Advanced Algorithmic Techniques (COMP523)

Online Algorithms 2

Recap and plan

- **Last lecture:**
  - Online Algorithms
  - Competitive Ratio
  - Online load balancing
  - Paging
- **This lecture:**
  - Online algorithms for paging.
  - A Randomised online algorithm for paging.

Paging Algorithms

- LRU (*Least Recently Used*): Replace the page that was requested the least recently.
- FIFO (*First-In First-Out*): Replace the page that has been in the cache the longest.
- LIFO (*Last-In First-Out*): Replace the page that has been in the cache the shortest.
- LFU (*Least Frequently Used*): Replace the page that was requested the least frequently so far.
- MIN (*Offline Optimal*): Replace the page whose next request happens the furthest in the future.

Paging Algorithms

- Theorem: LRU and FIFO have competitive ratio k.

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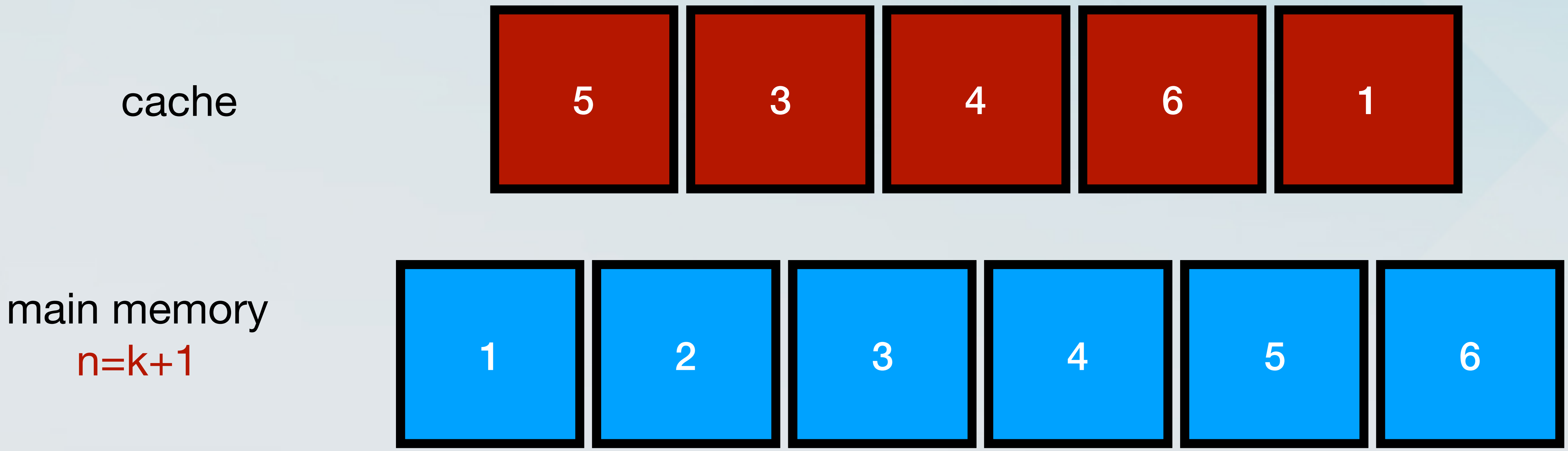
  - When all pages in the cache are marked, and a request for an unmarked page occurs, the phase ends.

Marking algorithm



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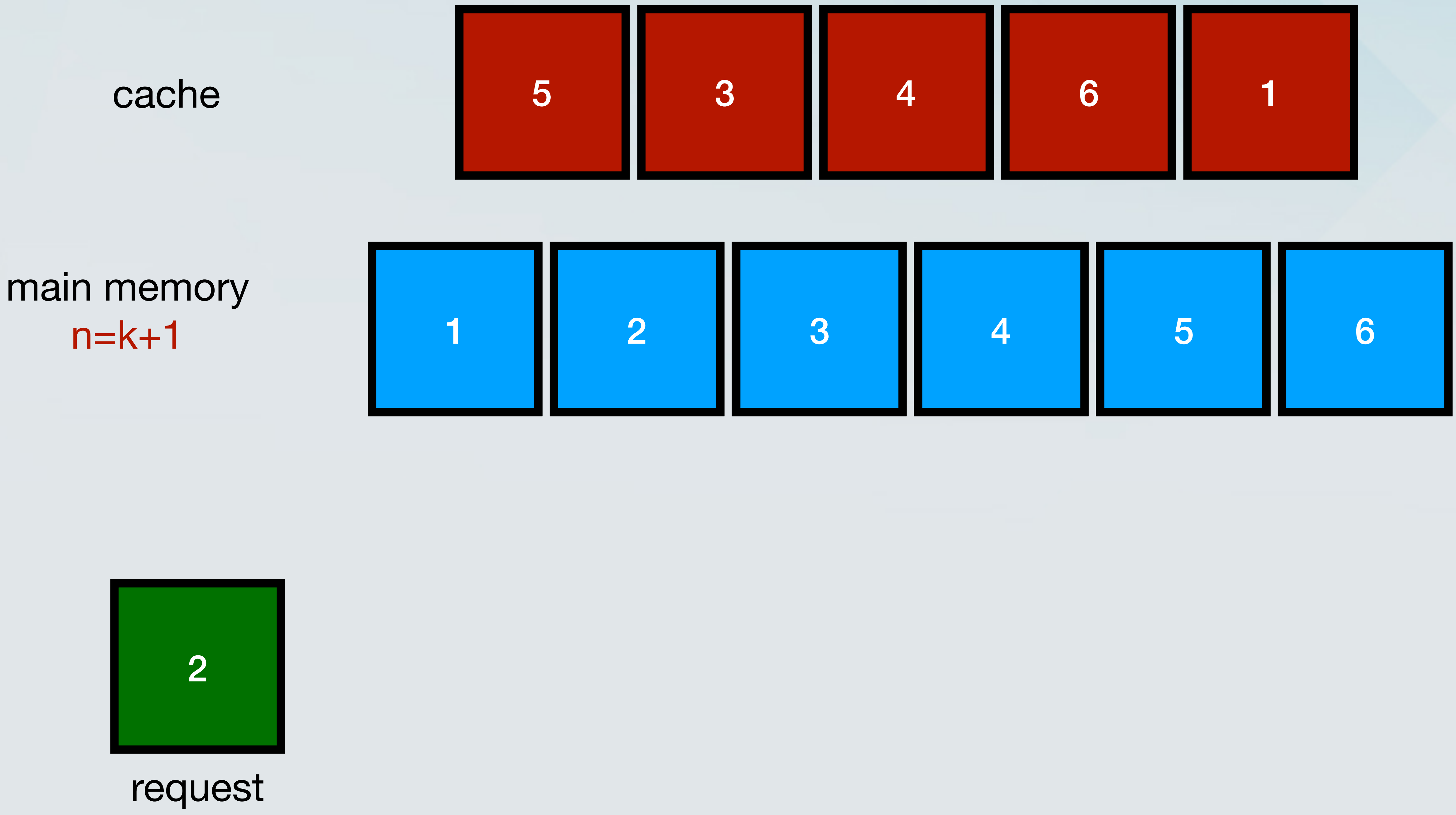

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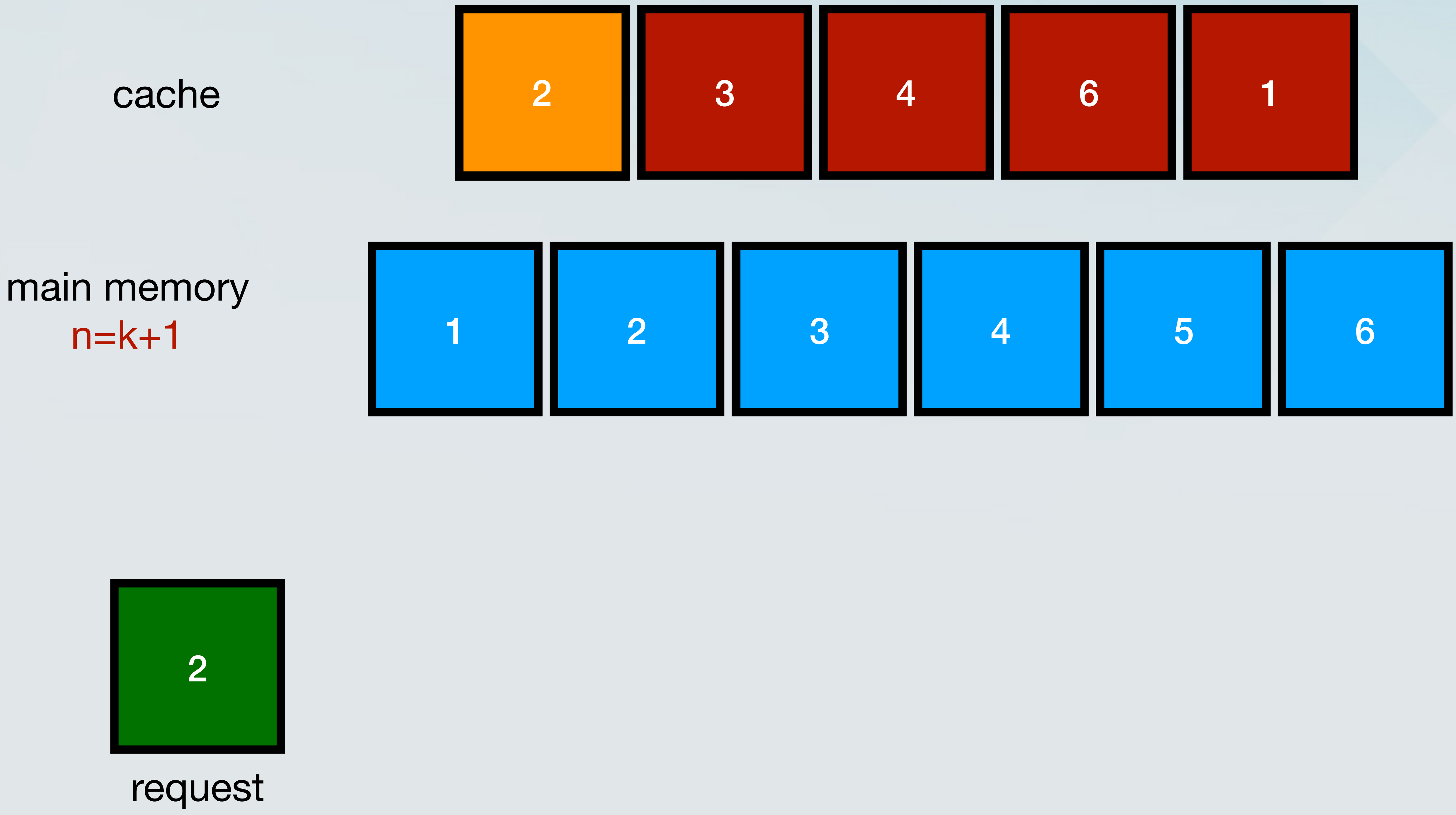

