Advanced Algorithmic Techniques (COMP523)

Randomised Algorithms 3

Recap and plan

- **Previous lecture:**
  - Randomised global cuts in multi-graphs.
- **This lecture:**
  - Types of randomised algorithms
  - Randomised approximation algorithms.
    - Applications: MAX-SAT, MAX-3SAT, MAX-CUT

Types of algorithms

- There are two (main) types of randomised algorithms:
  - Monte Carlo algorithms: The algorithm *computes the correct solution with high probability*, and the algorithm *always terminates*.
  - Las Vegas algorithms: The algorithm *always computes the correct solution*, and *its running time is a random variable with bounded expectation.*

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    - Las Vegas algorithms: The algorithm *always computes the correct solution*, and *its running time is a random variable with bounded expectation.*
        - *i.e., it might fail to terminate with some small probability.*

Examples

- Example of Monte Carlo algorithm:
  - The global minimum cut algorithm on multi-graphs.
- Examples of Las Vegas algorithms:
  - Randomised Partition (runs in expected time O(n))
  - Randomised Quicksort (runs in expected time O($n \log n$))
    - These algorithms pick the pivot element *uniformly at random*.

Recall: The Partition procedure

Procedure Partition(**A**[$i,\ldots,j$])

Choose a **pivot element** **x** of **A**

$k = i-1$

For $h = i$ to $j-1$ do

If **A**[h] $\leq$ **x**

$k = k + 1$

Swap **A**[k] with **A**[h]

Swap **A**[$k+1$] with **A**[j]

Return $k+1$

Partition

2	8	7	1	3	5	4

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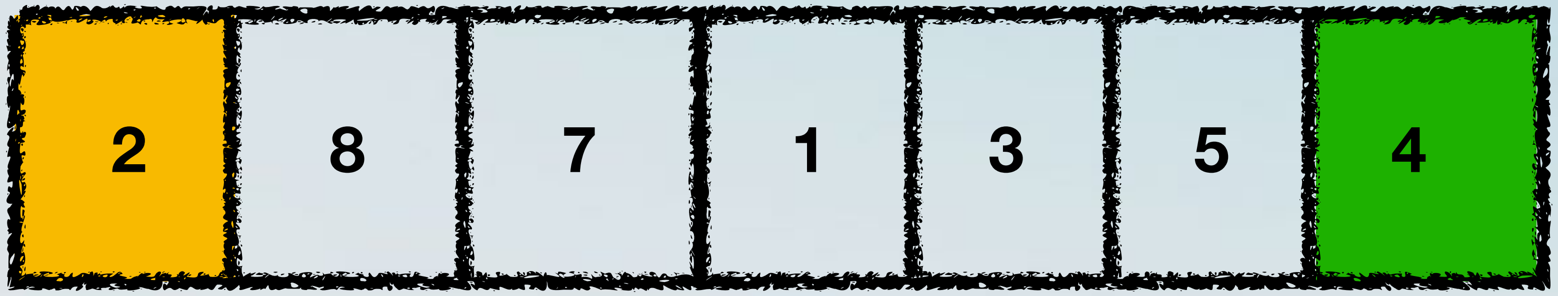

