Advanced Algorithmic Techniques (COMP523)

Recursion and Divide and Conquer Techniques #2

Recap and plan

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- **Last lecture:**
    - Asymptotic Complexity.
    - Searching in logarithmic time.
    - Finding majority in an array.

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  - Asymptotic Complexity.
  - Searching in logarithmic time.
  - Finding majority in an array.
- **This lecture:**
  - Sorting with the **MergeSort** algorithm.
  - Sorting with the **QuickSort** algorithm.
  - The limitations of comparison-based sorting.

Sorting with Mergesort

Merging two sorted arrays

- Given two sorted arrays **A**[$1,\ldots,n$] and **B**[$1,\ldots,m$], produce a sorted array **C**[$1, \ldots, n+m$] containing all the elements of **A** and **B**.

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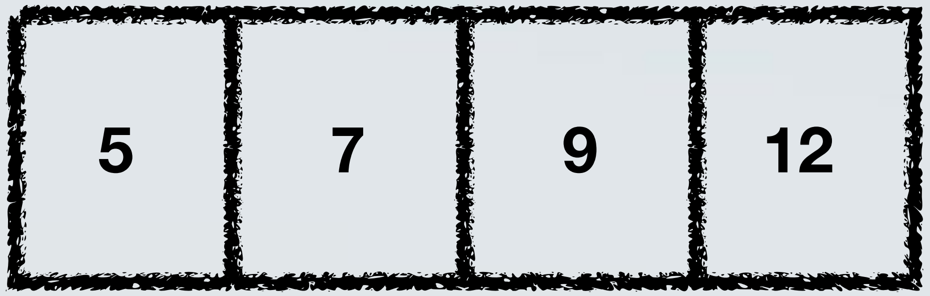

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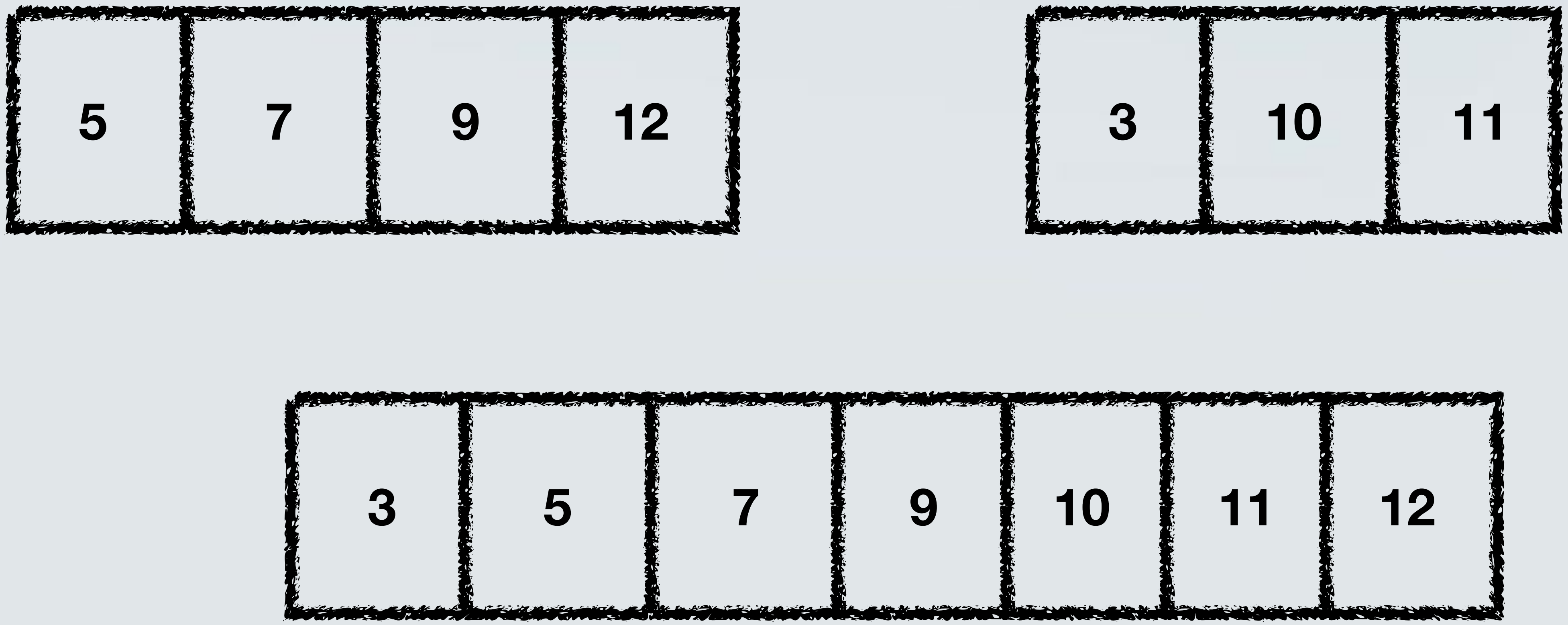

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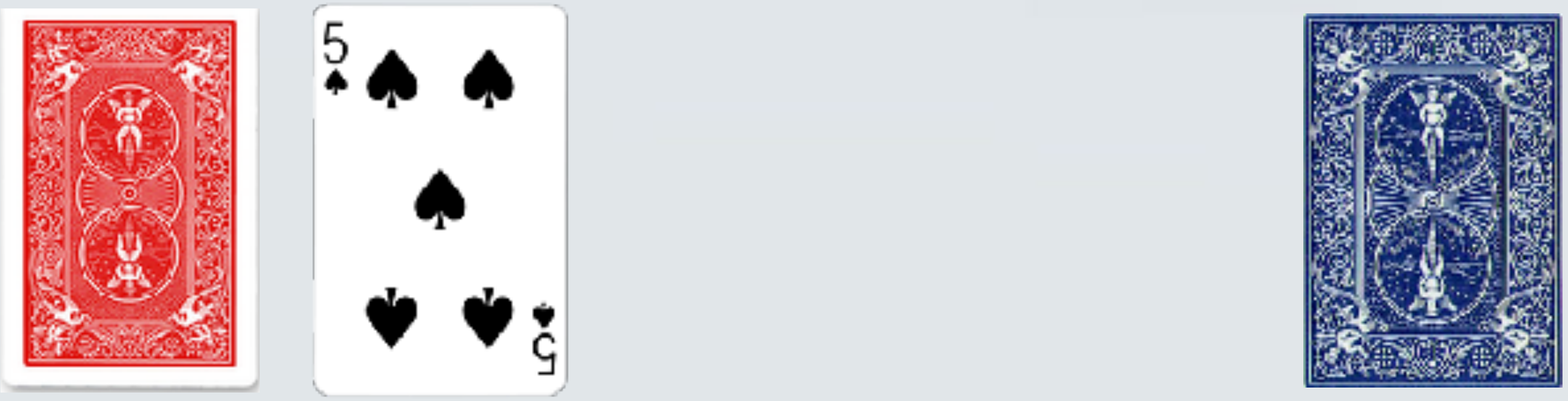

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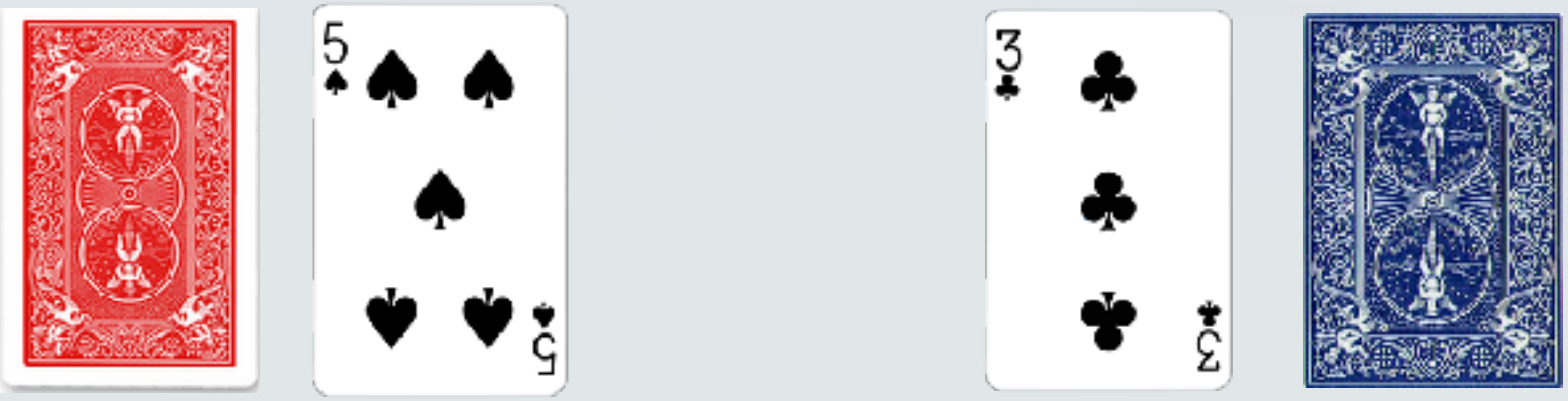

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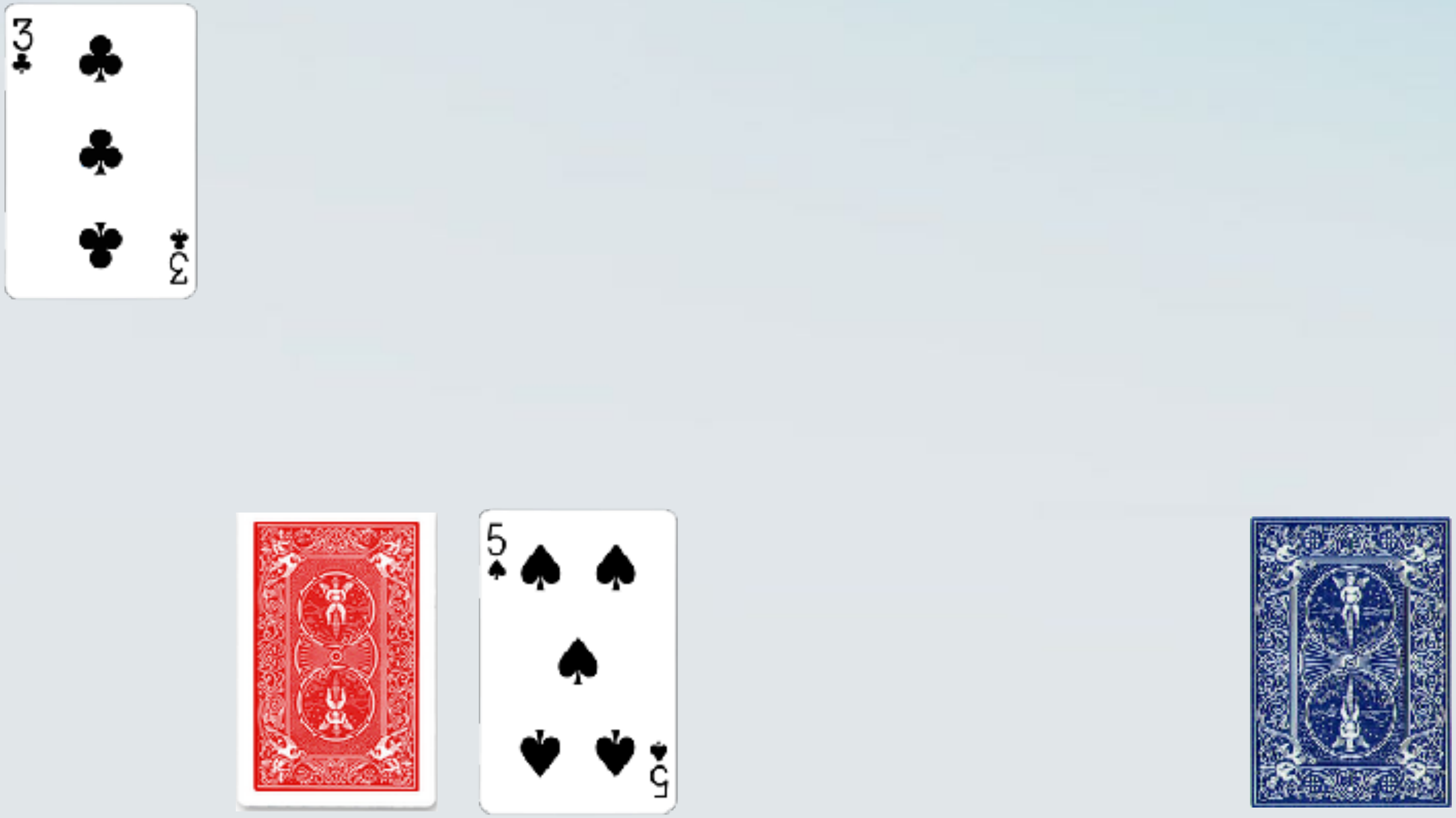