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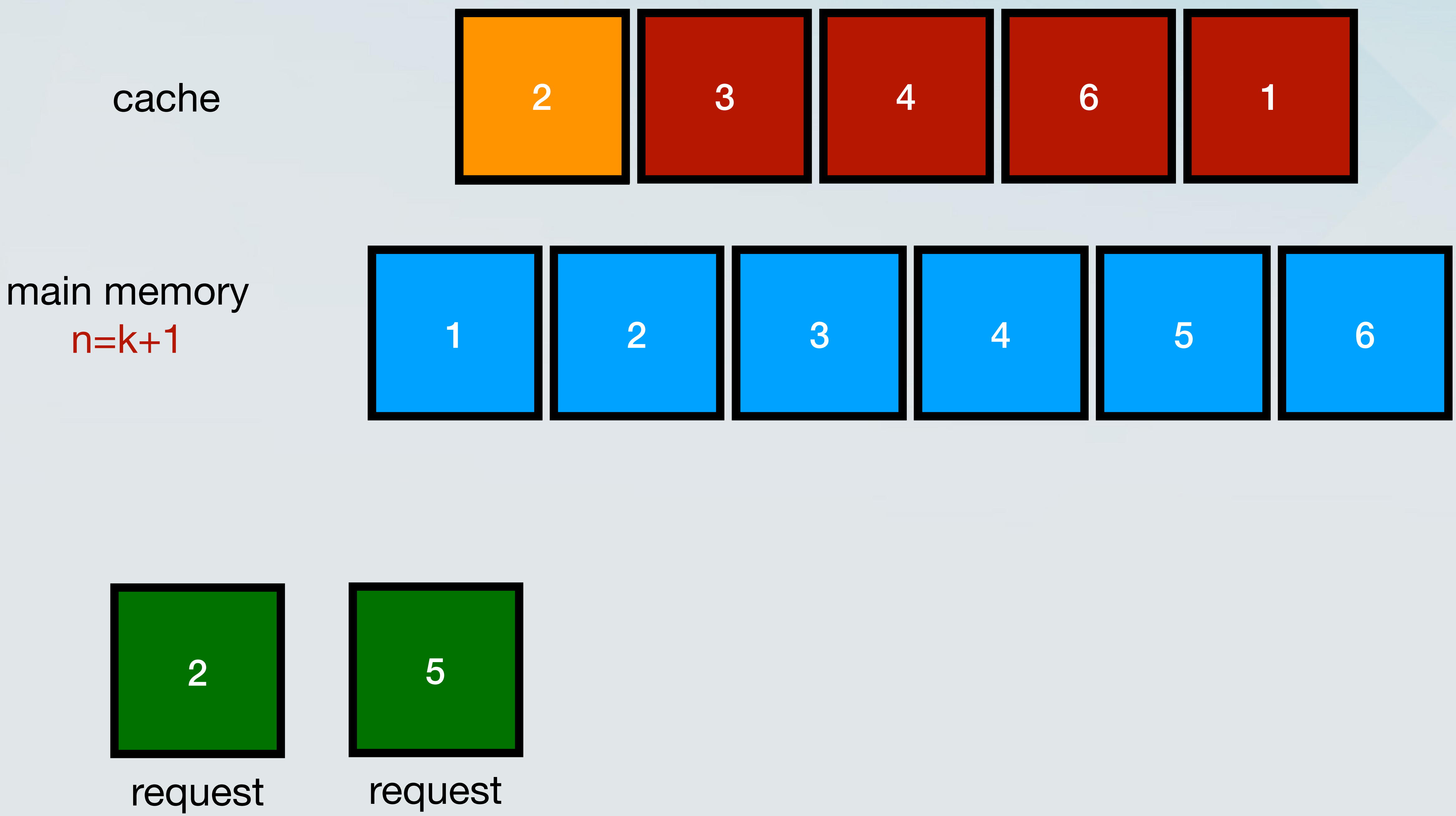

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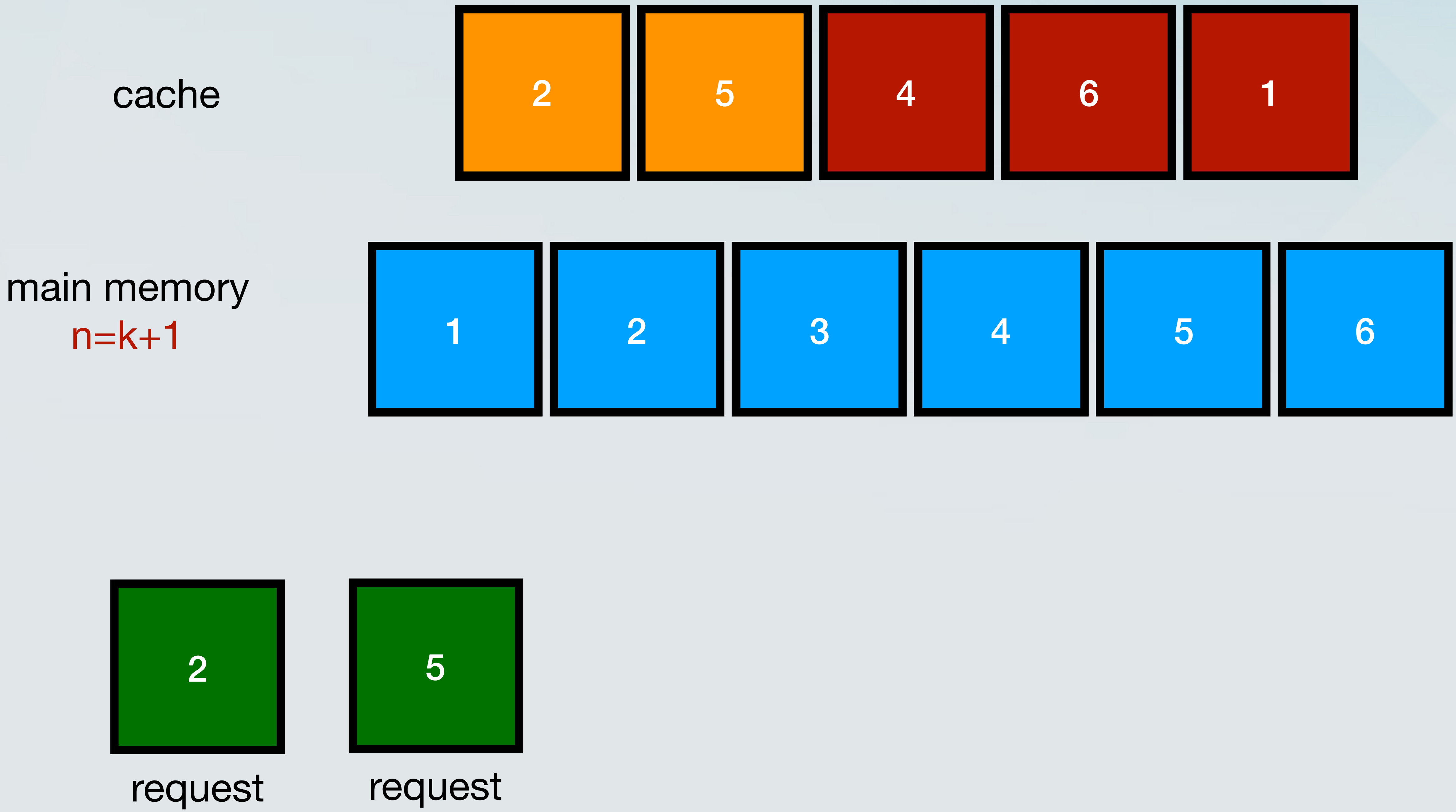

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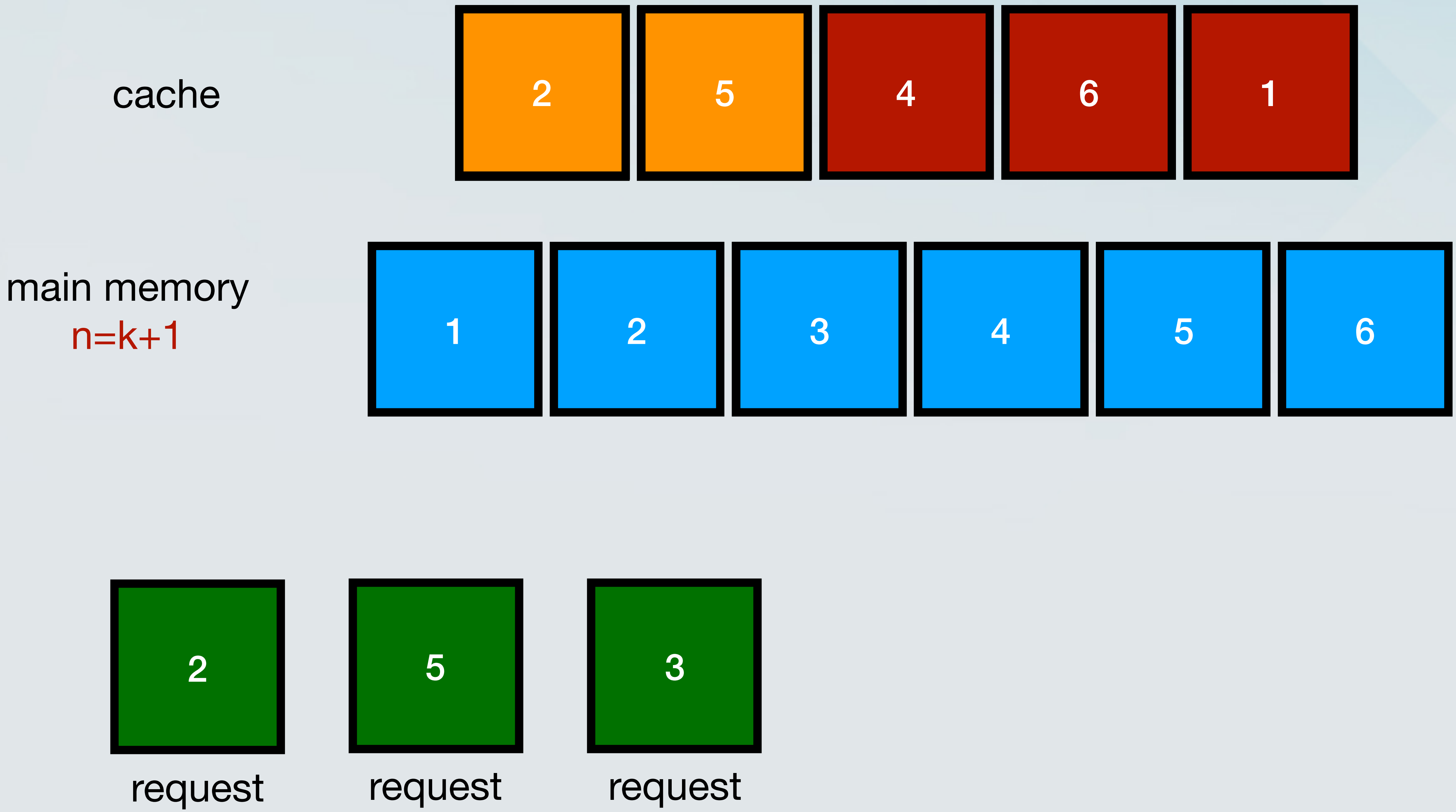