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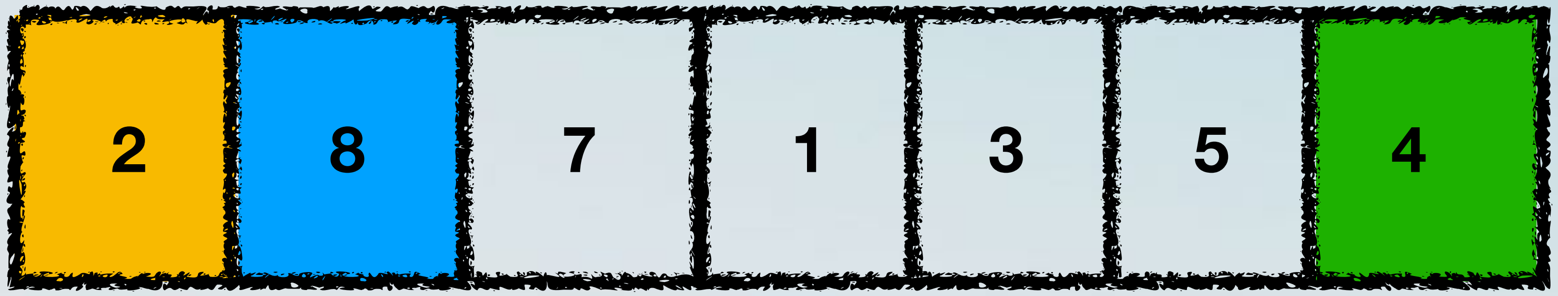

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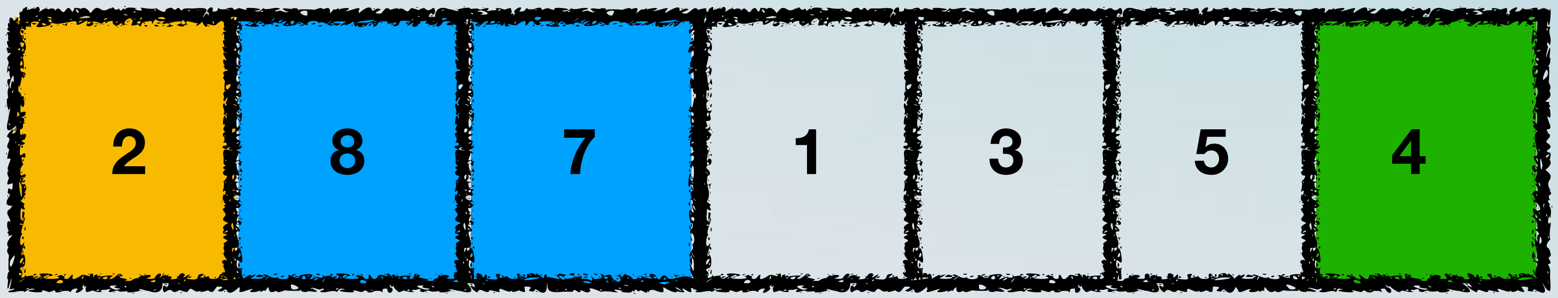

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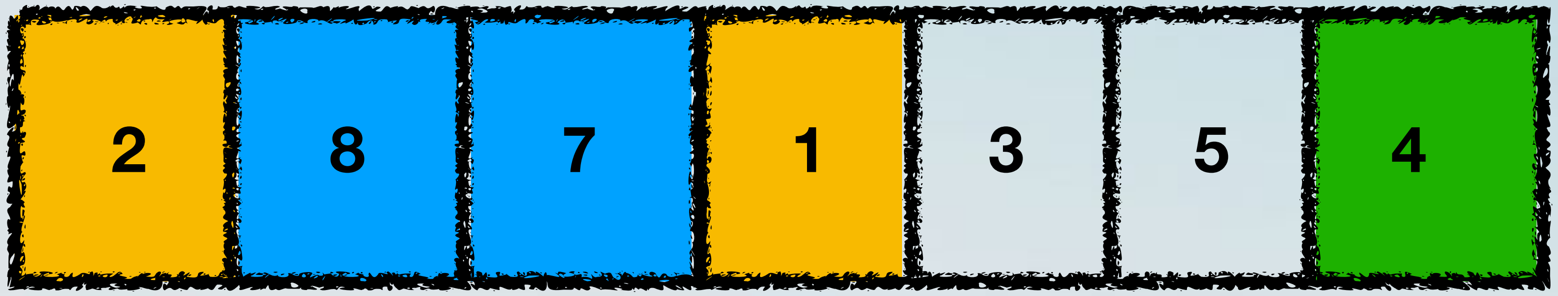