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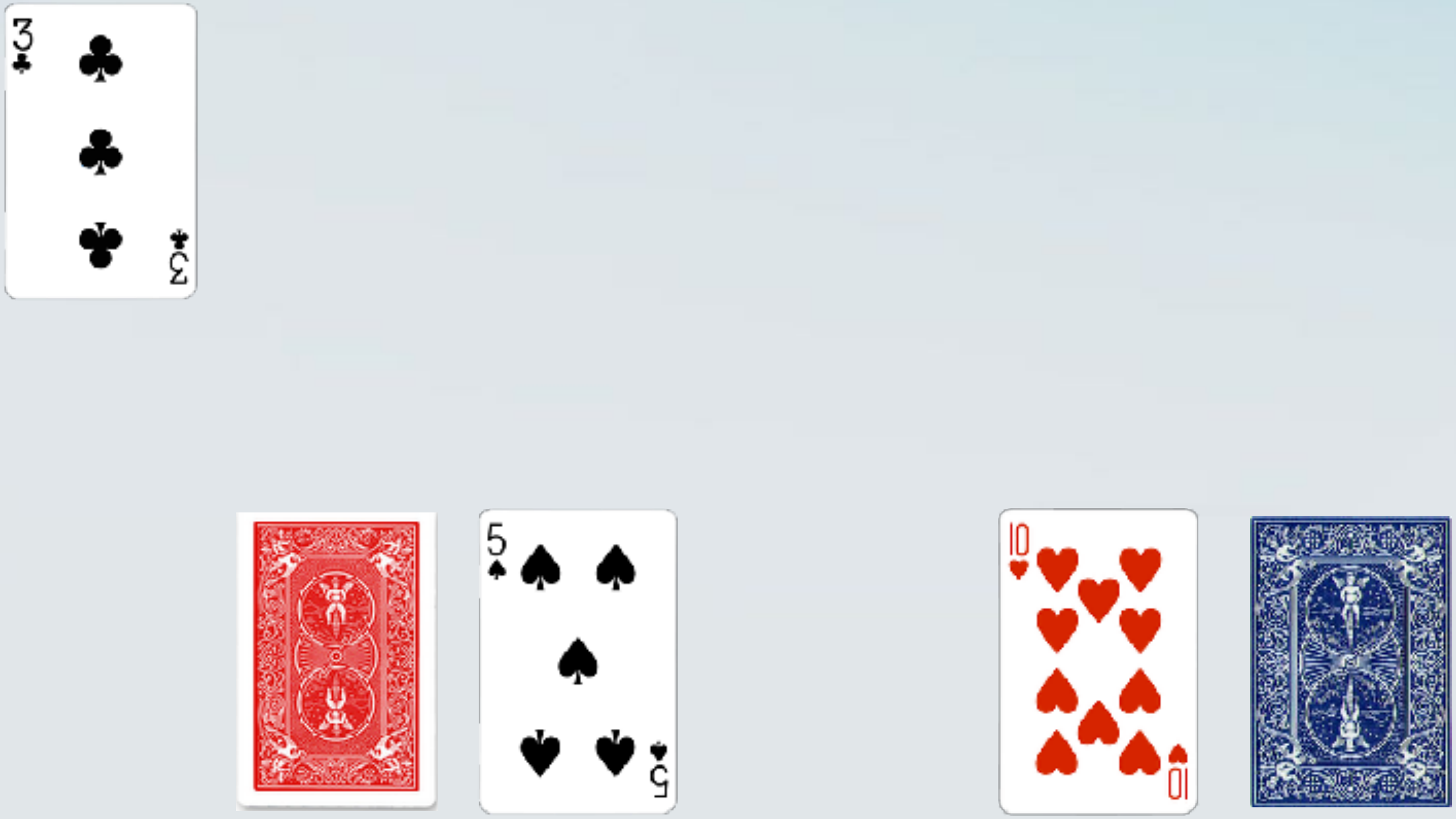

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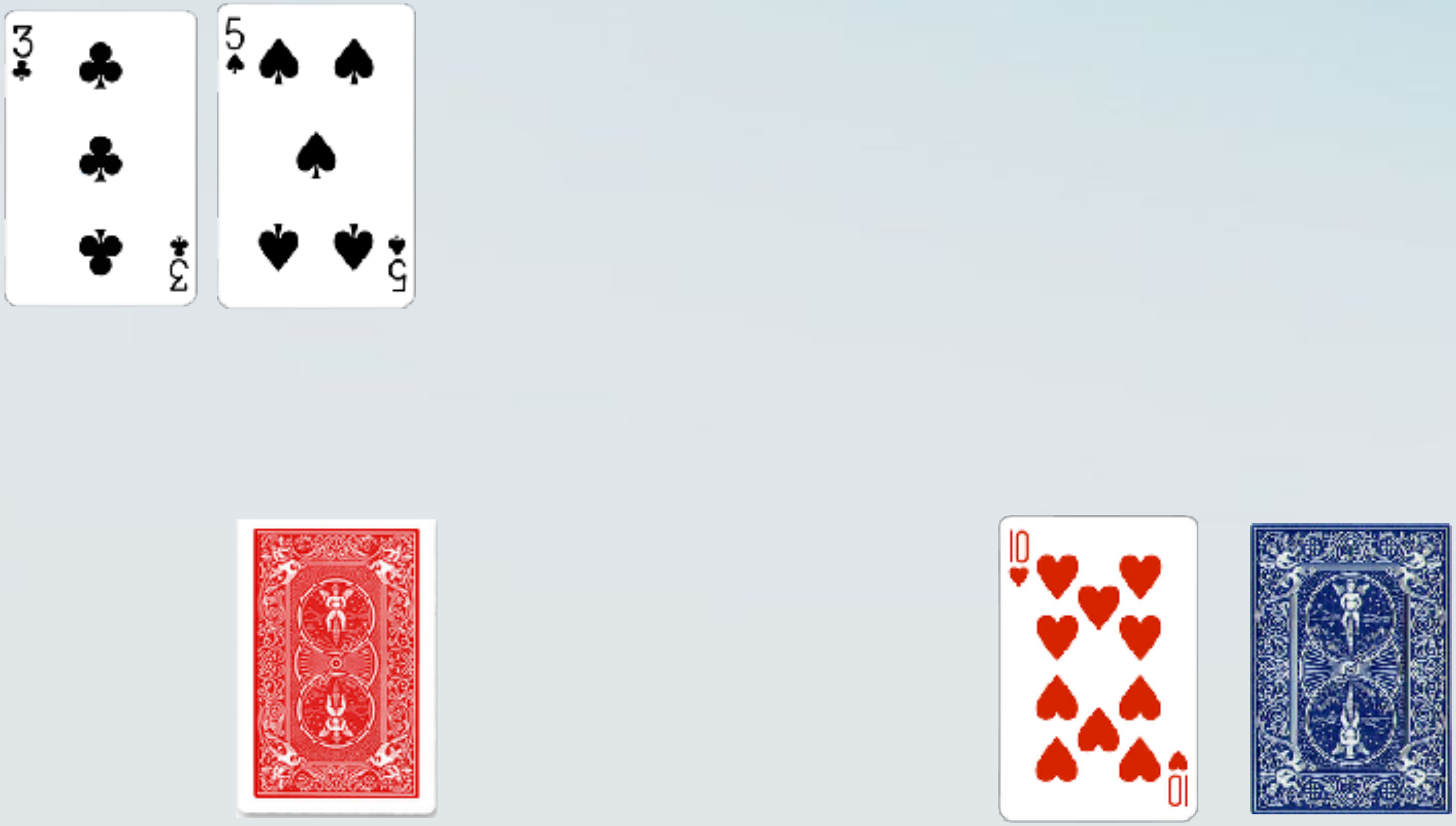

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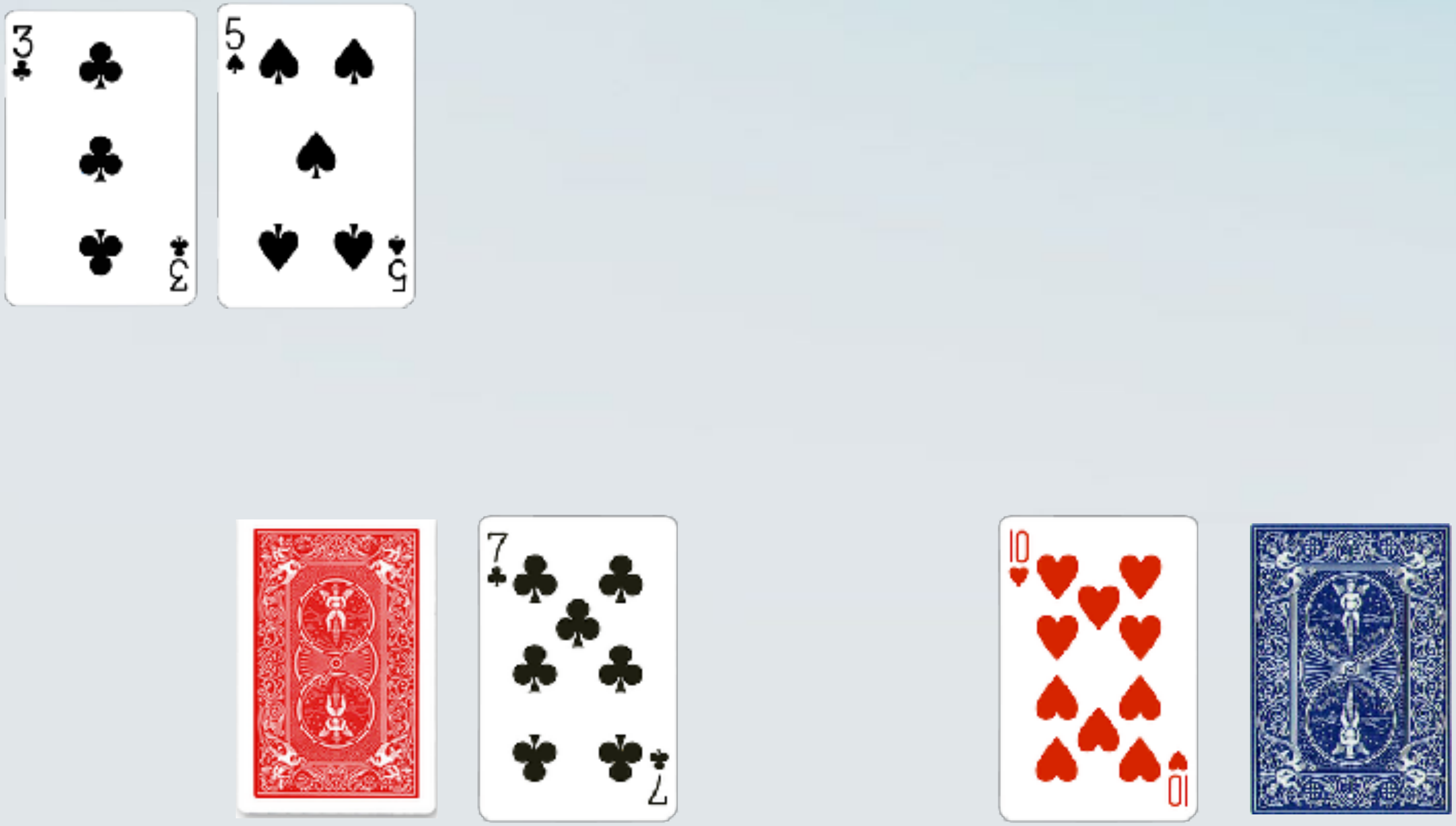

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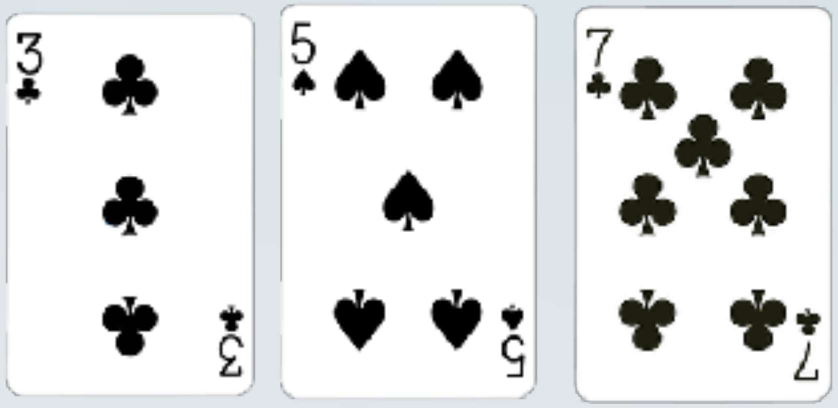