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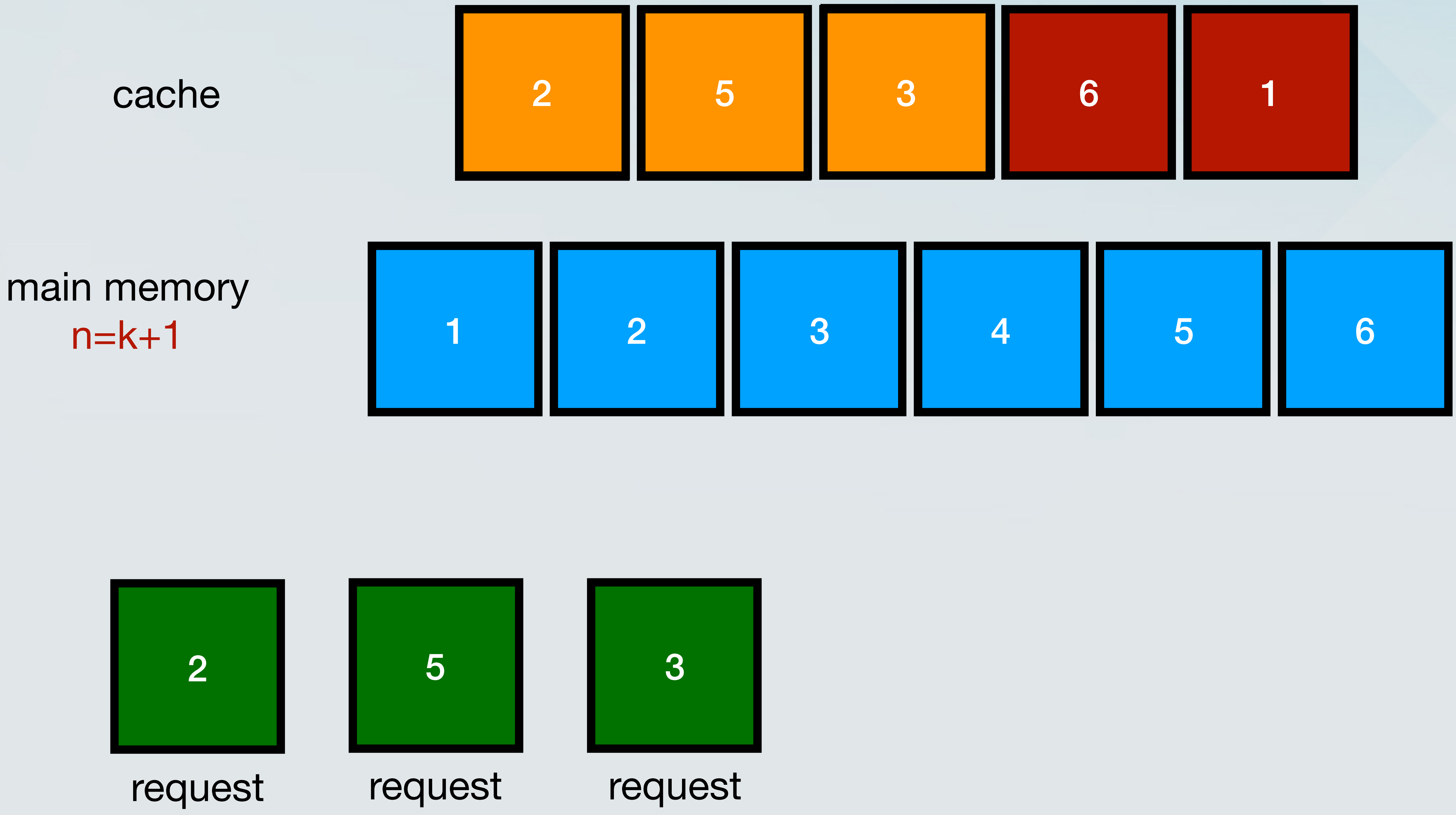

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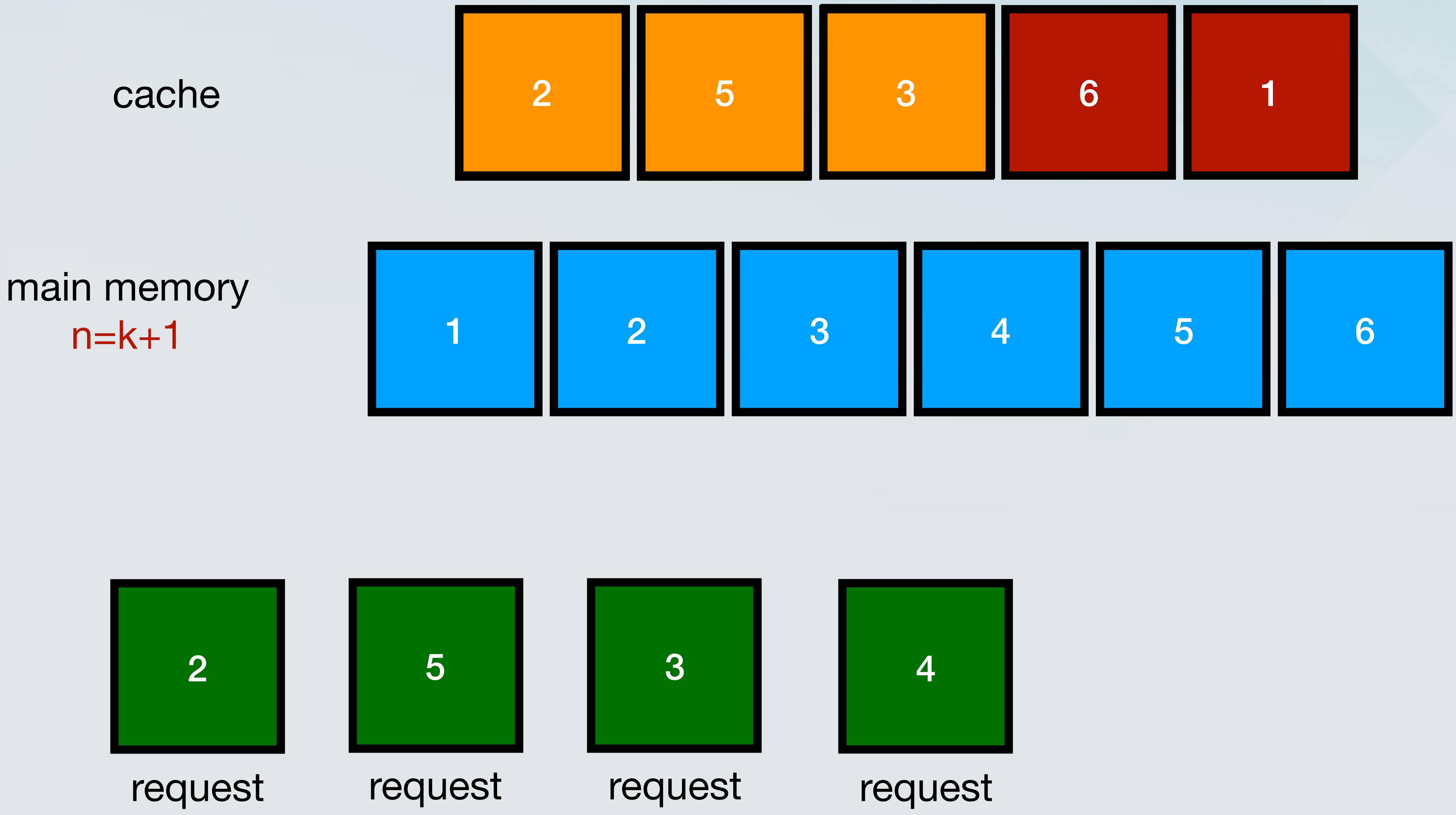

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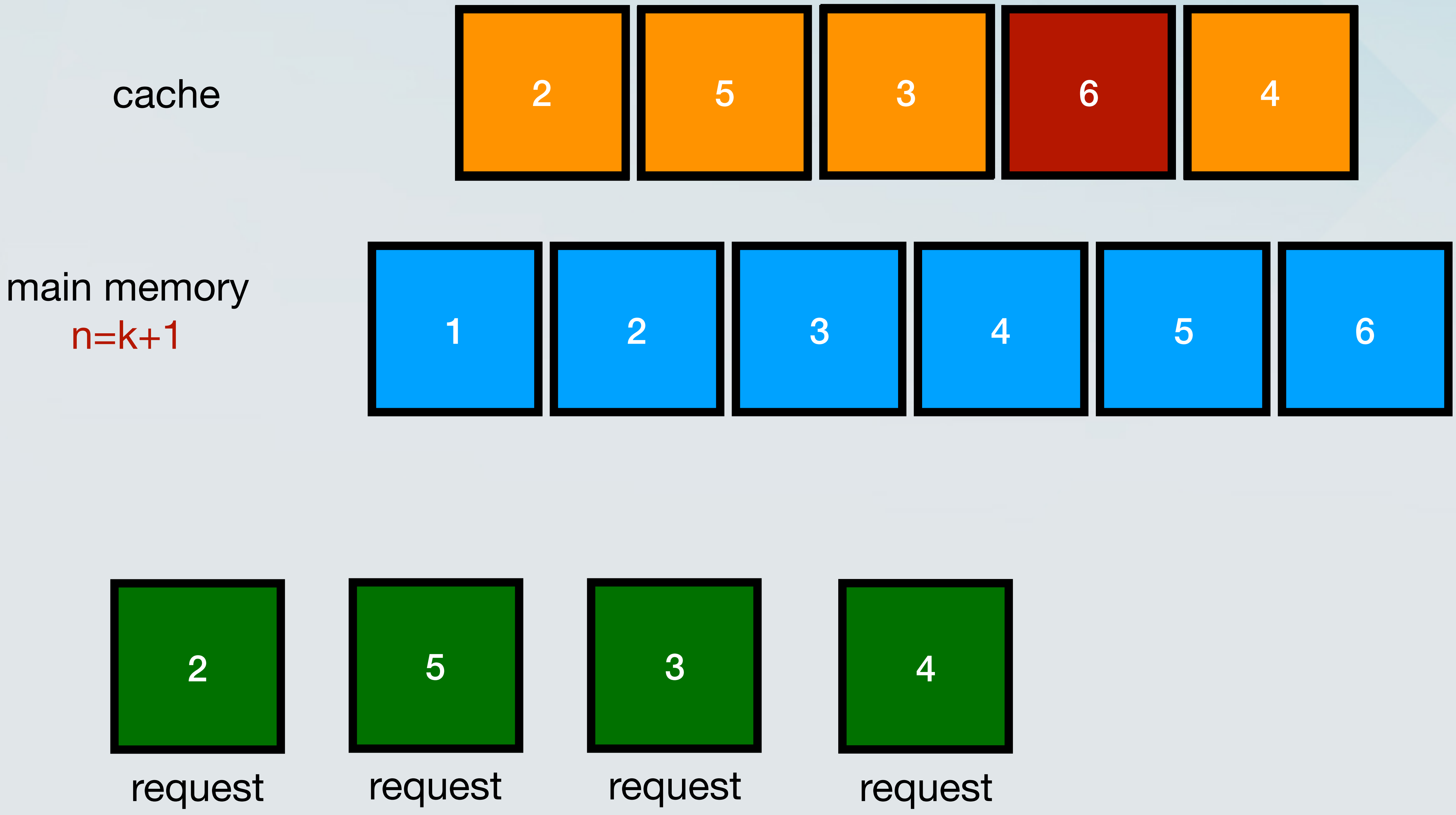