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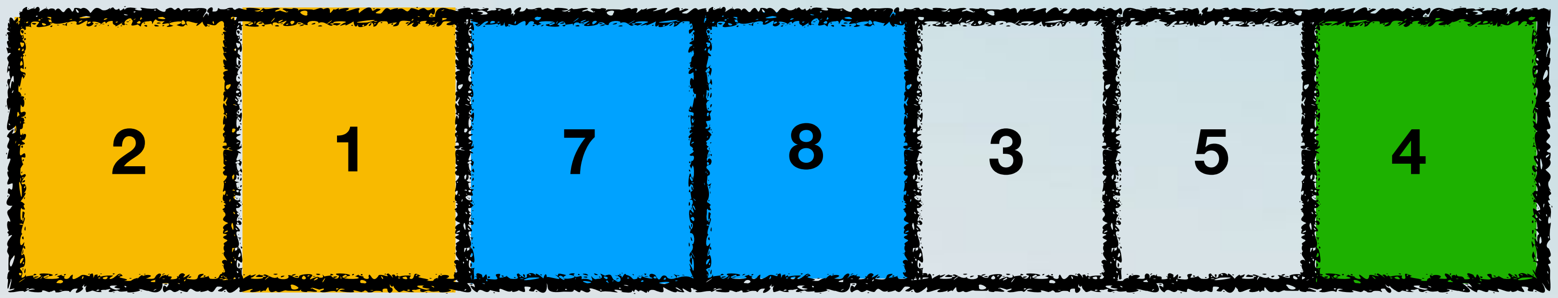

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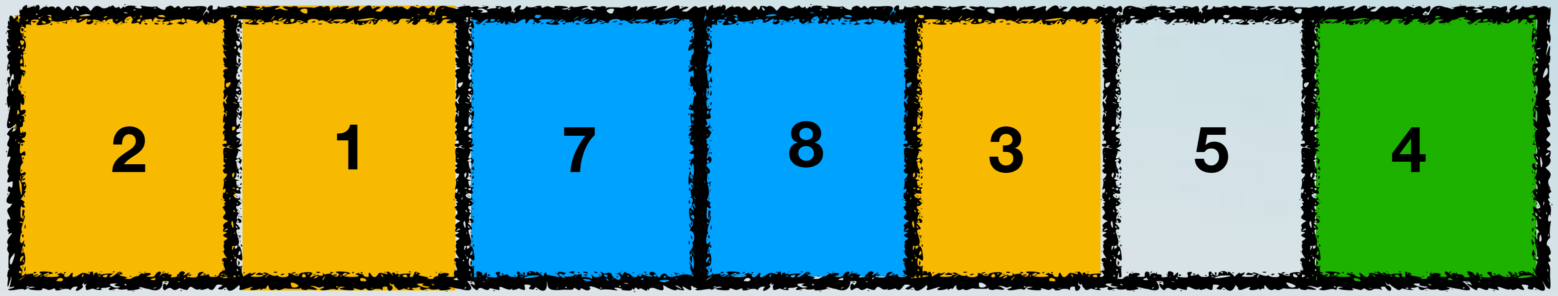