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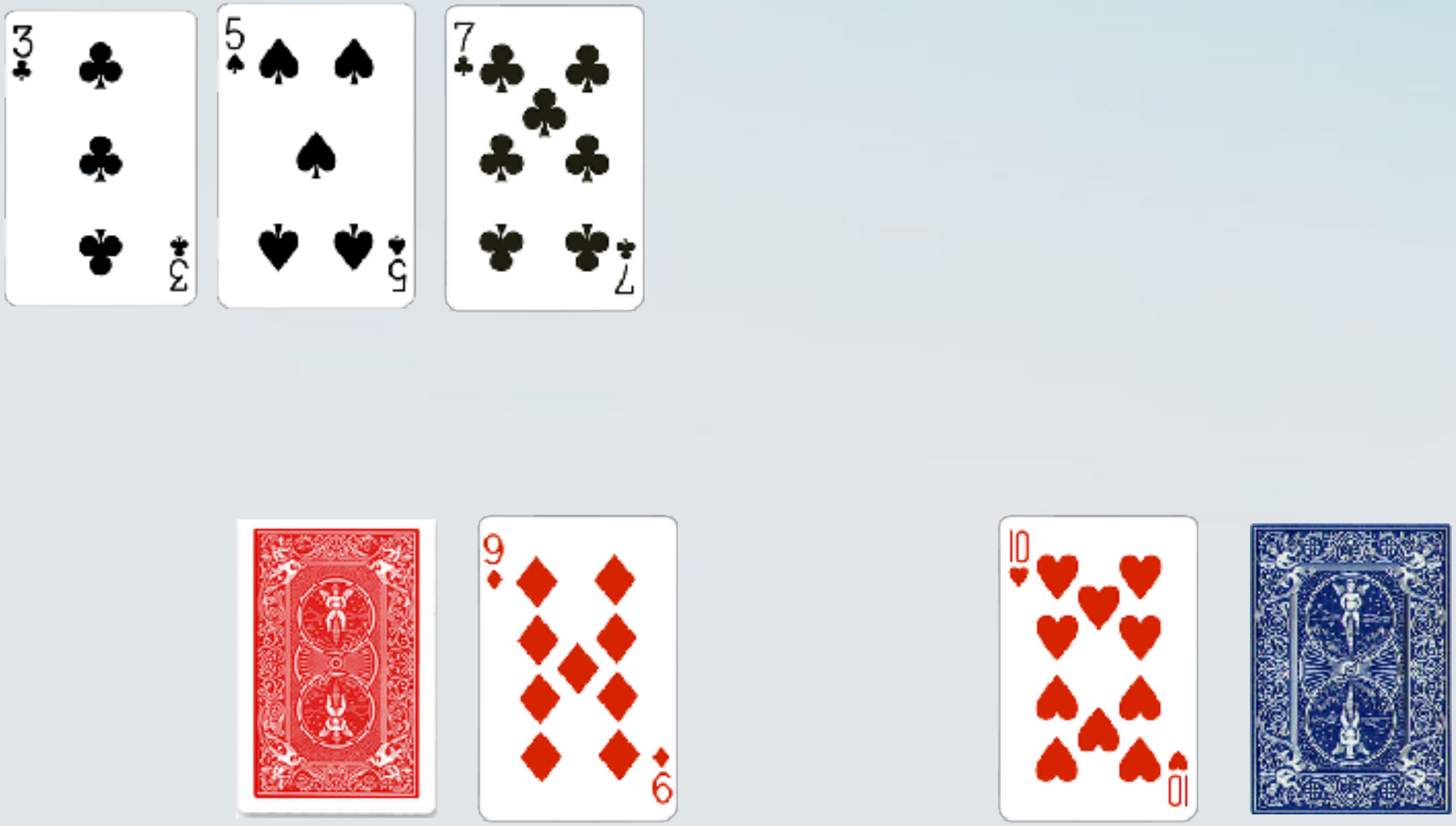

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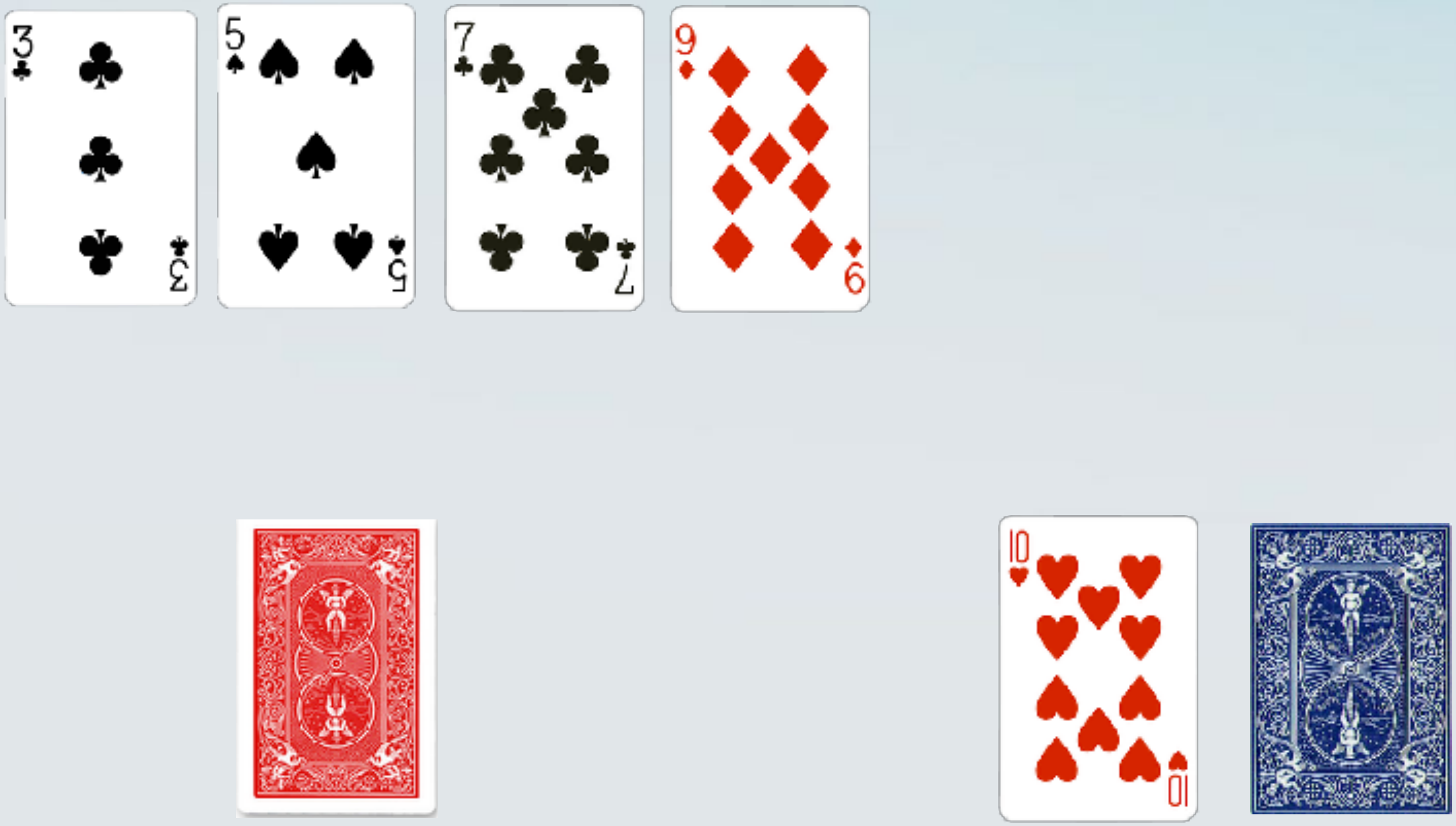

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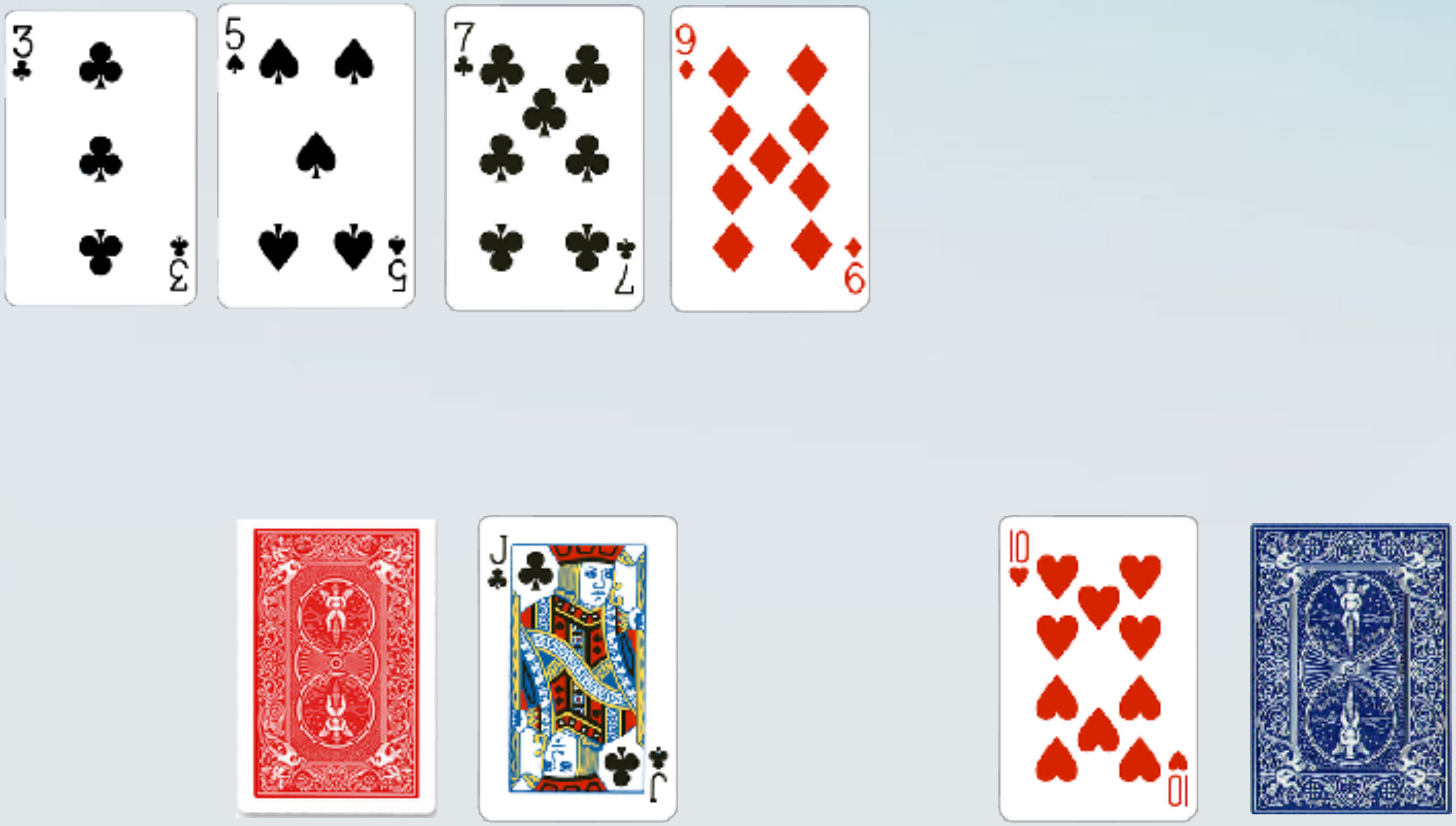

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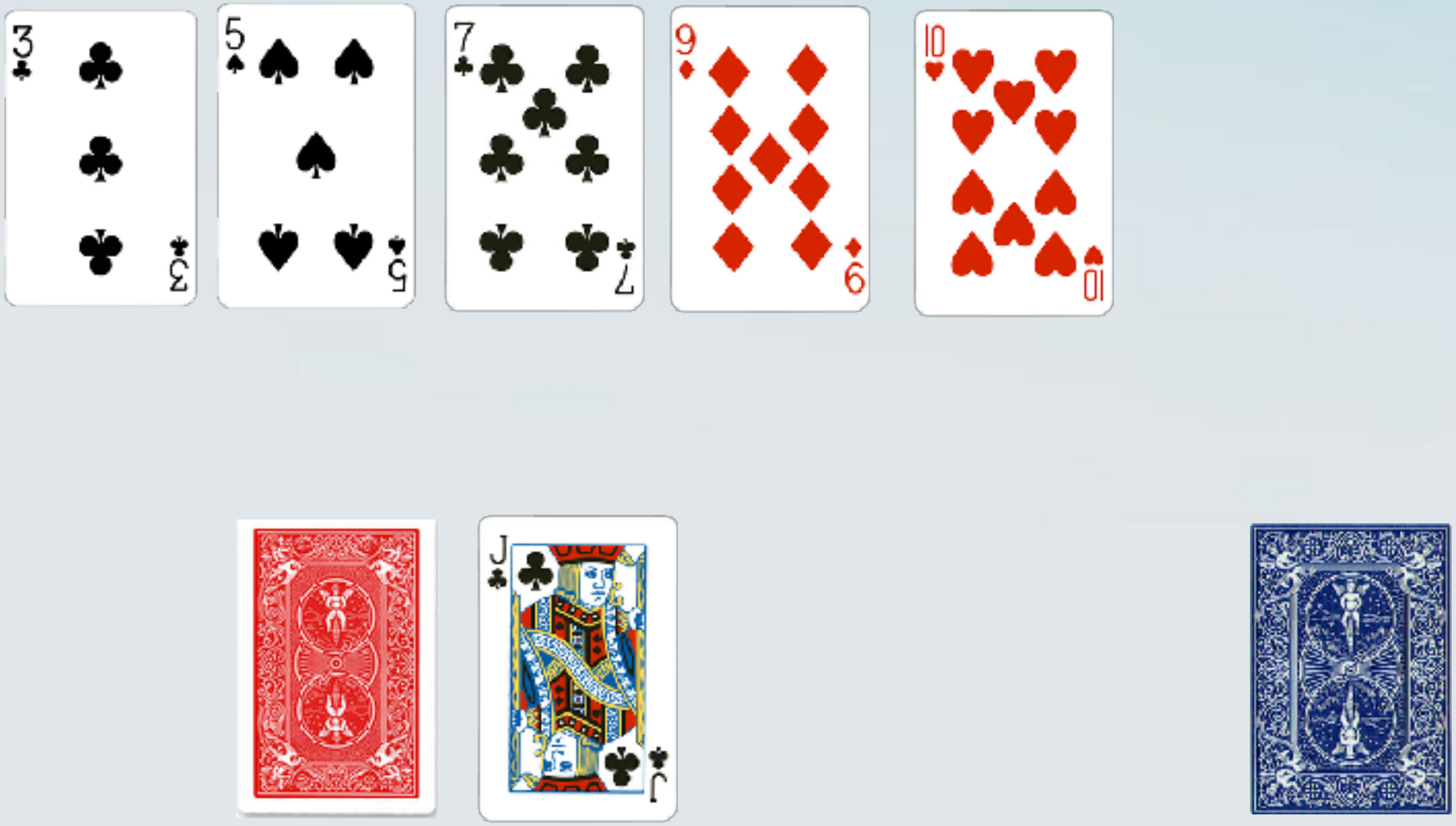