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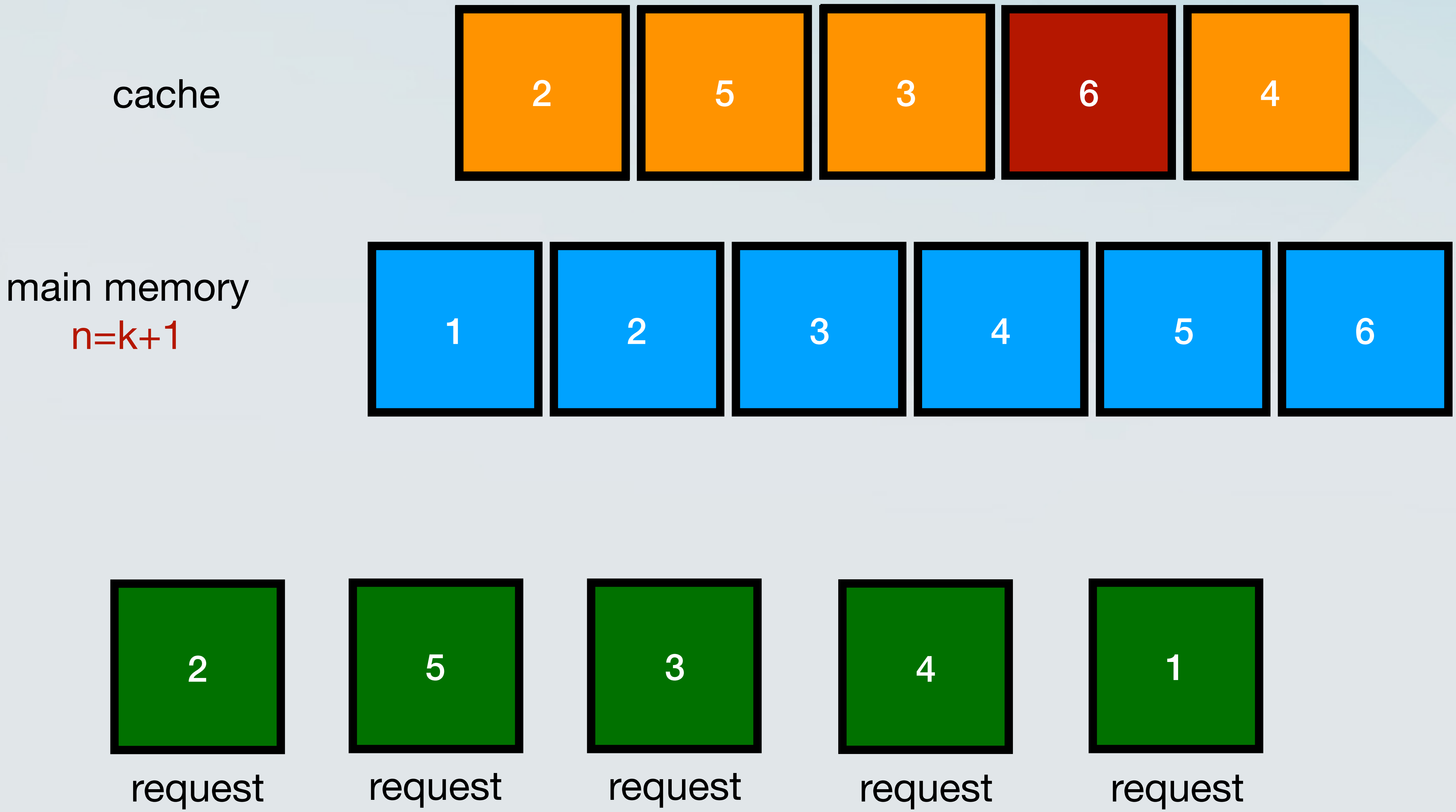

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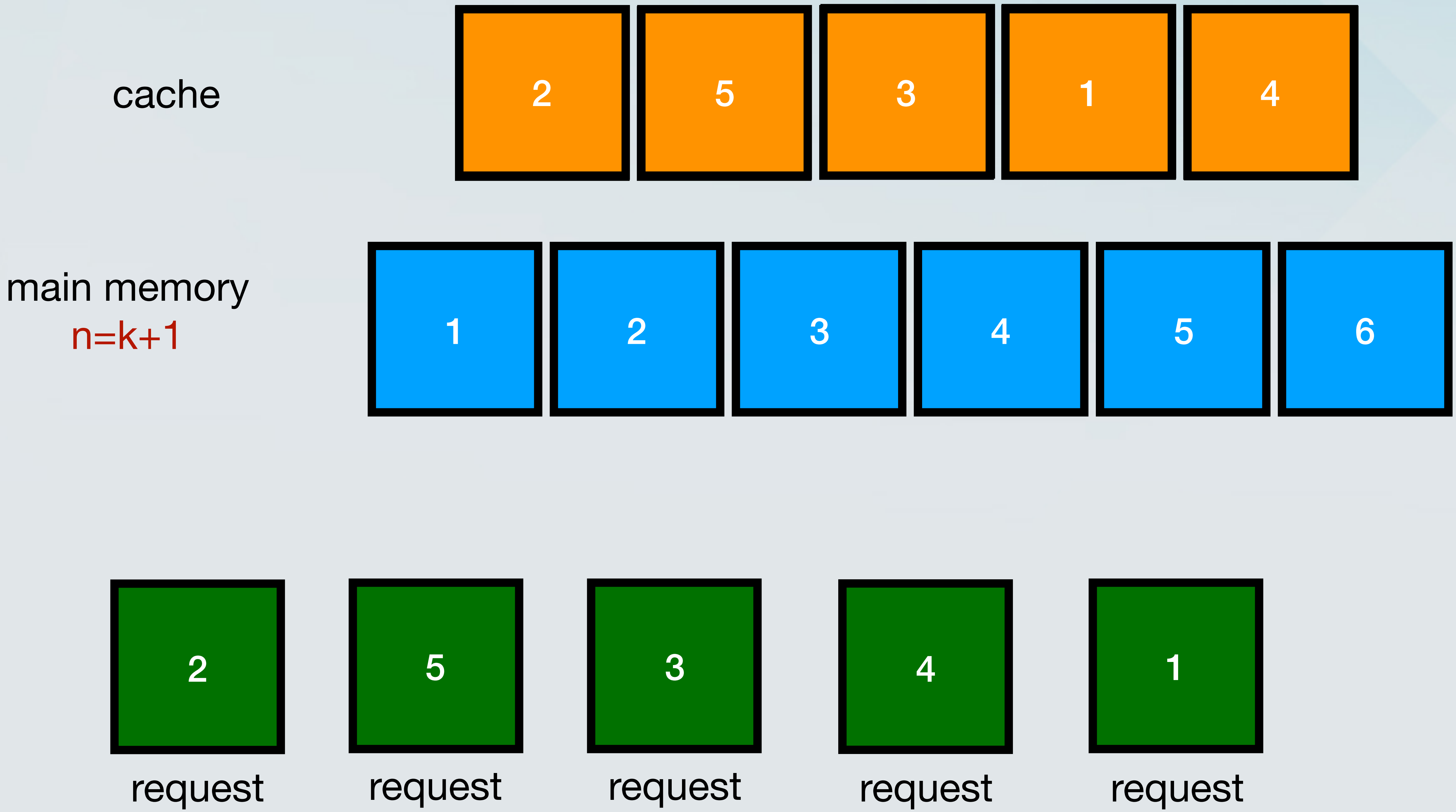

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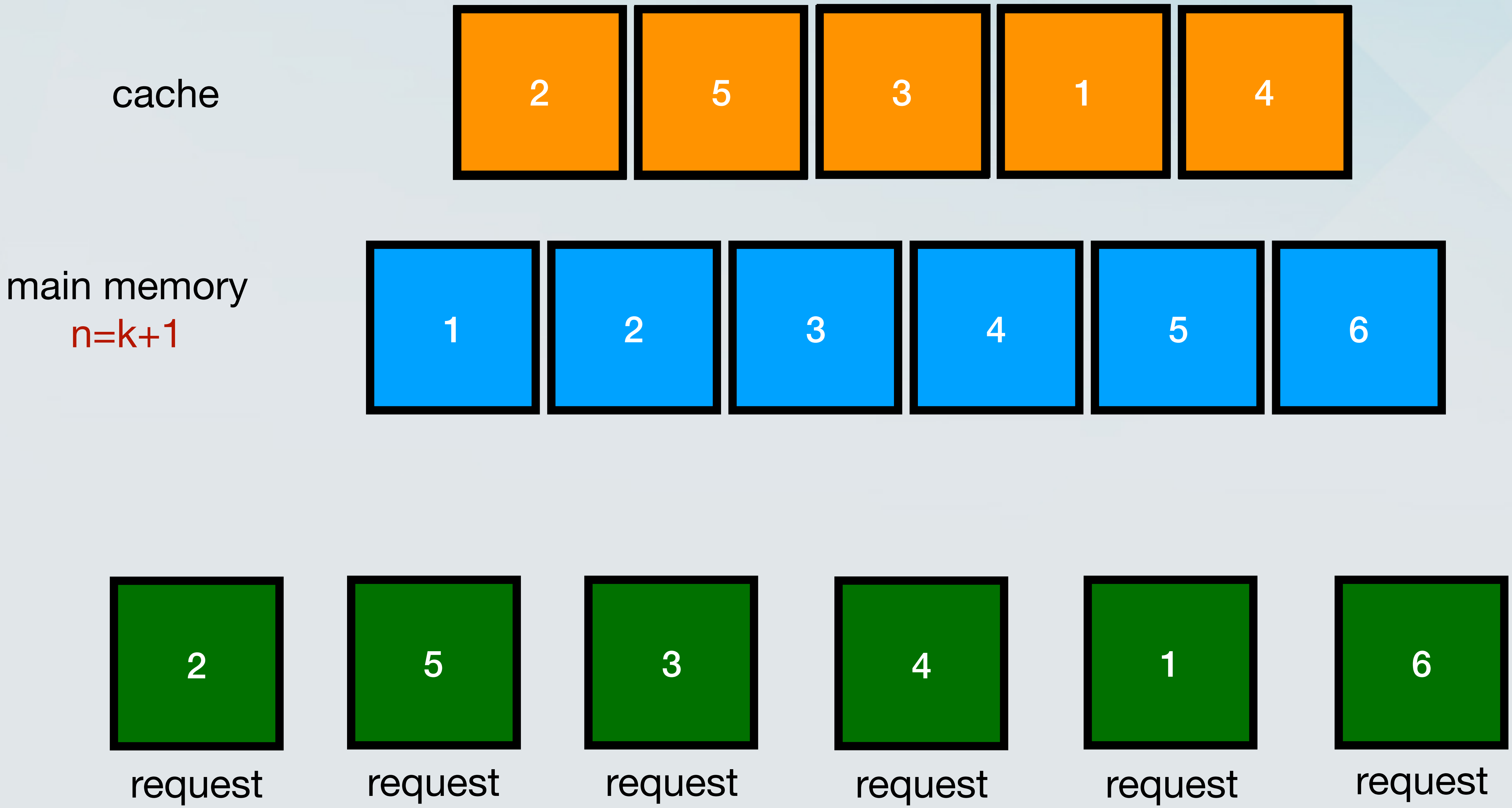