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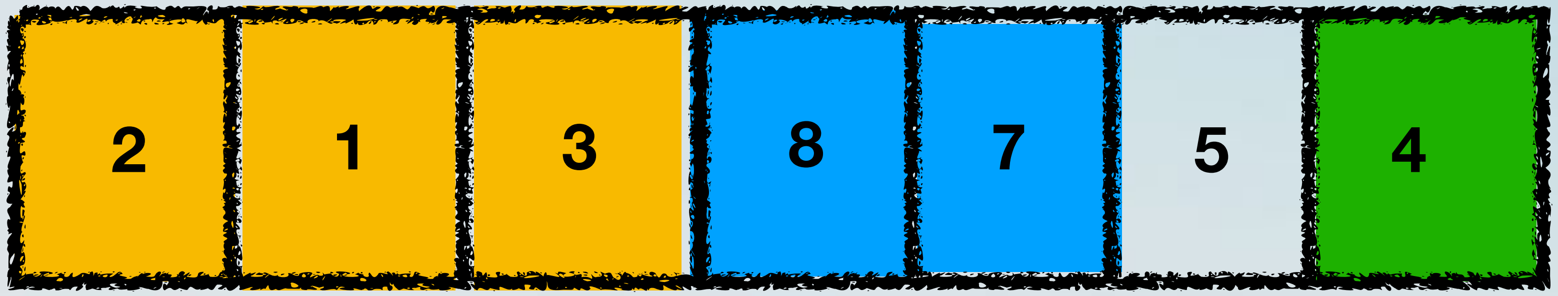

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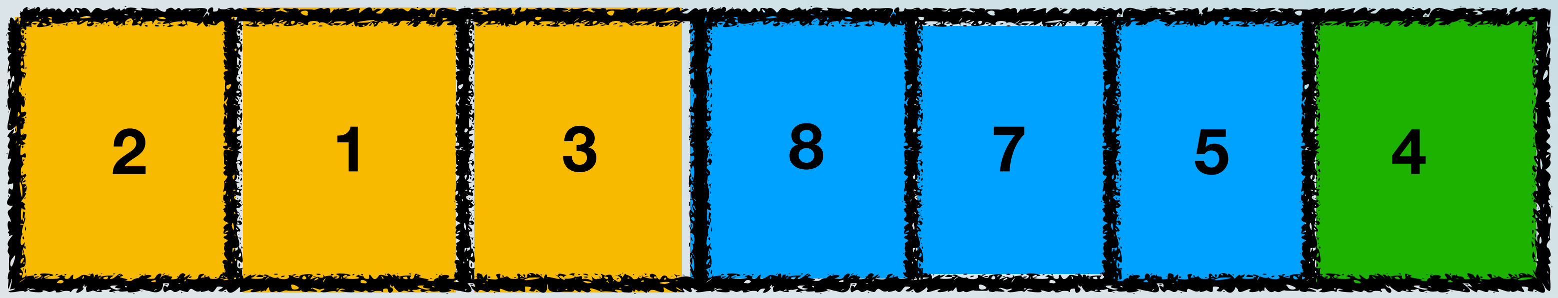