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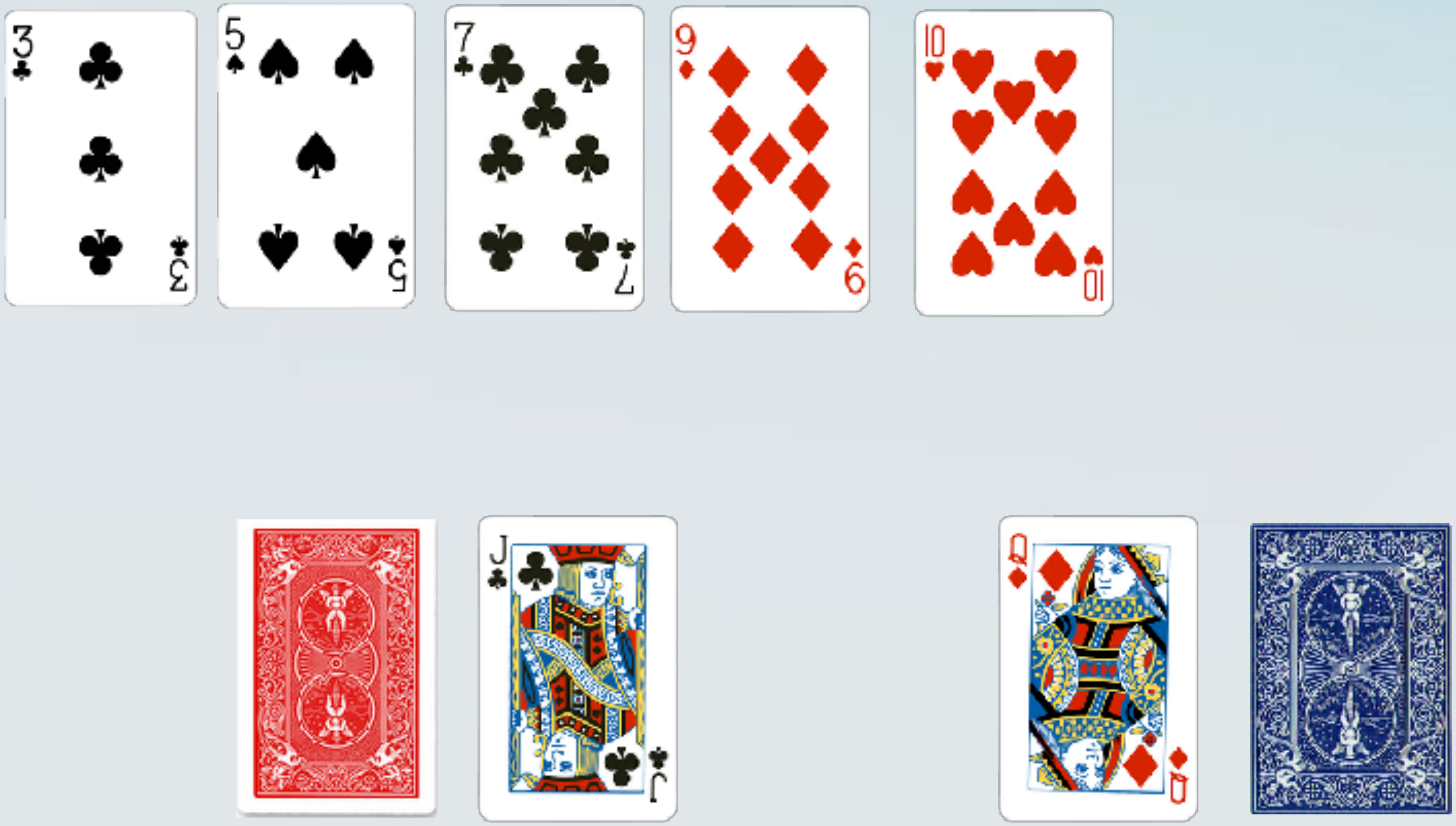

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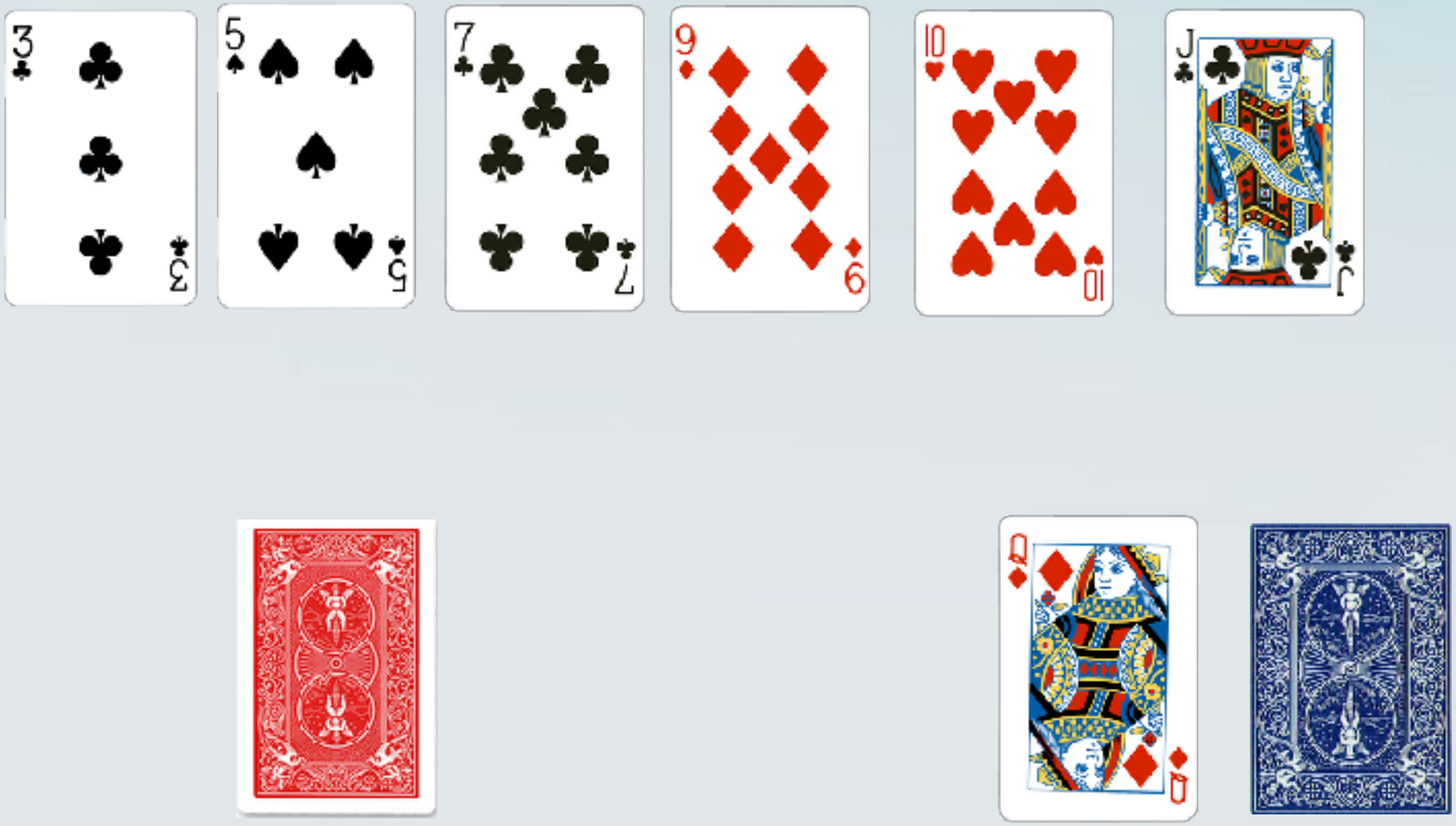

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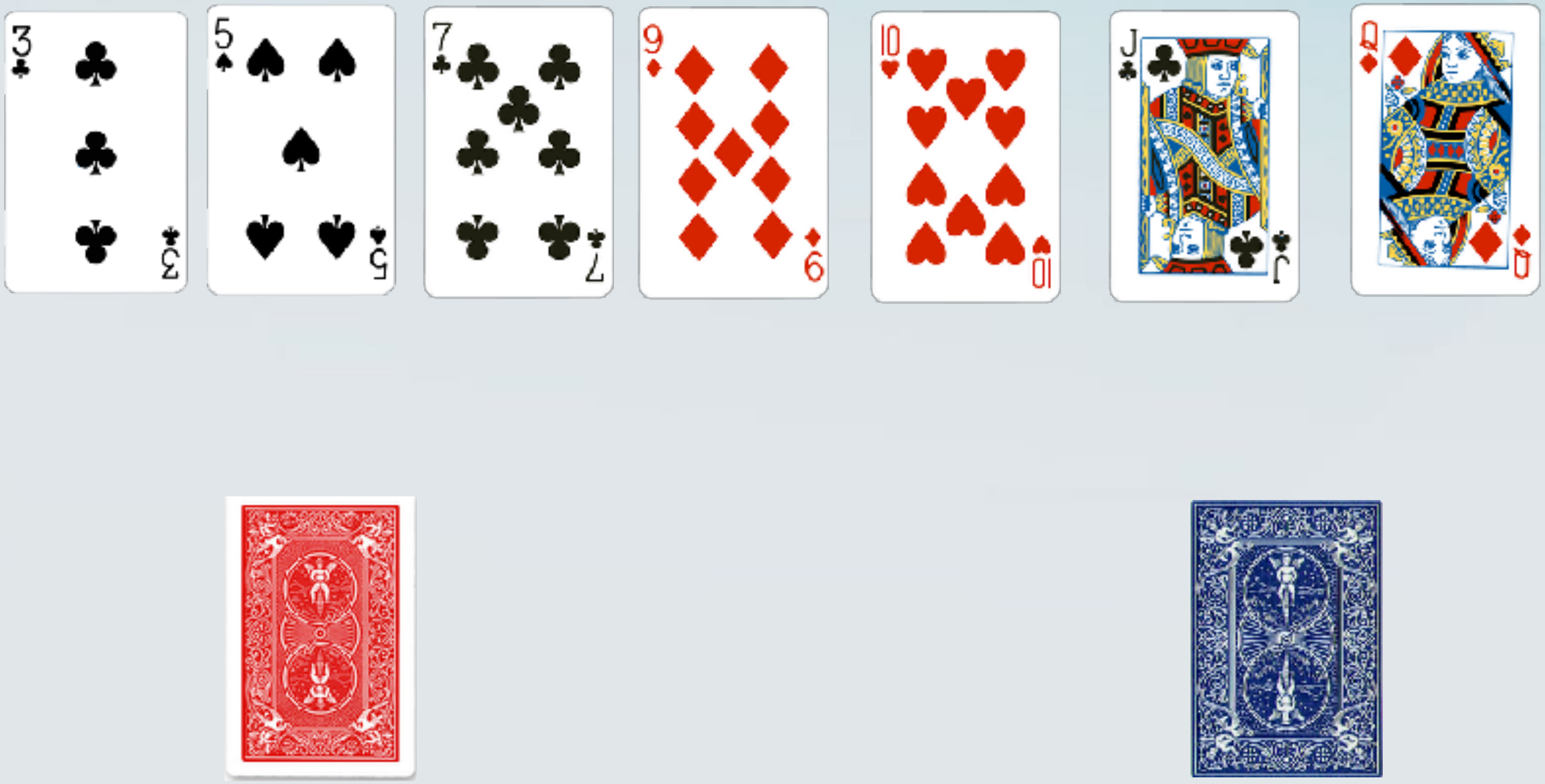

Procedure Merge

```
Procedure Merge(A, B)
/* Recall that |A| = n and |B| = m */

Initialise array C of size n+m

i=1, j=1

For k=1, … , m+n-1

    If A[i] ≤ B[j]
        C[k] = A[i]
        i=i+1
    Else
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O(m+n)

The **Mergesort** algorithm

- Divide and conquer algorithm.
- Split the array **A**[$1,\dots,n$] to two subarrays, **A**[$1,\dots,n/2$] and **A**[$n/2+1, \dots, n$]
- Sort each subarray using **Mergesort**.
  - Stop the recursion when the subarray contains only one element.
- Merge the sorted subarrays **A**[$1,\dots,n/2$] and **A**[$n/2+1, \dots, n$] using the **Merge** procedure.

Mergesort pseudocode

```
Algorithm Mergesort($\mathbf{A}[i,\ldots,j]$)

If $i=j$, return $i$

$q=(i+j)/2$

$\mathbf{A}_{left}$=Mergesort($\mathbf{A}[i,\ldots,q]$)
$\mathbf{A}_{right}$=Mergesort($\mathbf{A}[q+1,\ldots,n]$)
return Merge( $\mathbf{A}_{left}$ , $\mathbf{A}_{right}$ )
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Initial call: **Mergesort($\mathbf{A}[i,\ldots,n]$)**