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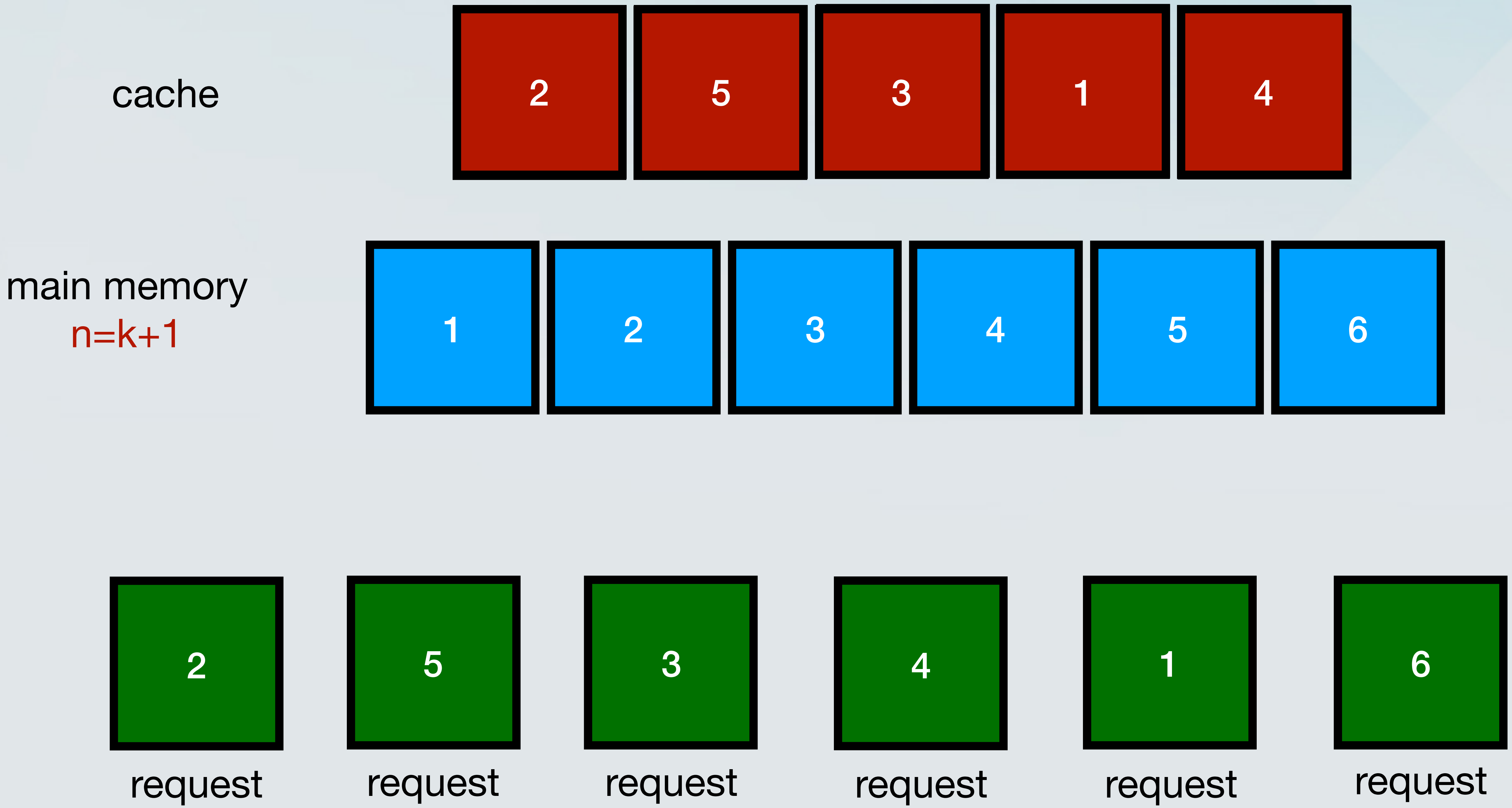

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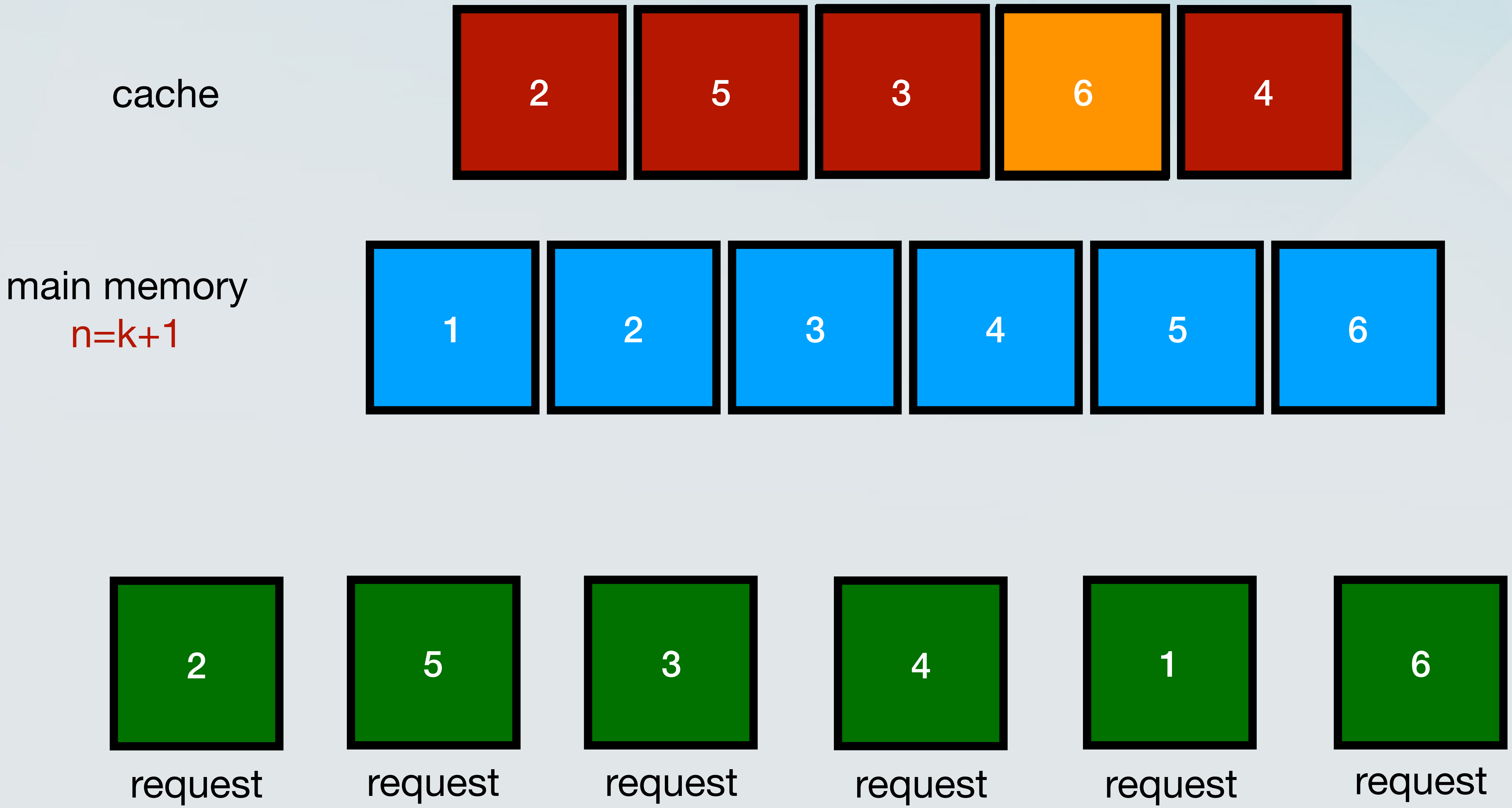

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    - Every time it fails, the requested page is marked.
    - If all pages in the cache are marked and a new page is requested, then the phase changes.
  - The optimal offline algorithm "*faults*" at least once in every phase.
    - The phase ends when k+1 different pages are requested.
    - The optimal offline algorithm can only keep at most k of those in the cache.

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- Proof: LRU and FIFO are marking algorithms.
- Corollary: LRU and FIFO are the best online algorithms for the paging problem.

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- We have to make a distinction, when it come to the power of the adversary:
  - Oblivious Adversary: The adversary fixes an input sequence in advance.
  - Adaptive Adversary: The adversary can change the input sequence based on the realisations of randomness of the choices of the algorithm.

Marking algorithm

- Consider the following algorithm:
  - The algorithm proceeds in *phases*.
  - At the beginning of a phase, all the pages are unmarked.
  - Whenever a page is requested, it is marked.
  - When a “*fault*” occurs, the algorithm replaces an unmarked page.
  - When all pages in the cache are marked, and a request for an unmarked page occurs, the phase ends.

Randomised Marking algorithm

- Consider the following algorithm:
  - The algorithm proceeds in *phases*.
  - At the beginning of a phase, all the pages are unmarked.
  - Whenever a page is requested, it is marked.
  - When a "*fault*" occurs, the algorithm replaces an unmarked page, *selecting one uniformly at random*.
  - When all pages in the cache are marked, and a request for an unmarked page occurs, the phase ends.

Randomised Marking algorithm

- Theorem: The Randomised Marking algorithm has competitive ratio $2H_k$ against *oblivious adversaries*.

The proof

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  - call the remaining $k-m_i$ pages "*old*".

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- Every time an "*old*" page is requested, we may have "*fault*".

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- Every time a "*new*" page is requested, we have a "*fault*".
- Every time an "*old*" page is requested, we may have "*fault*".
  - The "*fault*" happens if we replaced the "*old*" page with a "*new*" one.

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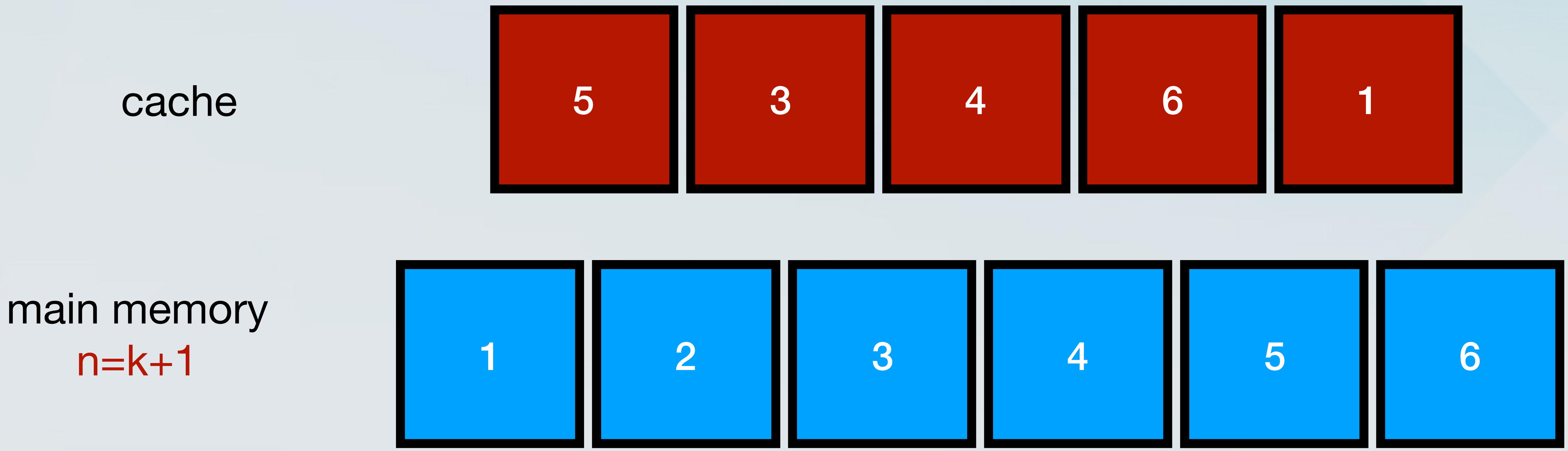