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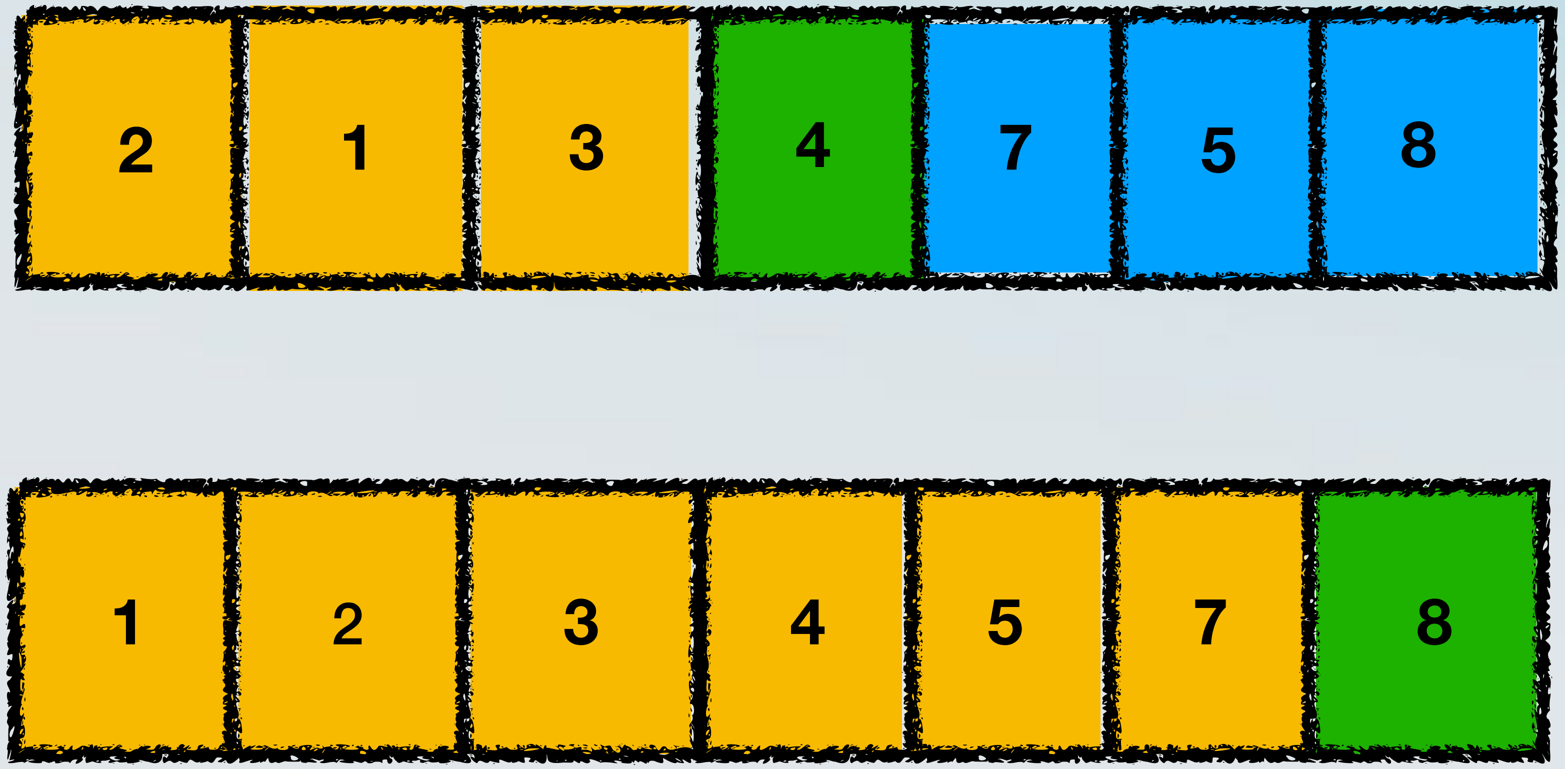

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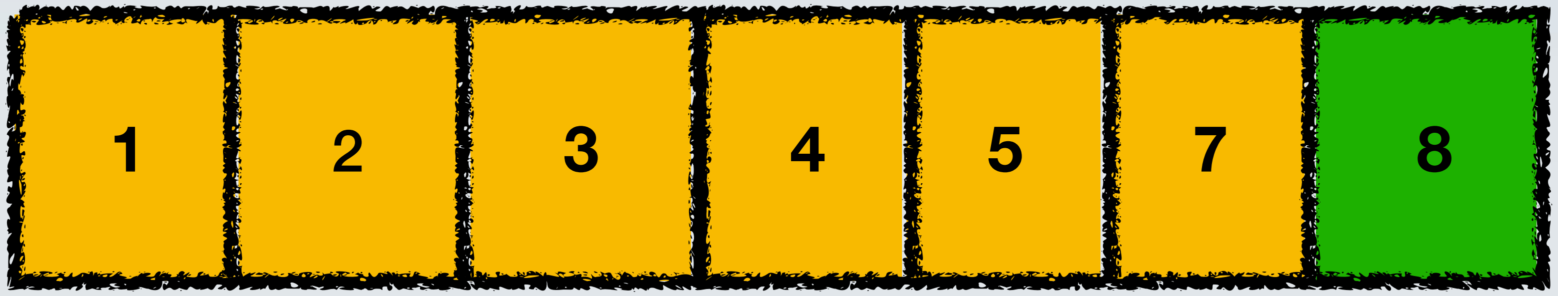