Mergesort example

6	4	8	9	2	1	3

Mergesort example

divide

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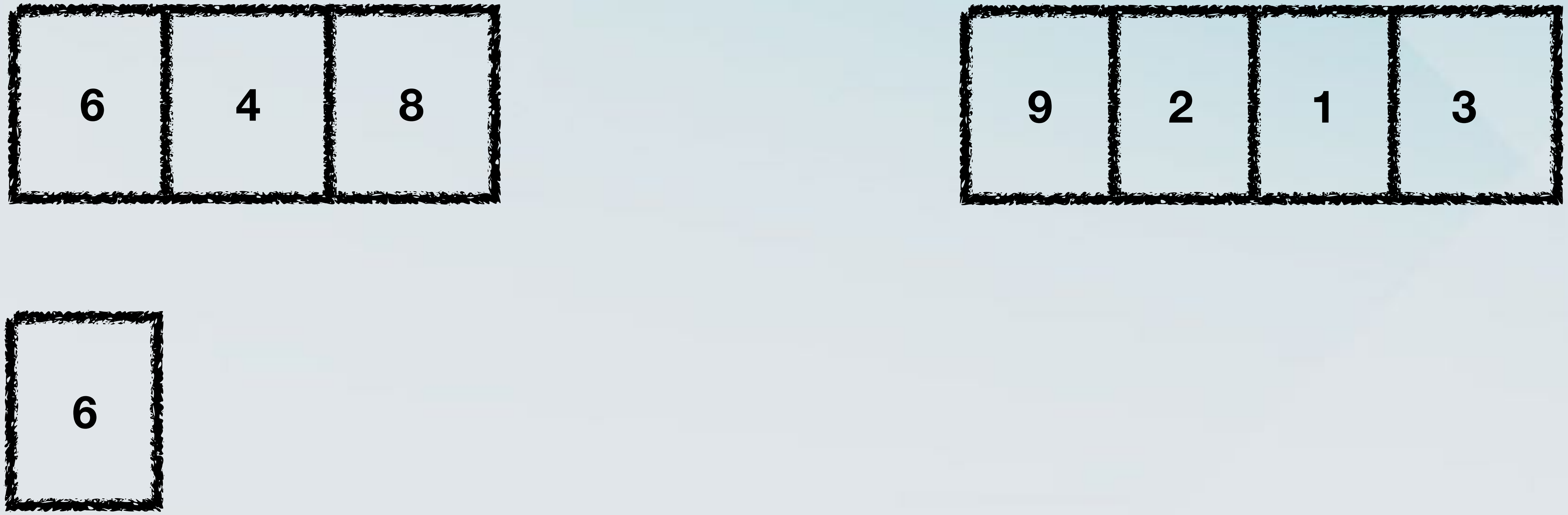

6	4	8

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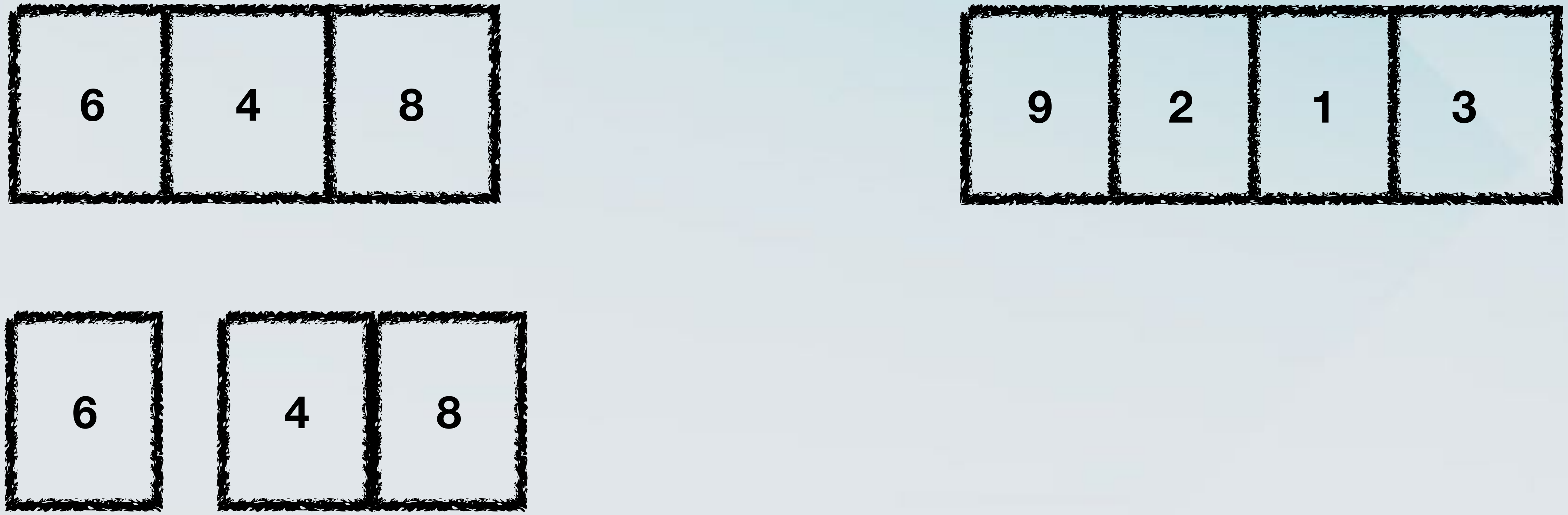
6

Mergesort example

divide



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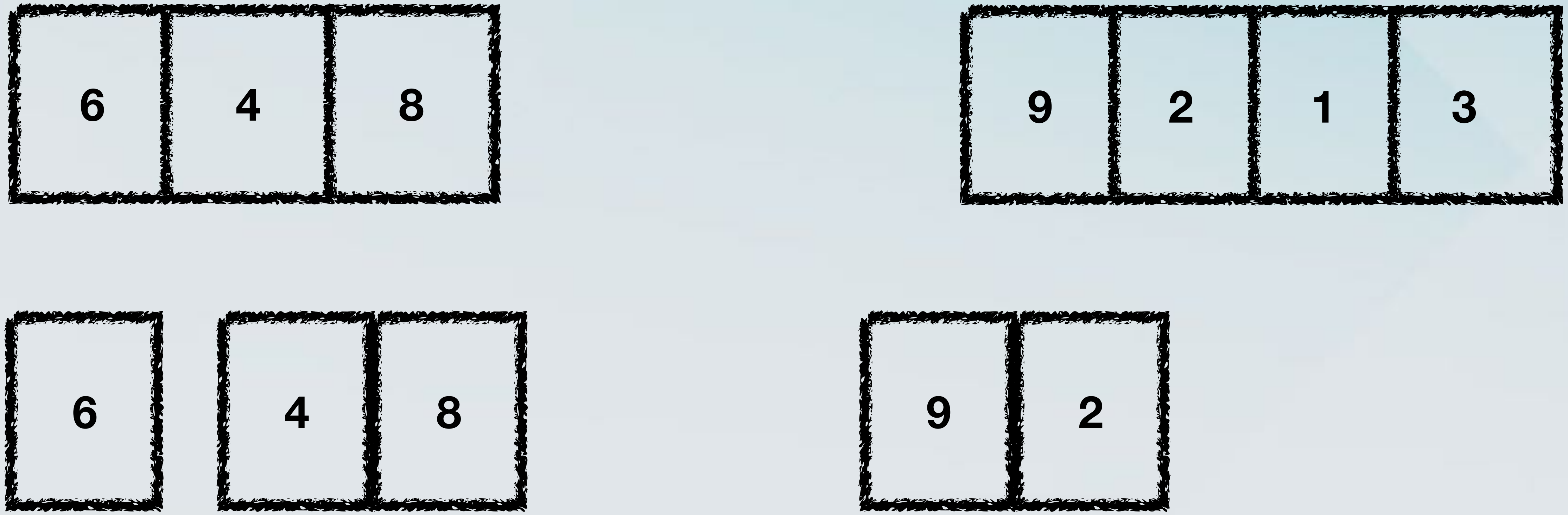

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9	2

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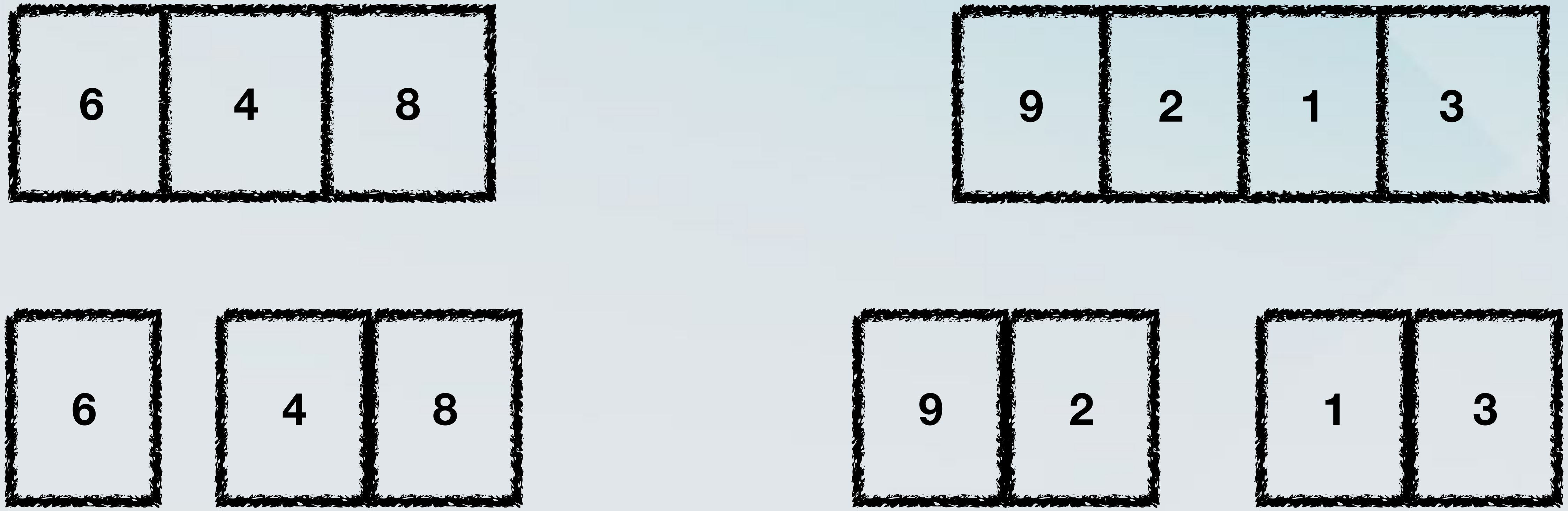

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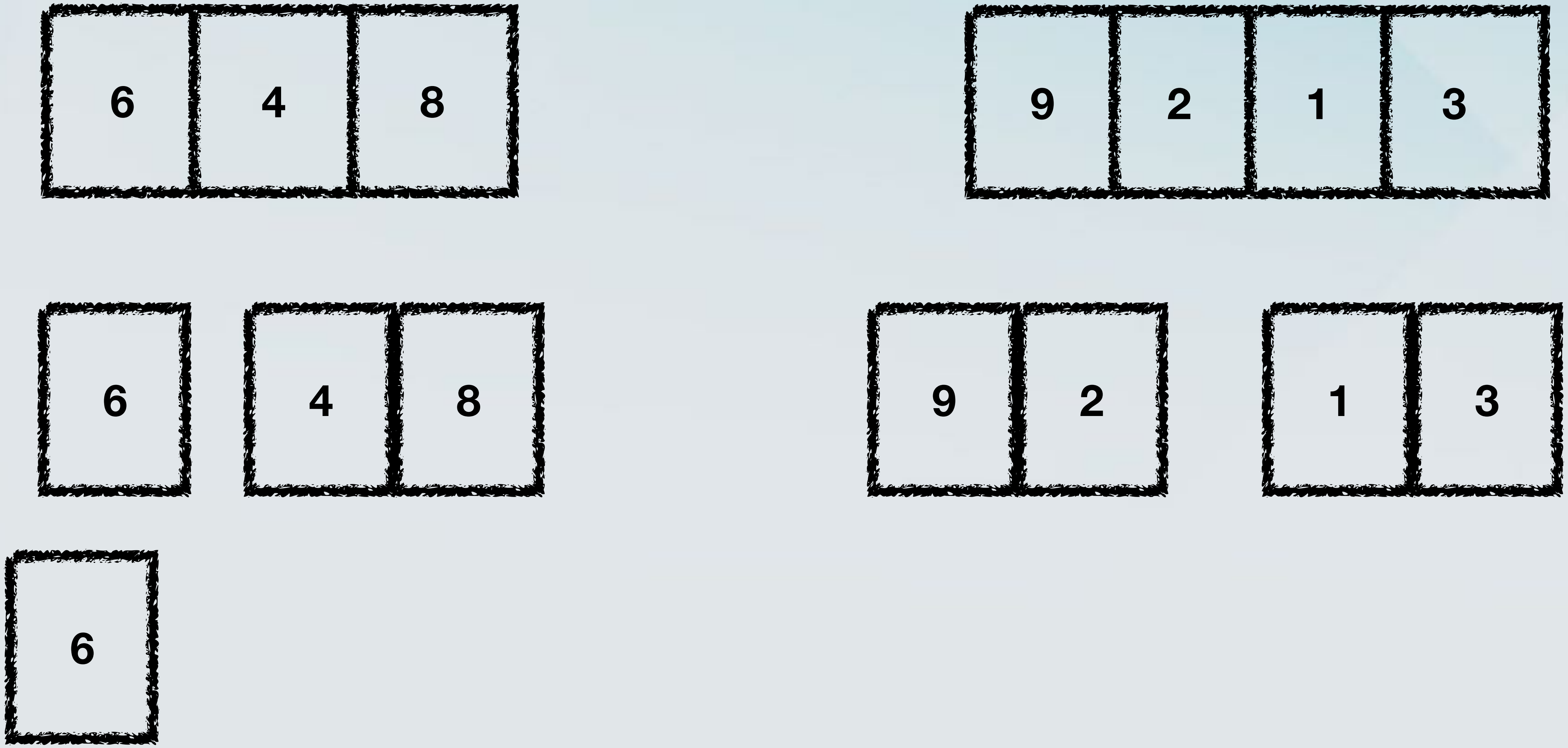