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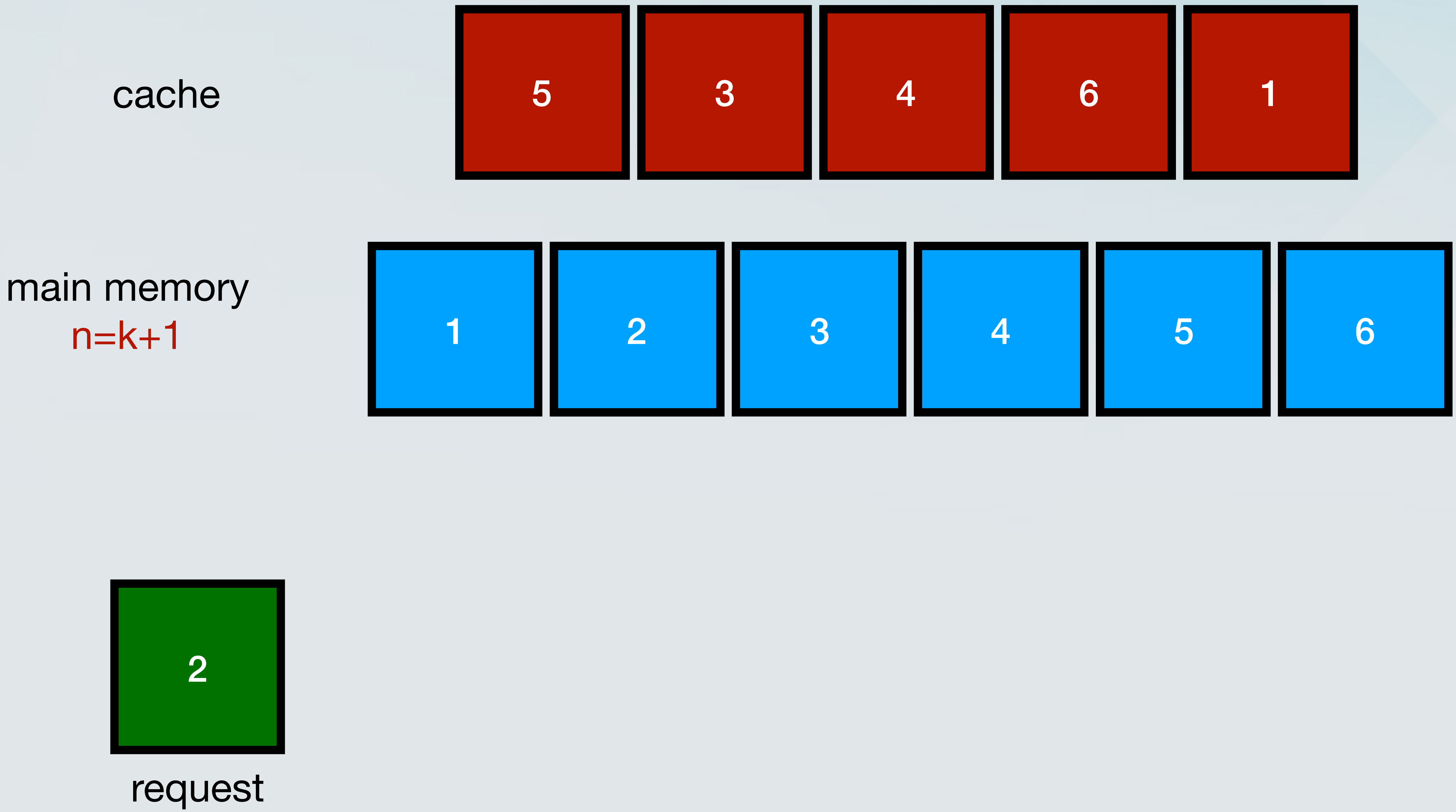

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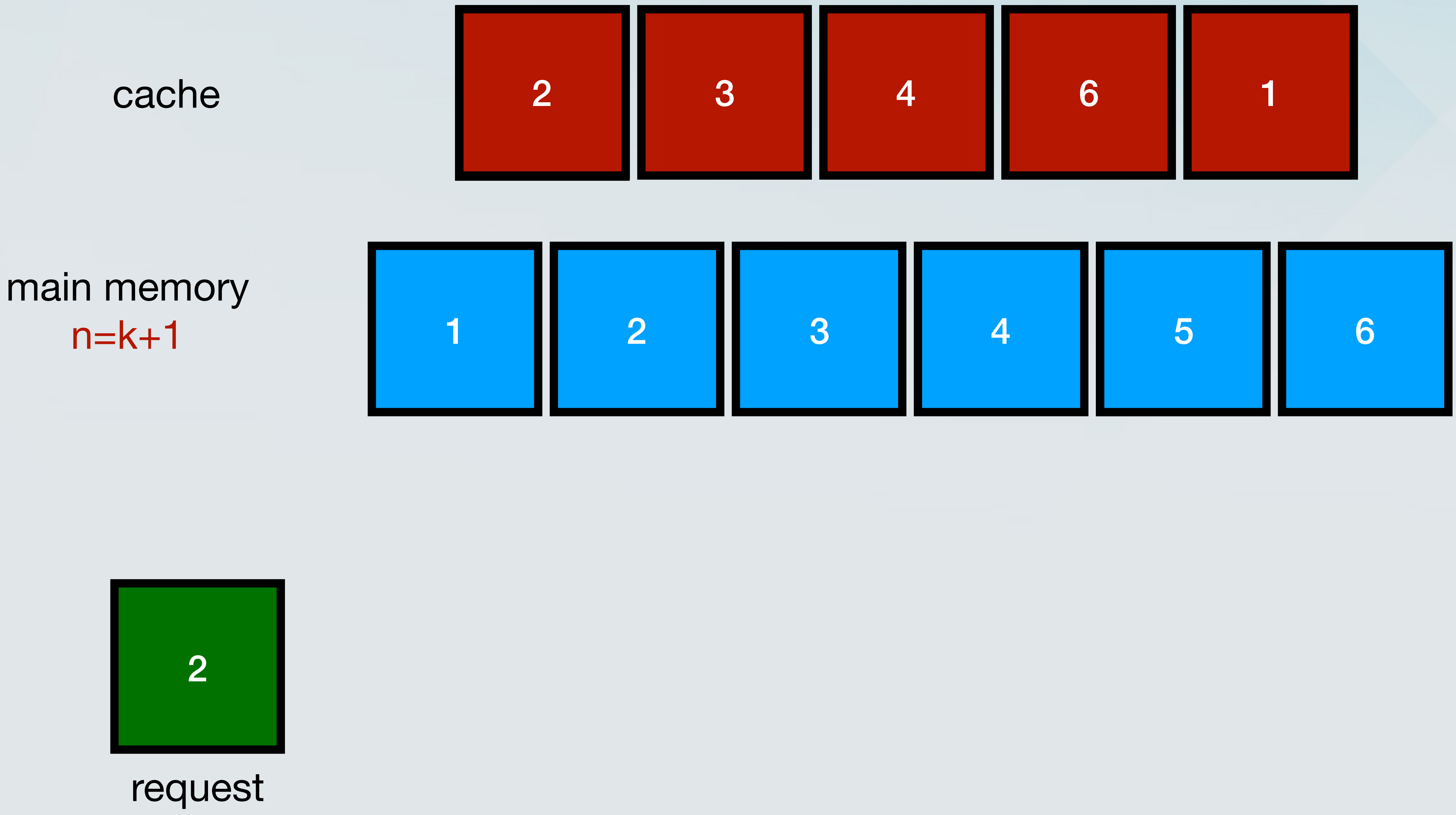

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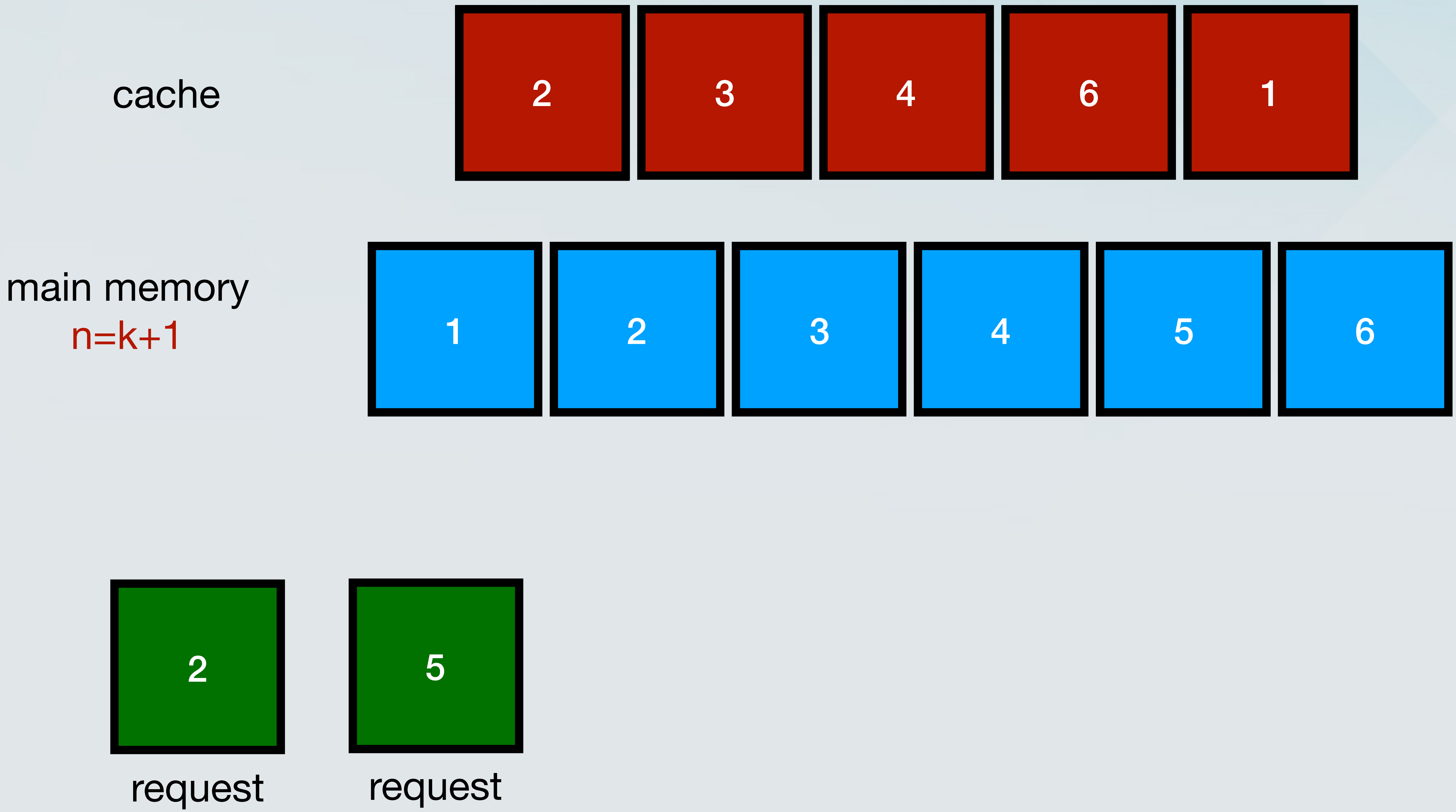