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Some give good partitions.
Some give bad partitions.
The running time is a random variable.
Its expectation can be calculated
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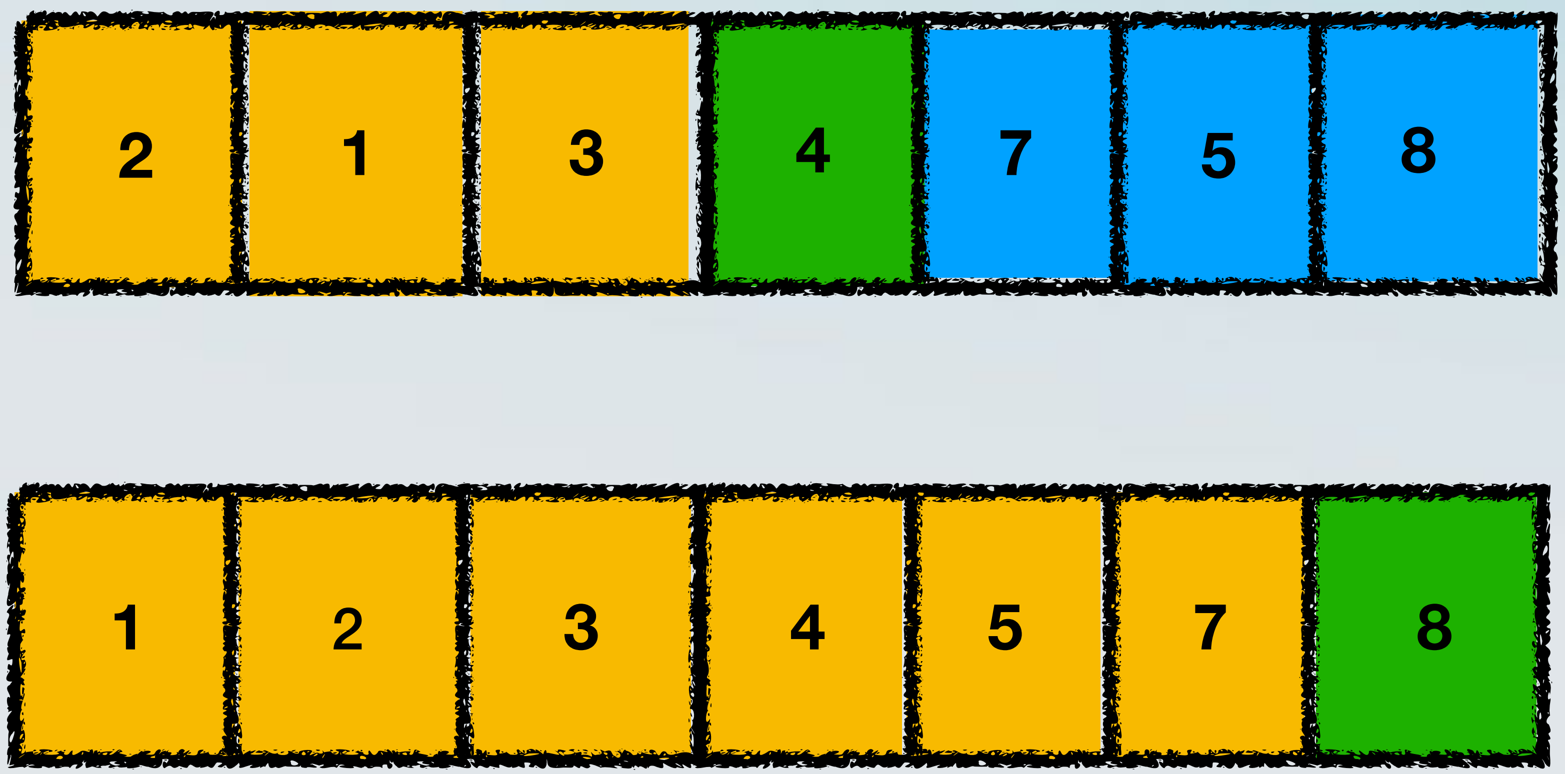

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More details: CLRS

Randomised Approximation algorithms

- We will use randomisation to design good approximation algorithms.
- These algorithms will always terminate.
- Their approximation ratio will be calculated with respect to their *expected outcome*.