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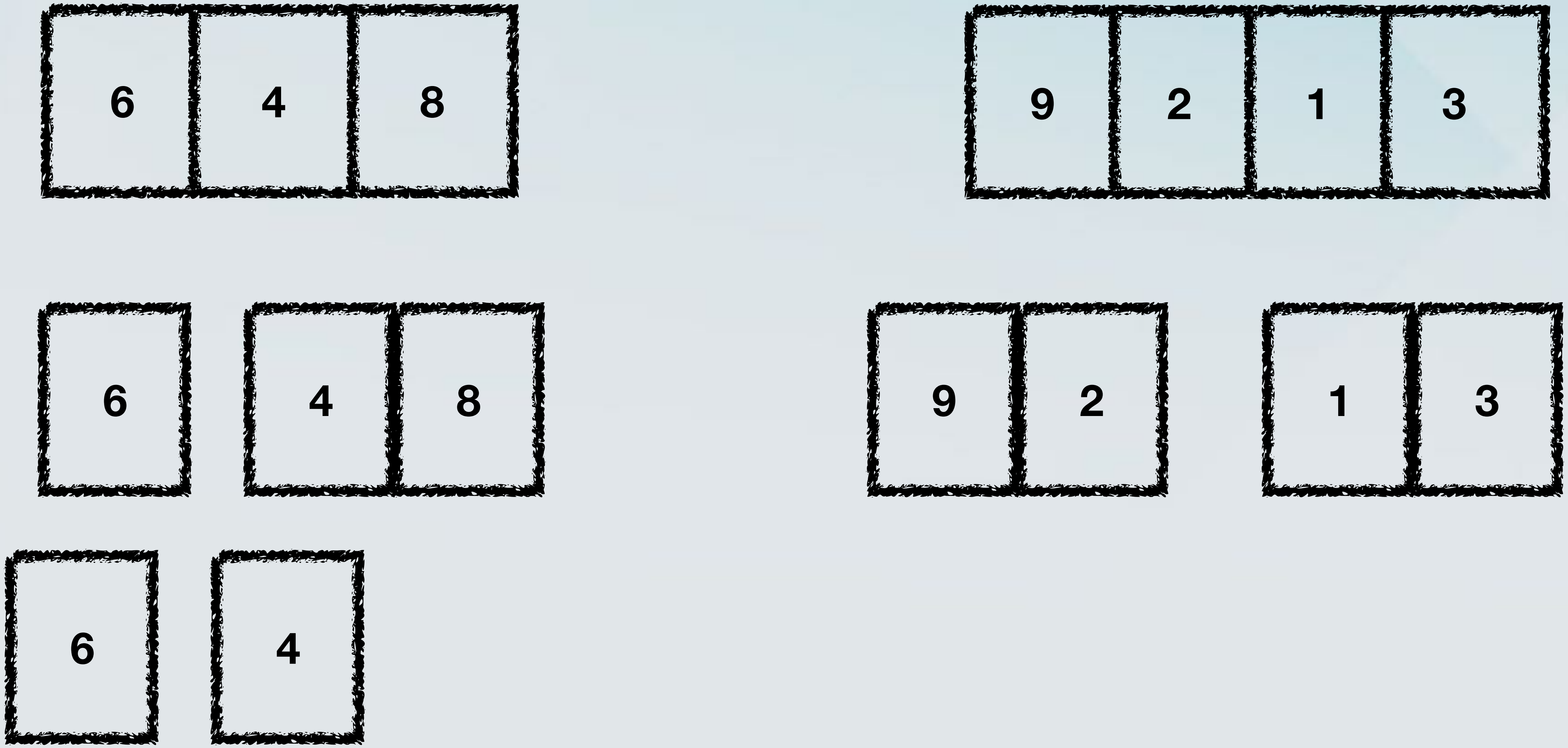

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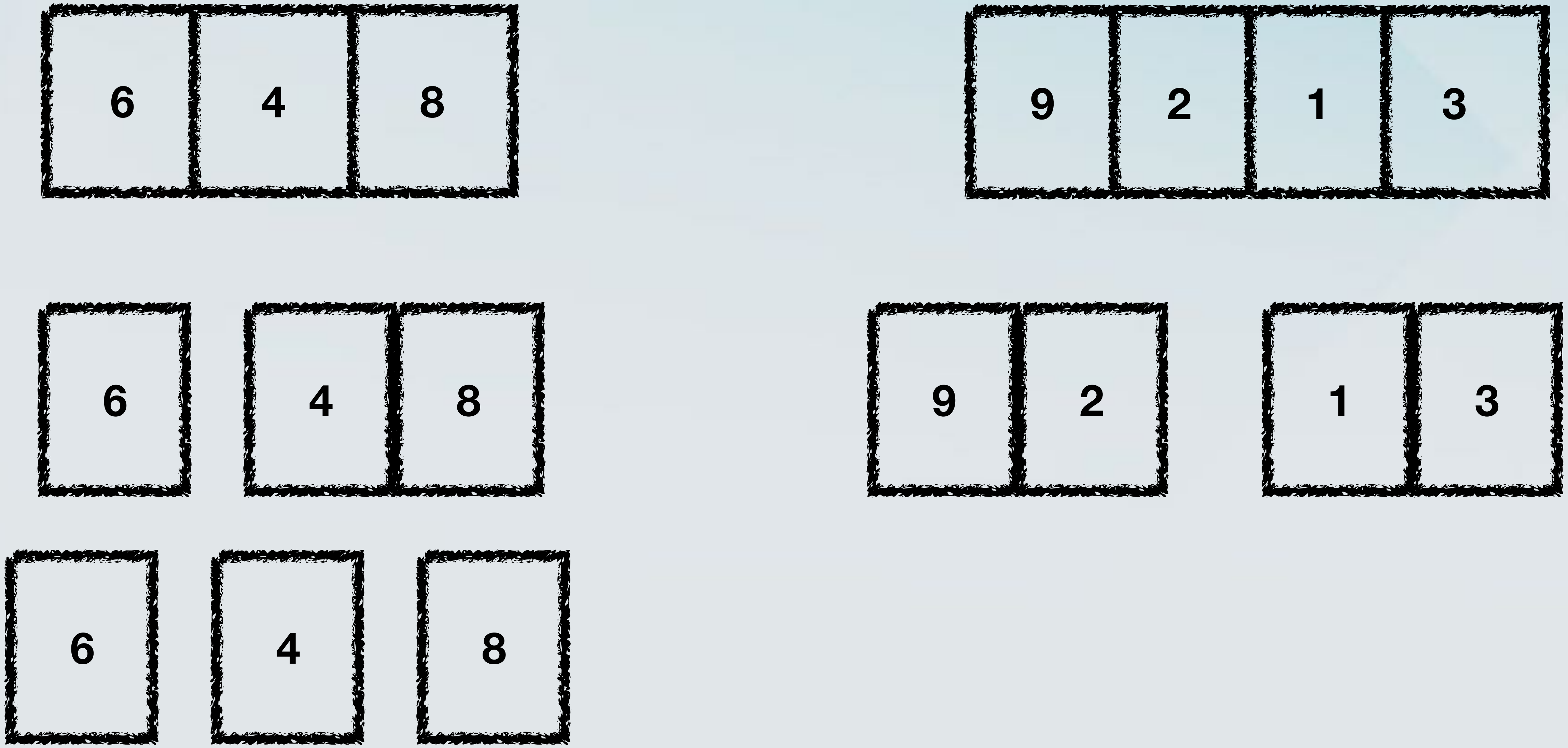

