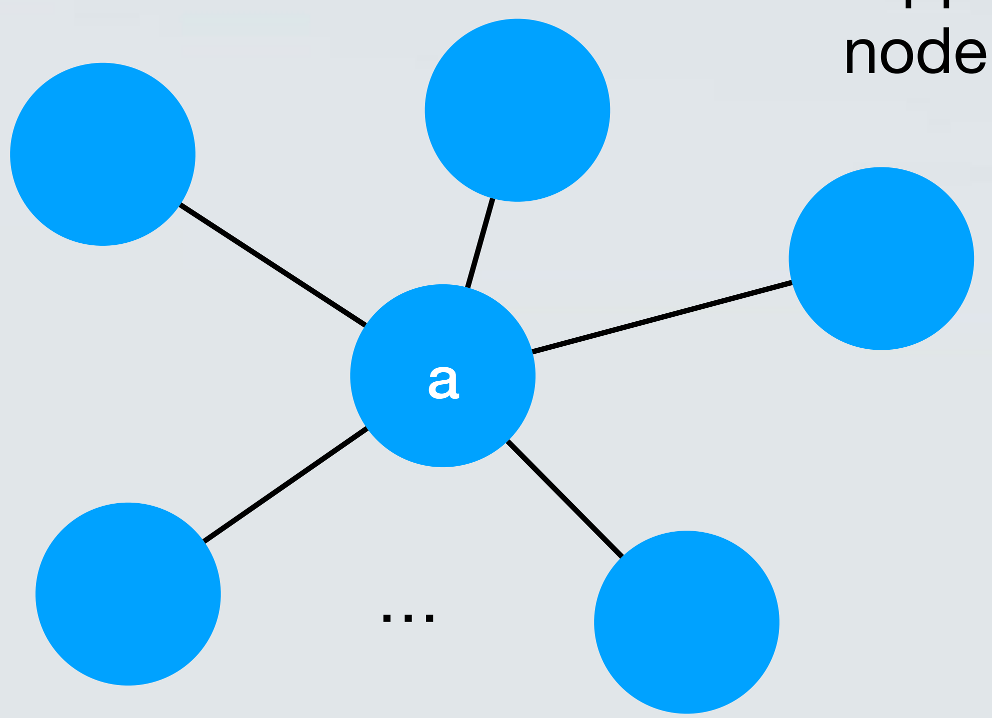

Is (A, B) a minimum cut?

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  - For an edge e in F, its endpoints lie in different "sides" of the cut.
  - If we contract e, then we can't possibly produce the cut (A, B).

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  - Claim: $|E| \geq (kn)/2$. (why?)
- The probability that an edge in F is contracted (*in the first round*) is at most *2/n*.

After round *j*

- Suppose that we have gone through *j* rounds and we have **not** contracted any edges in F yet.
- What is the probability that we contract an edge in F now?
- There are *n-j* super-nodes in the graph G'.
- A cut in G' is also a cut in G.
    - The degree of every super-node of G' is again at least k.
    - $|E_{G'}| \geq k(n-j)/2$.
    - The mistake probability is $k / |E_{G'}| = 2/(n-j)$.

Events

- Mistake: Contract an edge in F.
- Event E_j: The algorithm does not make a mistake in round j.
- We have shown:

$$\mathbf{Pr}[E_1] \geq 1 - \frac{2}{n}$$

$$\mathbf{Pr}[E_{j+1} \mid E_1 \cap E_2 \cap \ldots \cap E_j] \geq 1 - \frac{2}{n-j}$$

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$$= \frac{2}{n(n-1)} = \binom{n}{2}^{-1}$$

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Application

- Suppose that we repeat an experiment multiple times, and each time the probability of success is $p > 0$.
    - e.g., compute a minimum cut in a graph.

Success Amplification

- Run the algorithm independently X times.
- The probability that it fails is equal to

  the probability that it fails the 1st time x
  the probability that it fails the 2nd time x
  ...
  the probability that it fails the Xth time.

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Generally

- We can run the algorithm independently a number of times.
- This will decrease the error probability.
- This will increase the running time.
- There is a trade-off between the two.

Is there a simpler solution?

- We will use a randomised algorithm to solve the problem.
- The algorithm will be faster and simpler.
- It will produce the correct outcome *with high probability*.
  - Sometimes it might make a mistake!
  - But not too often.

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- To get high success probability, we need a lot of repetitions, so does not seem faster.
  - One can do clever optimisations to the way in which multiple runs are performed to improve the running time considerably.