Approximation ratio

- For maximisation problems, we define

$$\max_x \text{opt}(x) \,/\, \text{obj}(A(x))$$

  - i.e., the worst case ratio of the optimal value of the objective over the value of the objective achieved by the algorithm, over all possible inputs to the problem.
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3 SAT

- A CNF formula with m clauses and k literals.

  $\phi = (x_1 \vee x_5 \vee x_3) \wedge (x_2 \vee x_6 \vee \neg x_5) \wedge \ldots \wedge (x_3 \vee x_8 \vee x_{12})$

- ("An AND of ORs").

- Each clause has three literals.

- Truth assignment: A value in {0,1} for each variable x_i.

- Satisfying assignment: A truth assignment which makes the formula evaluate to 1 (= true).

- Computational problem 3SAT : Decide if the input formula ϕ has a satisfying assignment.

MAX 3SAT

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A 2-approximation algorithm for MAX-SAT

- Algorithm: For each variable x_i, set x_i to *1* with probability 1/2 and to *0* with probability 1/2.

Analysis

- Let Y_j be a random variable such that:

  $Y_j = 1$, if clause j is satisfied.
  $Y_j = 0$, otherwise.

- Let X be a random variable, which is equal to the number of satisfied clauses.

  - By definition: $X = \sum_{j=1}^{m} Y_j$

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- We have that:

$$E[X] = E\left[\sum_{j=1}^{m} Y_j\right] = \sum_{j=1}^{m} E[Y_j] = \sum_{j=1}^{m} \textbf{Pr}[\textbf{clause } j \textbf{ is satisfied}]$$

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    - f(j) is the number of literals in clause *j*.

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- If we use the trivial upper bound of m on the value of the optimal, we get the 2-approximation.

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Can we use the same idea to get
an approximation algorithm for MAX 3SAT?

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Global Minimum Cut

- We are given an *undirected* graph G=(V, E).
- A *cut* of G is a partition of the nodes of the graph into two sets, A and B.
- The size of a cut (A, B) is *the number of edges* with one endpoint in A and one endpoint in B.
- A *global minimum cut* is a cut of minimum size.

Maximum Cut

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  - *Almost always correctly*, by the contraction algorithm.
- Maximum Cut is NP-hard.
  - We will design an approximation algorithm for MAX-CUT.

MAX-CUT algorithm

- For every vertex v in V independently,
  place v in A with probability *1/2*,
  place v in B with probability *1/2*.

MAX-CUT algorithm

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- 7-10 minute exercise: Prove that the approximation ratio of this algorithm for the maximum cut problem is 2.

Analysis

- Let X_{ij} be a random variable such that:

  $X_{ij} = 1$, if edge (i, j) crosses the cut.
  $X_{ij} = 0$, otherwise.

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    - This happens with probability *1/2.*

$$E[Z] = \sum_{(i,j)\in E} \textbf{Pr}[\textbf{edge } (i,j) \textbf{ crosses the cut}] = \frac{m}{2}$$

Comparing solutions

- Assume that you have a deterministic algorithm that has a 2-approximation for a problem, and a randomised algorithm that has a 2-approximation for the problem.

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  - What if you only cared about the approximation ratio?

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- The deterministic algorithm works well *always*.

Comparing solutions

- The randomised algorithm works well *in expectation*.
- The deterministic algorithm works well *always*.
- What if things go horribly wrong?

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- Different methods for derandomisation.
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    - Next lecture!