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1	3

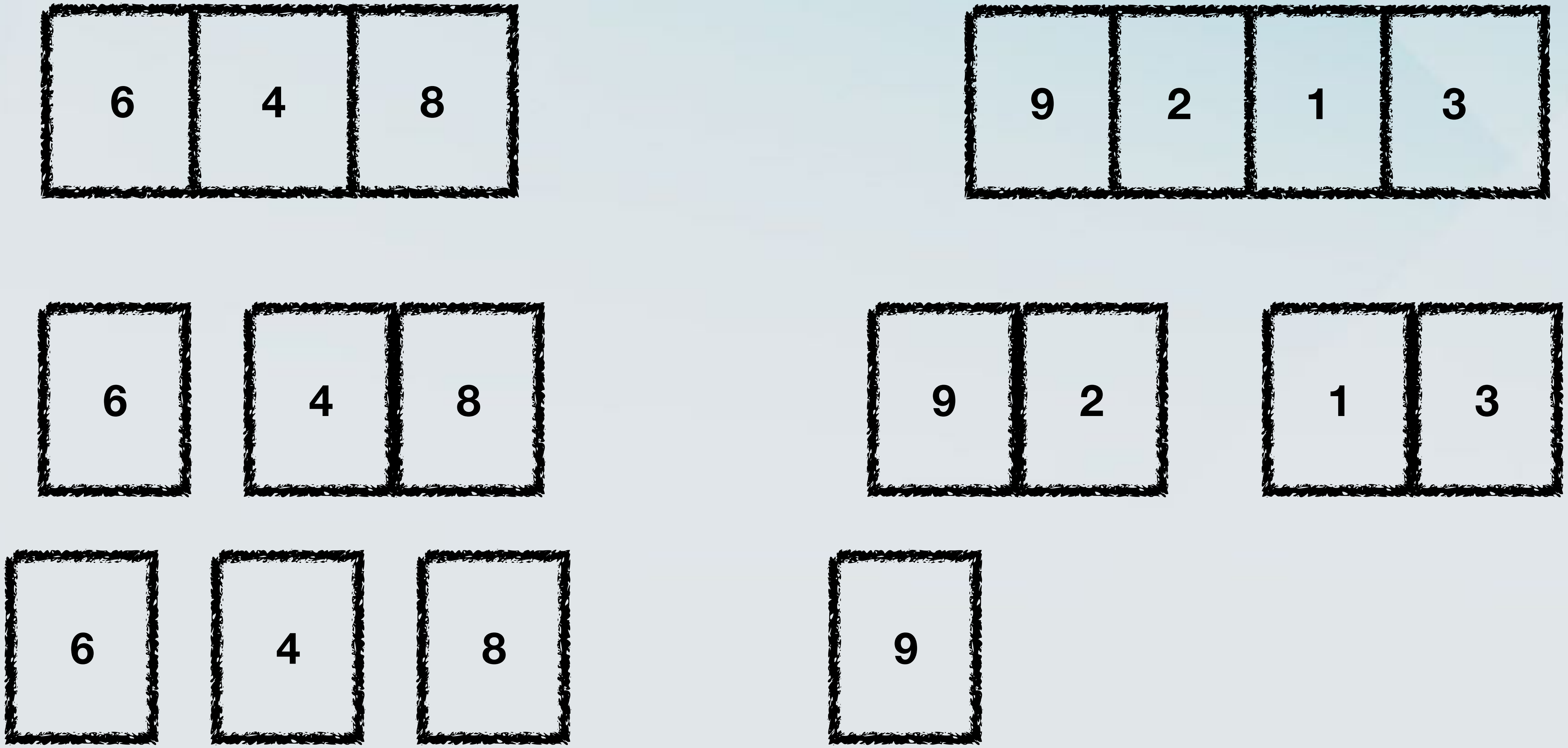
6

4

8

Mergesort example

divide



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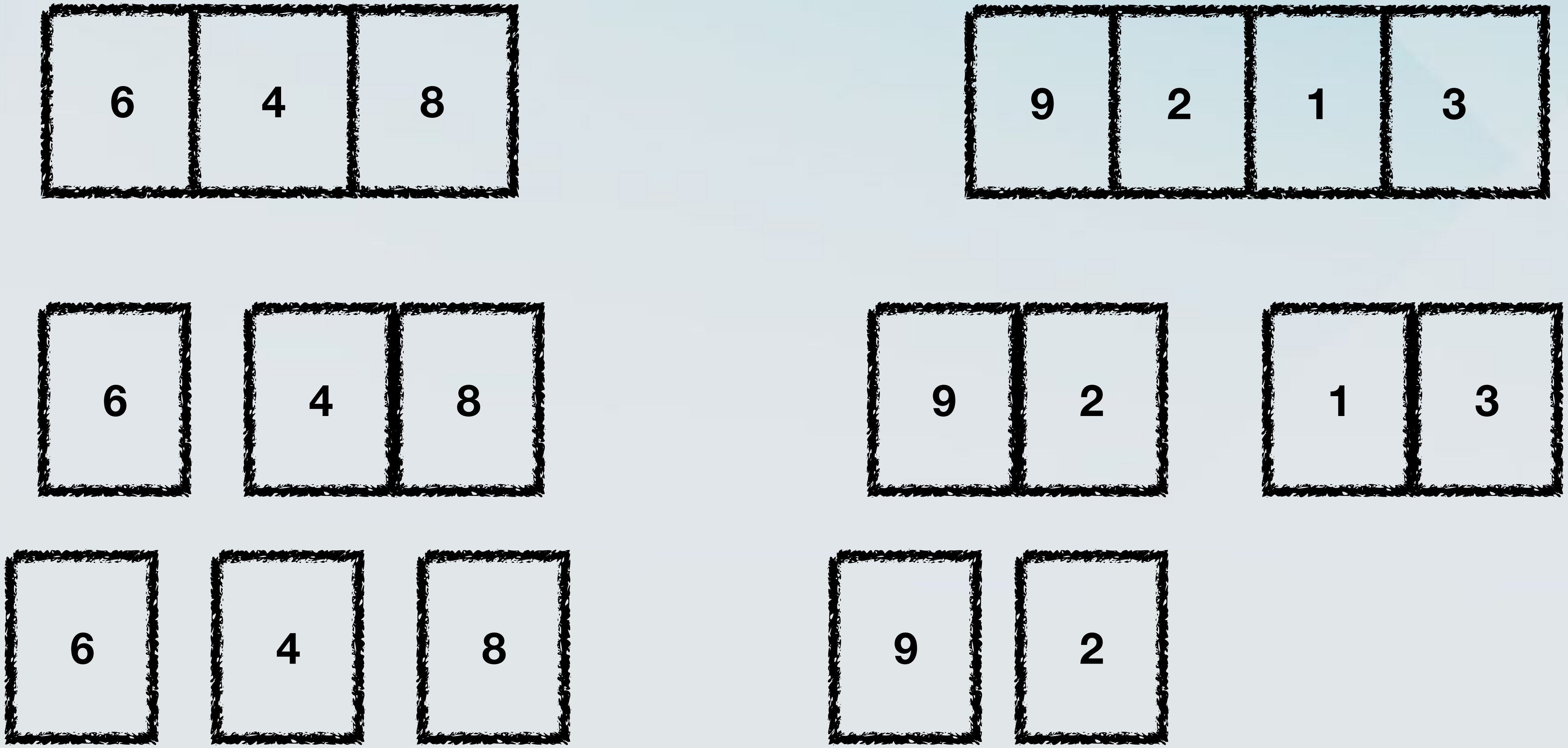

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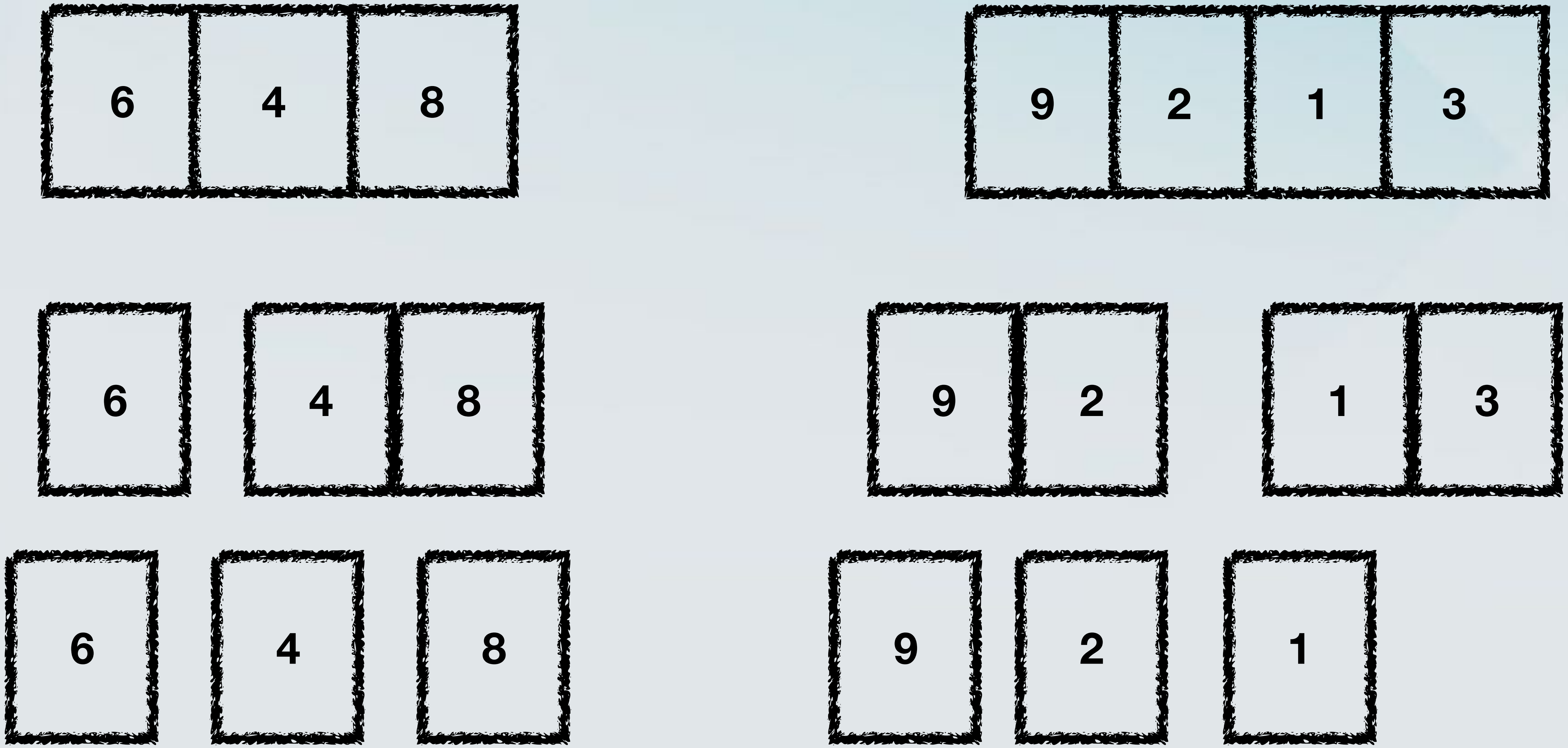

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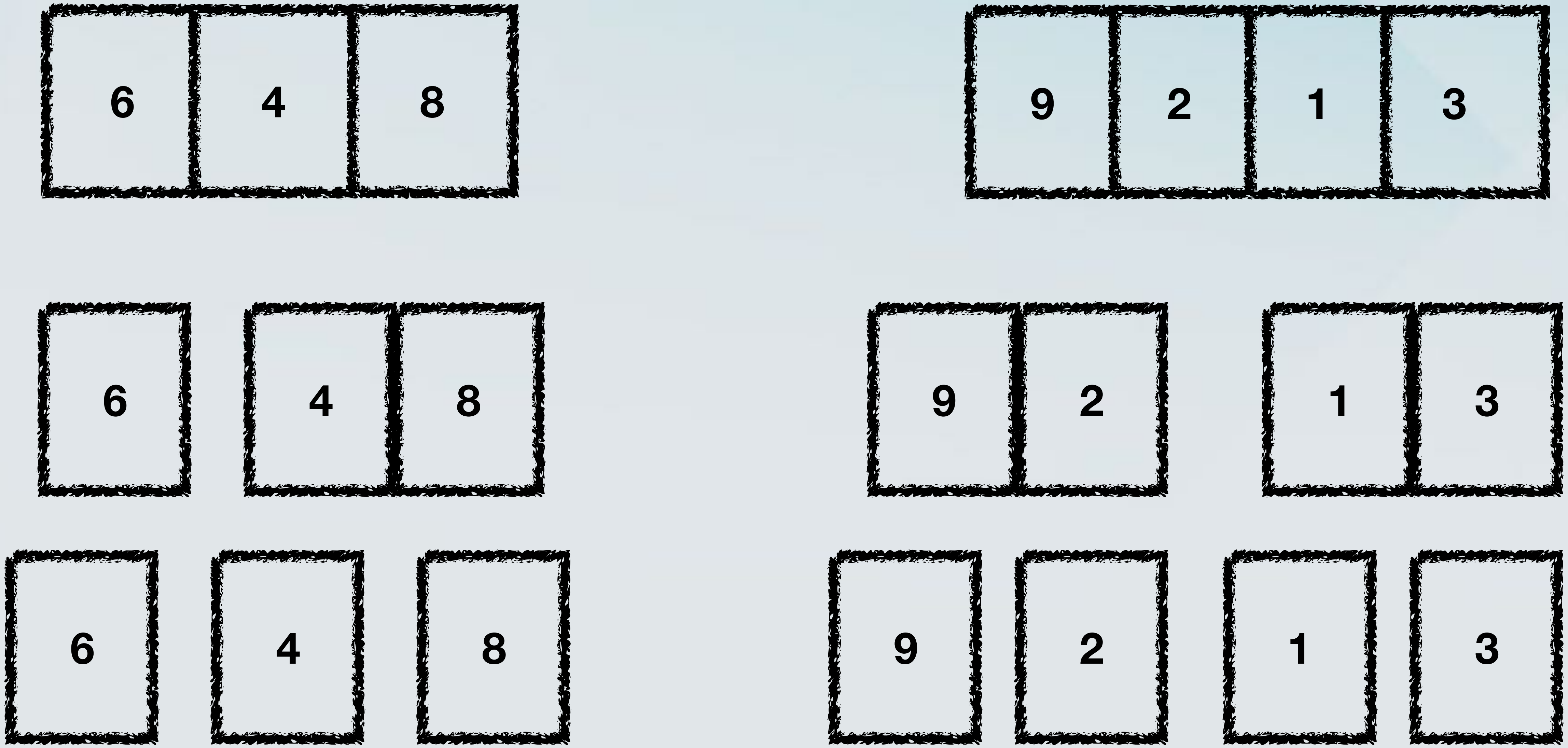