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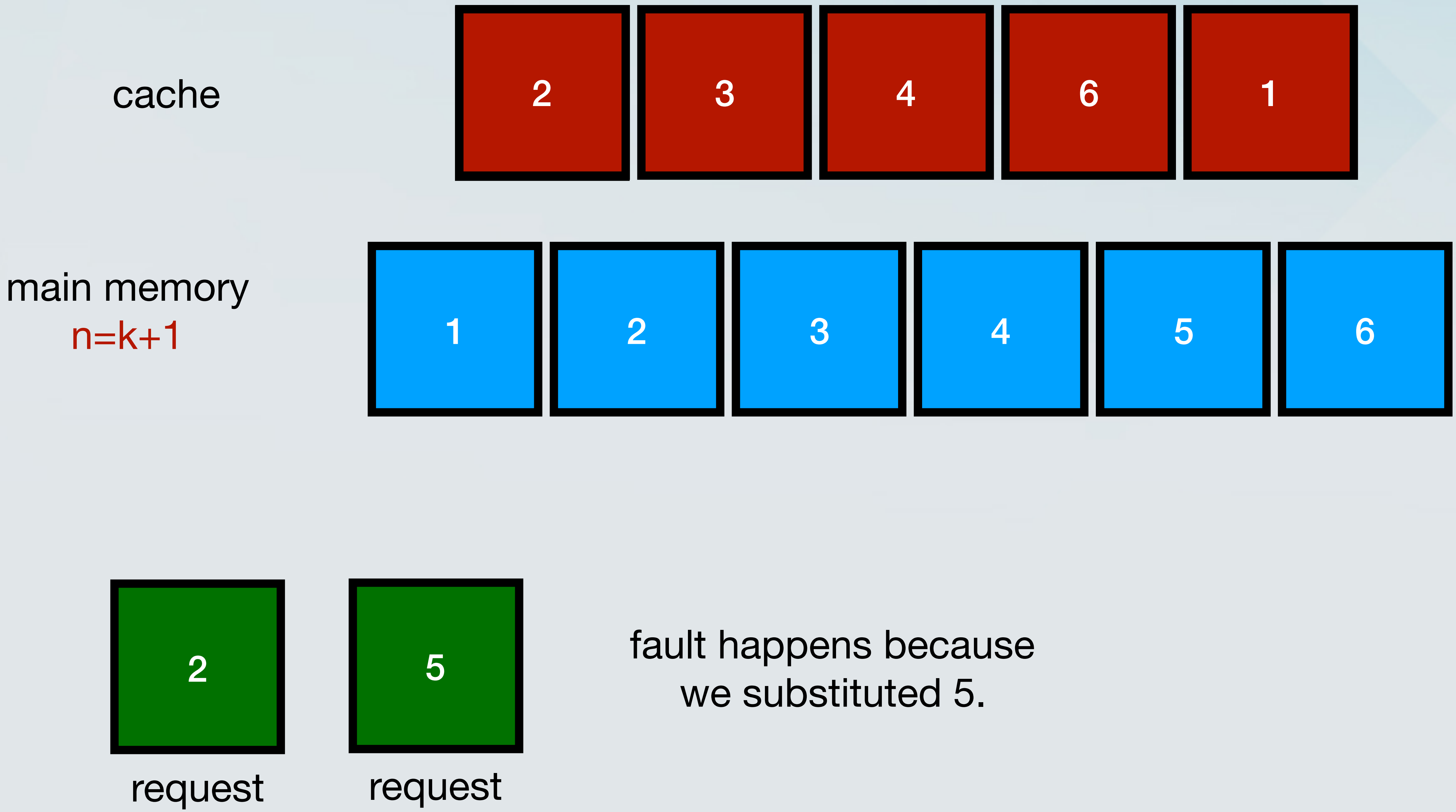

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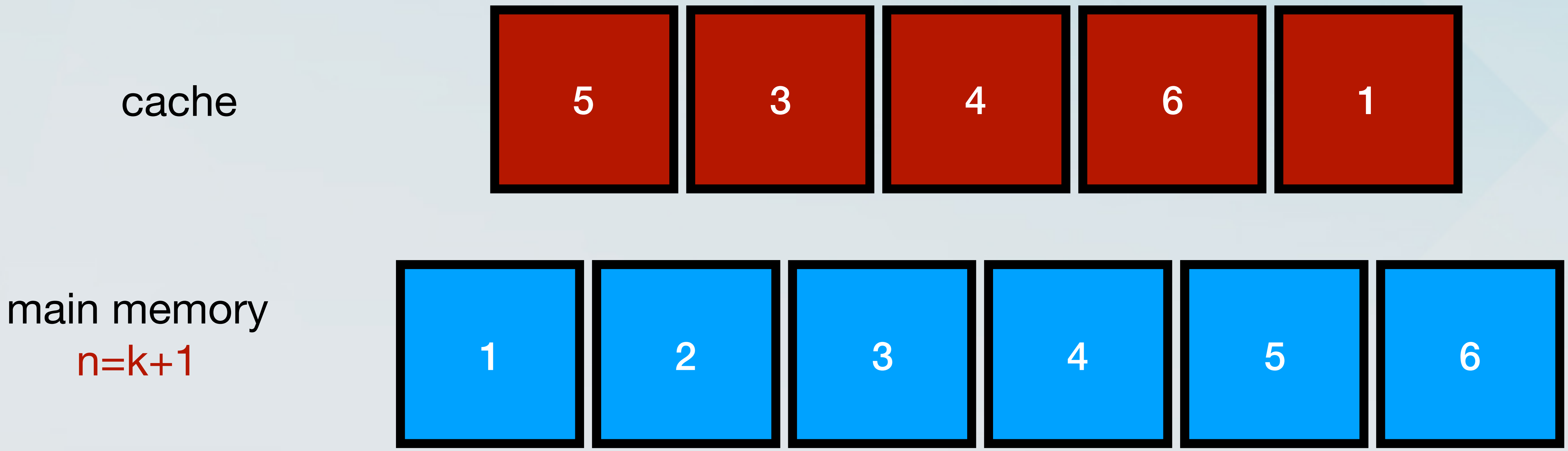

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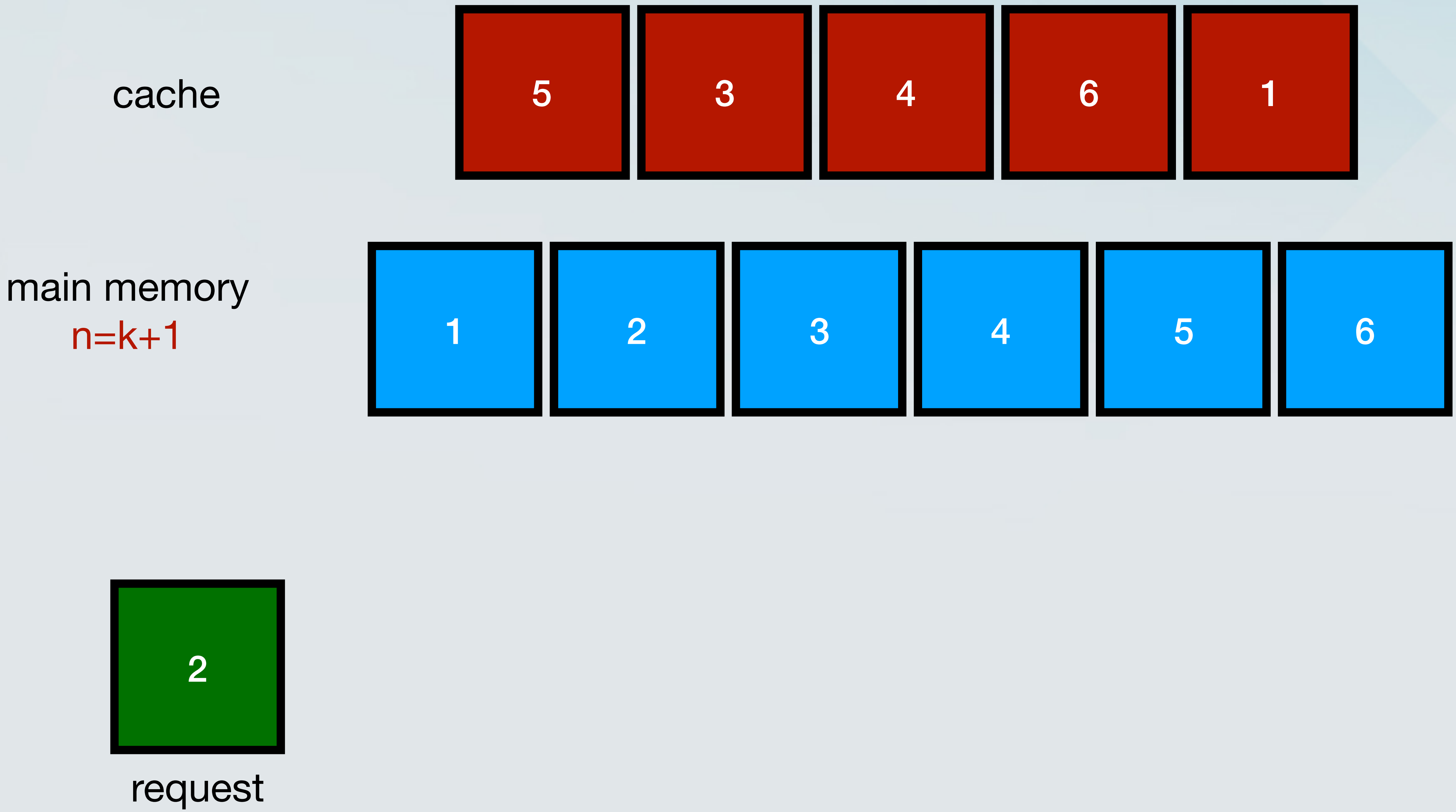

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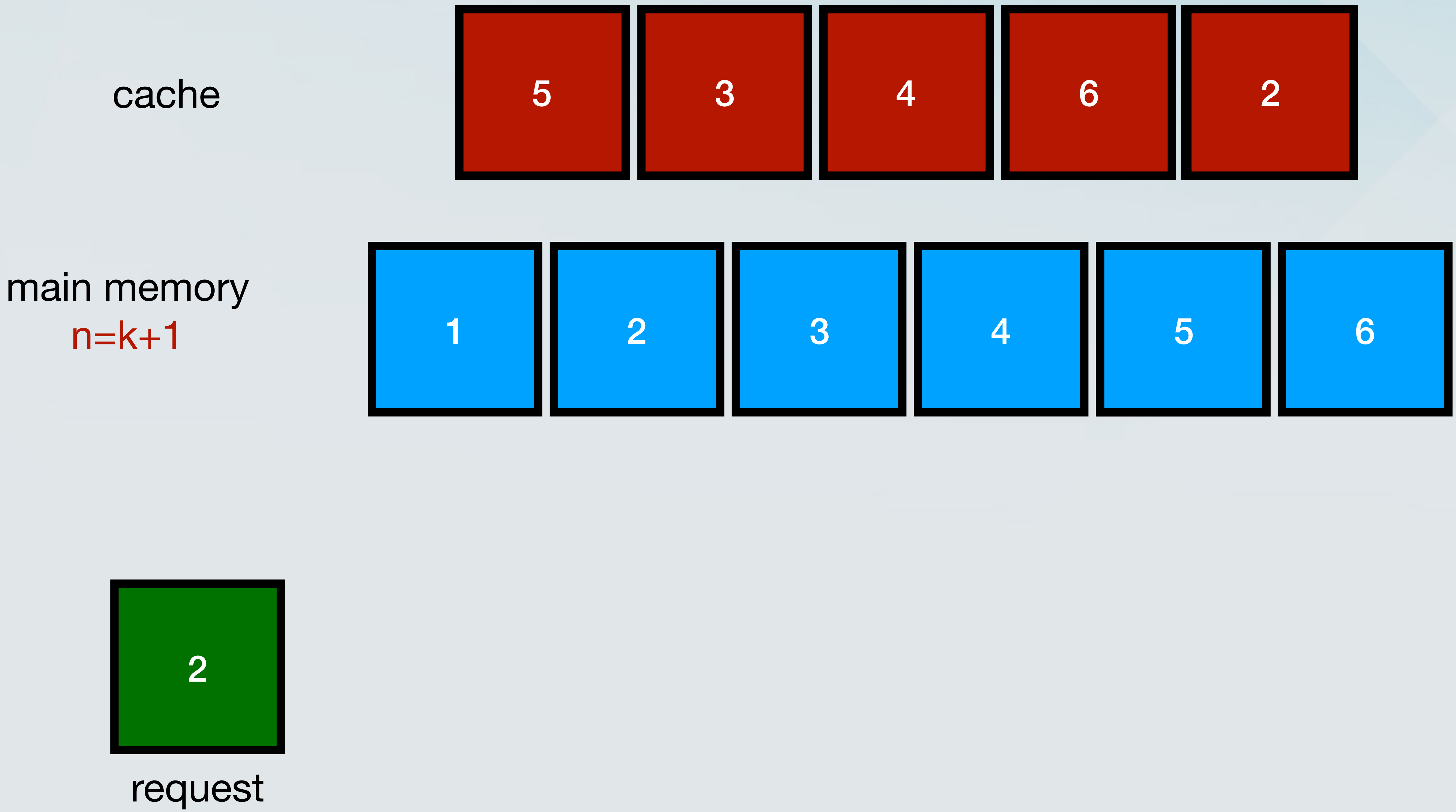

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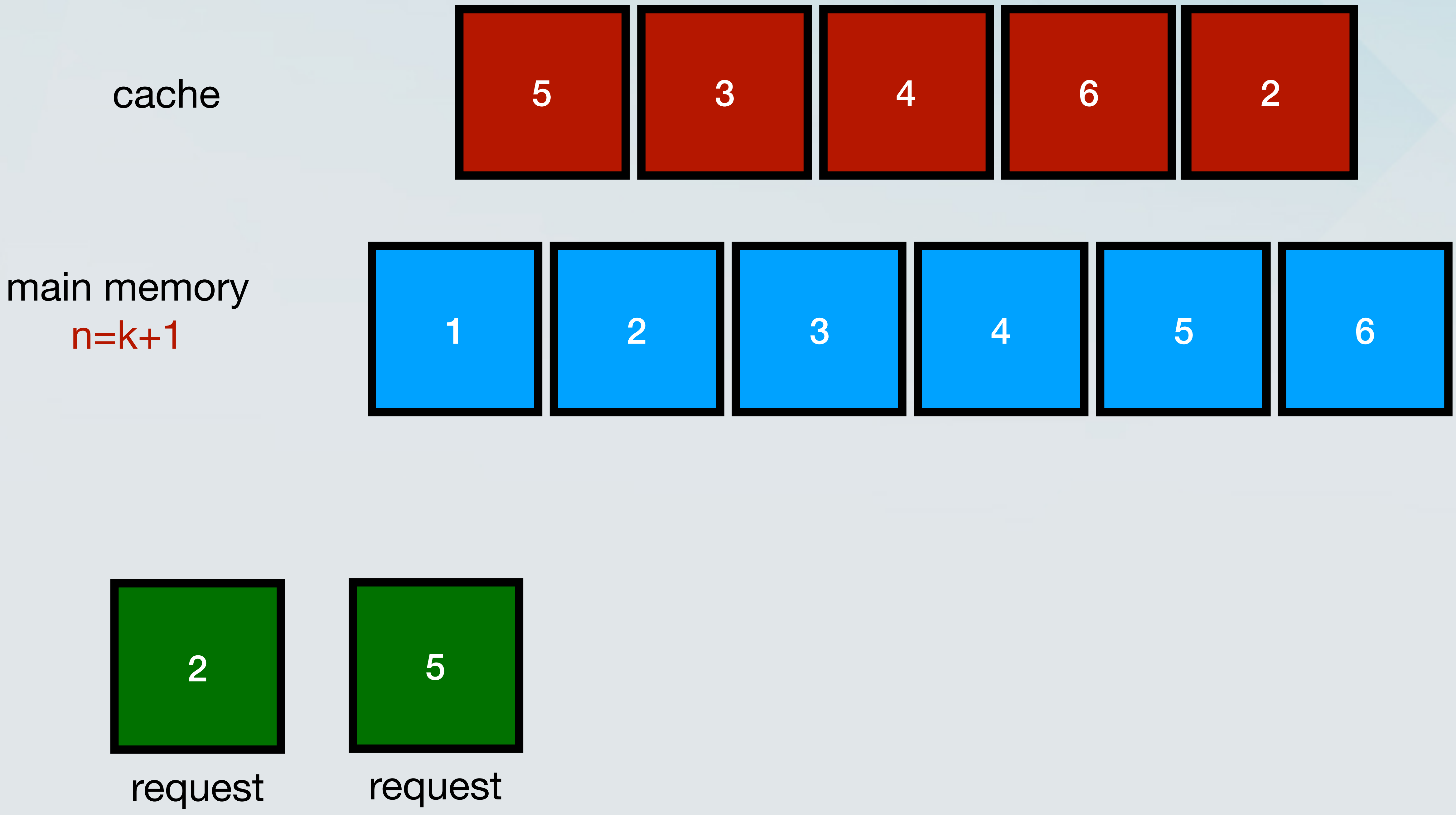

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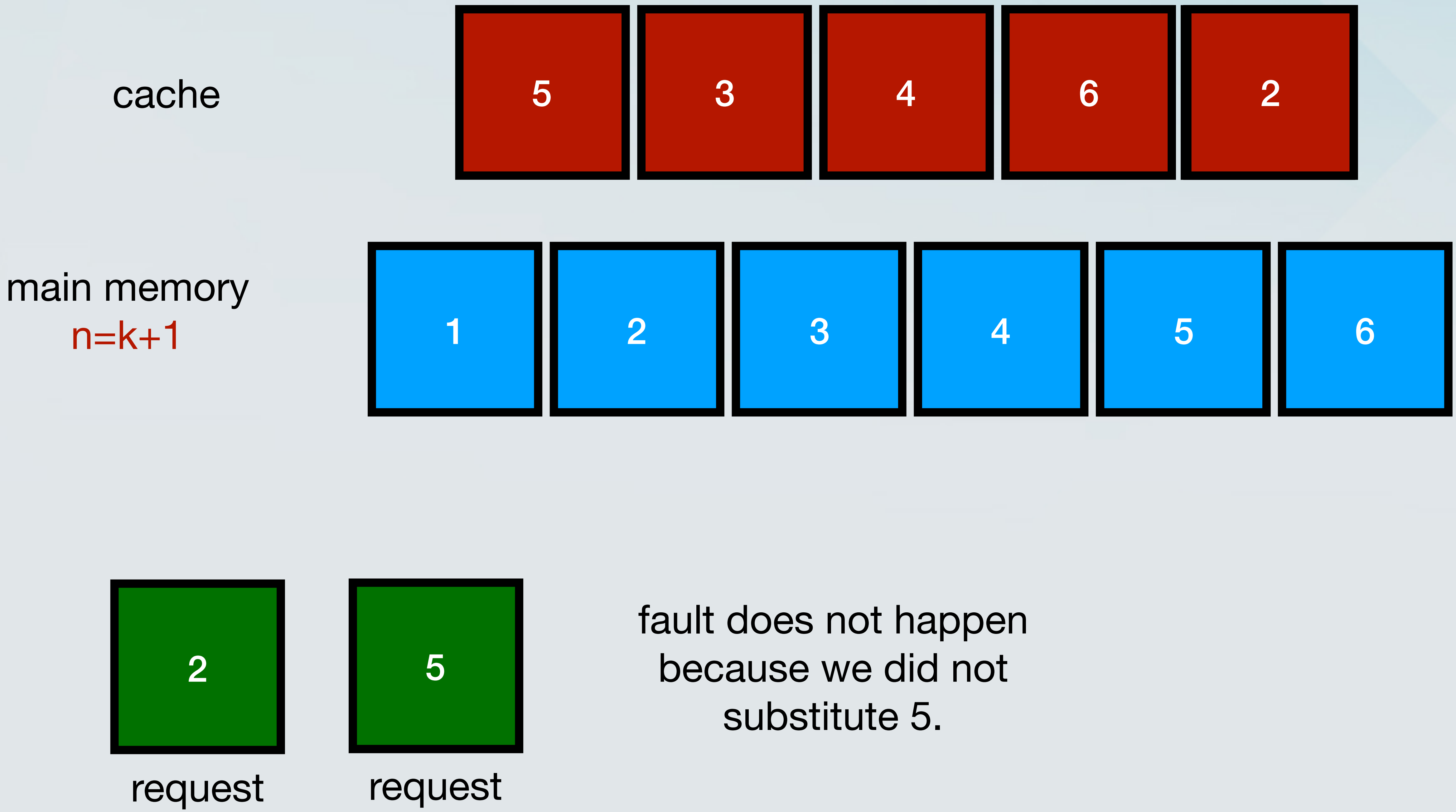

fault does not happen because we did not substitute 5.

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  - call the remaining $k-m_i$ pages “*old*”.
- Every time a “*new*” page is requested, we have a “*fault*”.
- Every time an “*old*” page is requested, we may have “*fault*”.
  - The “*fault*” happens if we replaced the “*old*” page with a “*new*” one.

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  - This is m_i/k.

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- The "*fault*" happens if we replaced the "*old*" page with a "*new*" one.
  - What is the probability of that happening?
- Assume (*wlog*) that the m_i requests for "*new*" pages come first and the $k-m_i$ requests for "*old*" pages follow.
  - What is the probability that on the second "*old*" page request, we make a "*fault*", given that we made a "*fault*" on the first "*old*" page request?

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  - This is $m_i/(k-1)$.

The proof

- The expected number of "*faults*" of our algorithm in phase i is

  $m_i + m_i/k + m_i/(k-1) + \ldots + m_i/(k-(k-m_i)+1 \leq m_i H_k$

- Summing up over all the phases, we have that:

$$\mathbb{E}[\textbf{cost of RMA}] \leq H_k \sum_{i=1}^{n} m_i$$

Arguing about the OPT

- What is the number of "*faults*" that the optimal offline algorithm makes?
- Let's look at two *consecutive* phases i-1 and i.
- Let n_{i-1} and n_i be the number of "*faults*" of OPT on those phases.
- By the definition of a phase, there are m_i new pages in phases i-1 and i.
    - It holds that $n_{i-1} + n_i \leq m_i$
    - Summing up, we get: $OPT \geq \frac{1}{2}\sum_{i=1}^{n} m_i$

Combining

$$\mathbb{E}[\textbf{cost of RMA}] \leq H_k \sum_{i=1}^{n} m_i \qquad OPT \geq \frac{1}{2} \sum_{i=1}^{n} m_i$$

The competitive ratio of RMA is $2H_k$