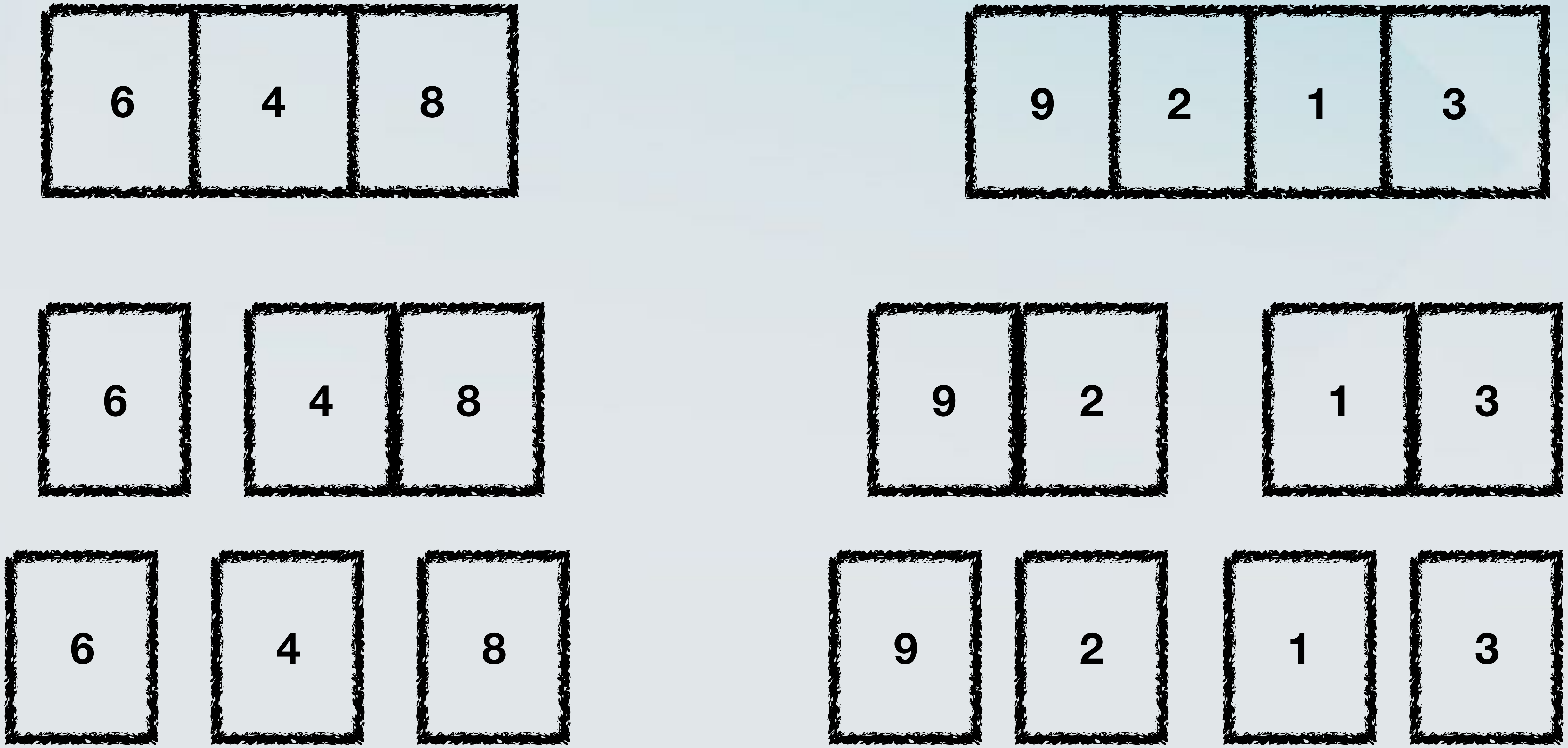
6 4 8   9 2 1 3

6 | 4 8   9 2 | 1 3

6 | 4 | 8   9 | 2 | 1 | 3

Mergesort example

divide merge



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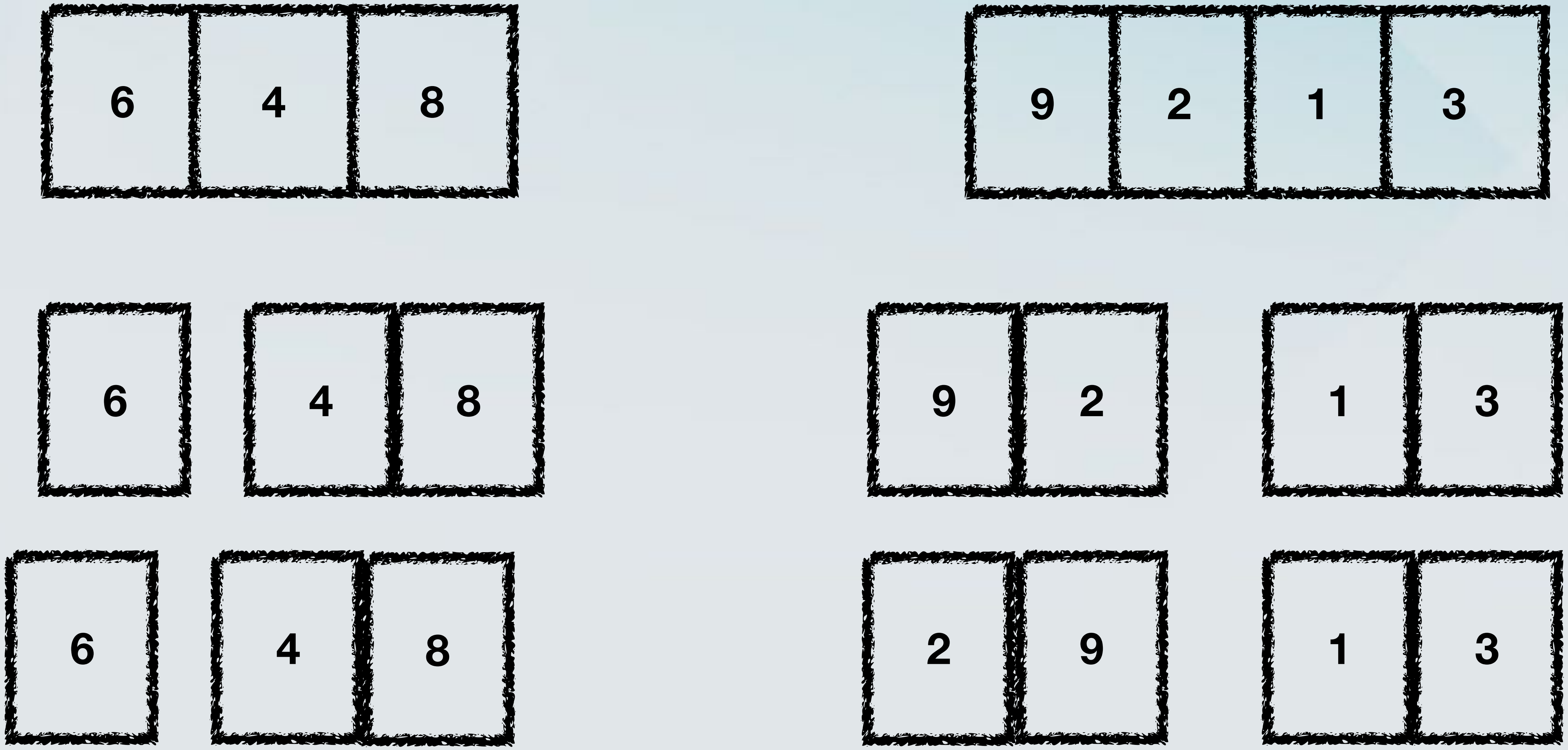

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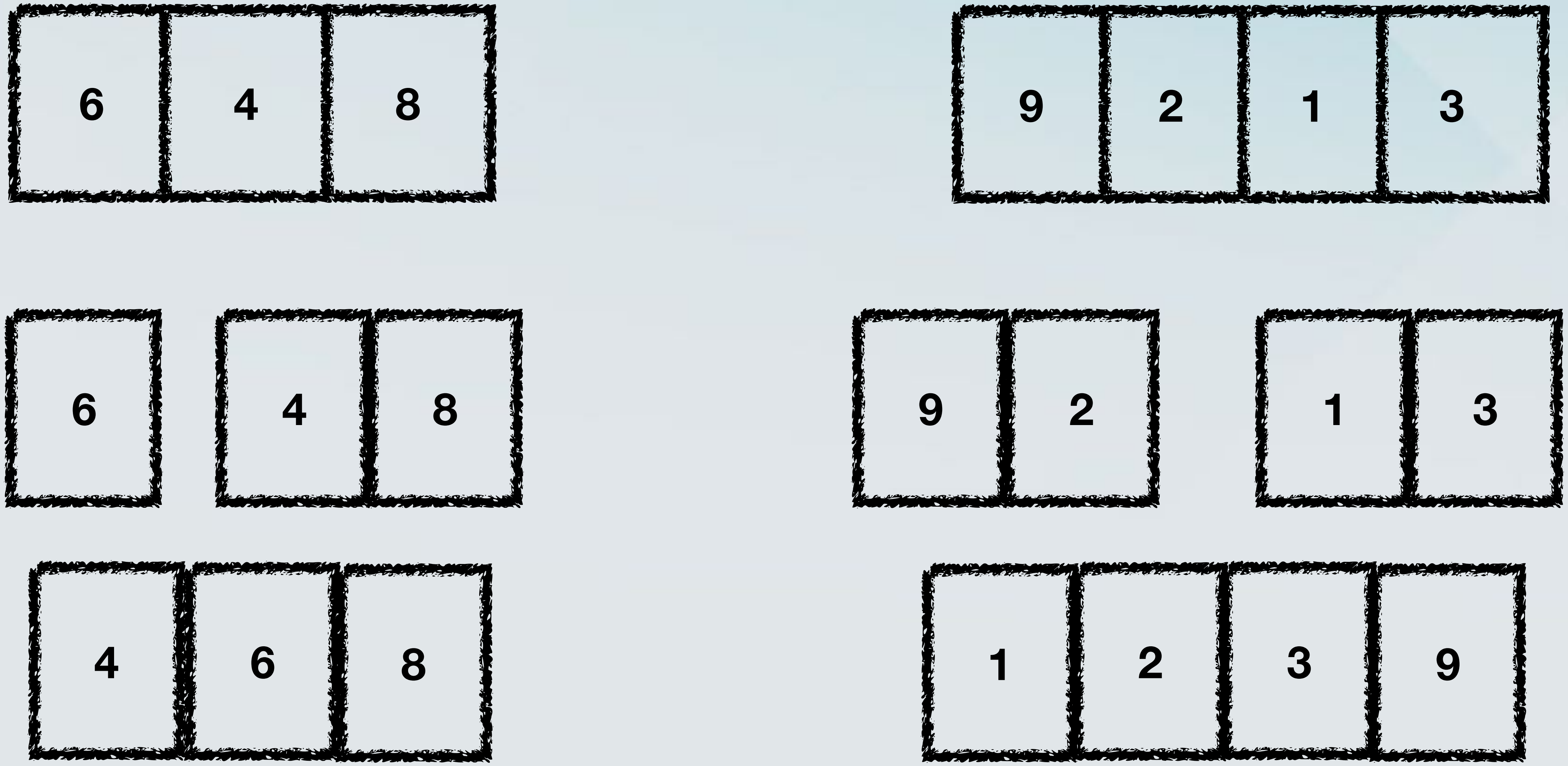

6	4	8

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6

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4	6	8

1	2	3	9

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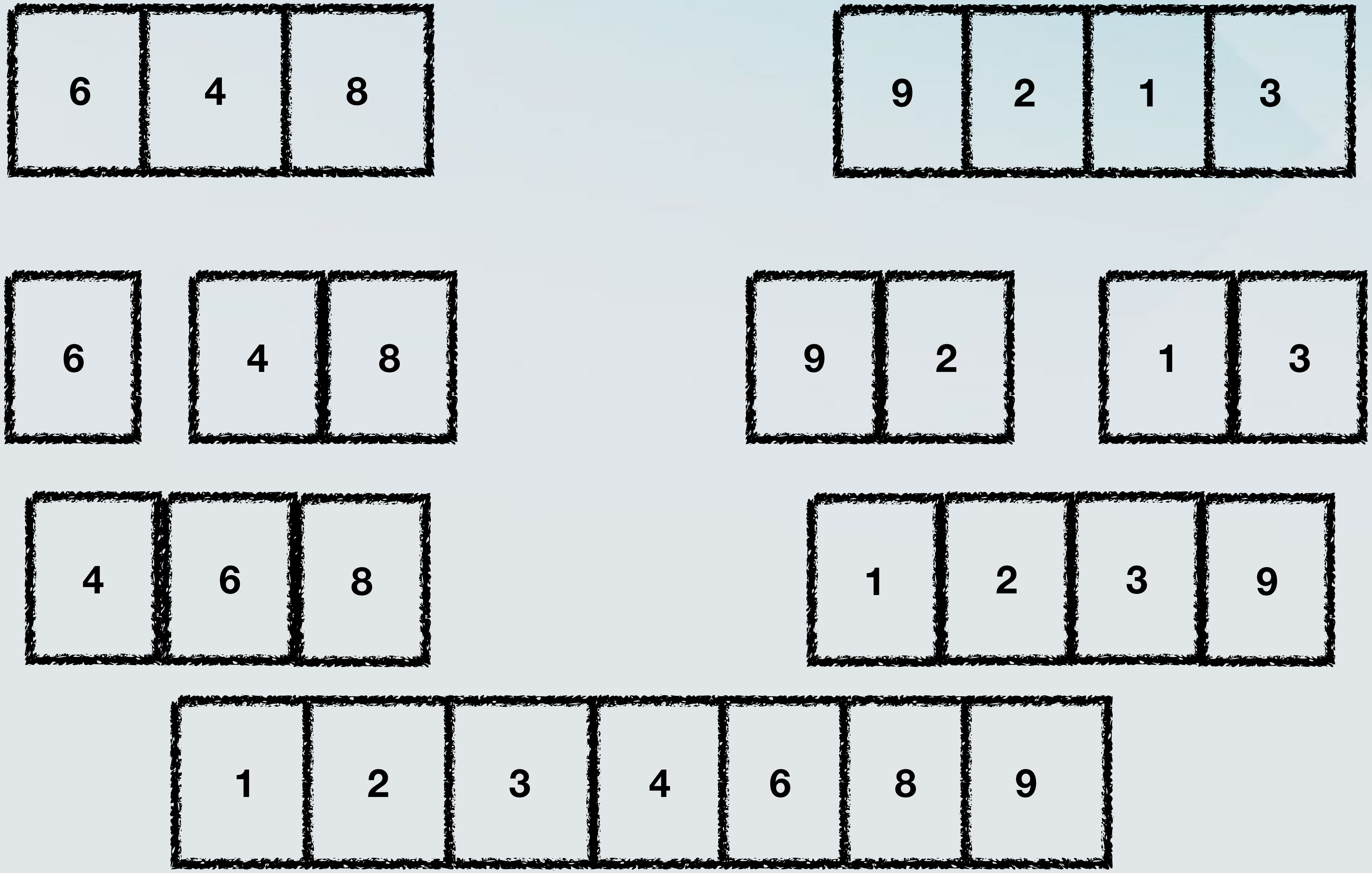

6 4 8   9 2 1 3

6   4 8   9 2   1 3

4 6 8   1 2 3 9

1 2 3 4 6 8 9

Running time of Mergesort

- We could guess the running time and prove it using induction, as we saw in the previous lecture.
- Instead, we will try to "figure out" what the running time should be.
- We will use the method of **recursion trees.**
- The running time is generally

$$T(n) = 2T(n/2) + f(n)$$

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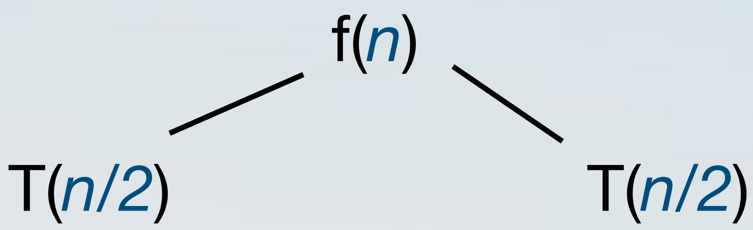

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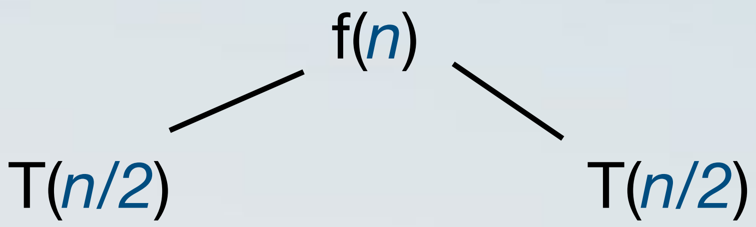
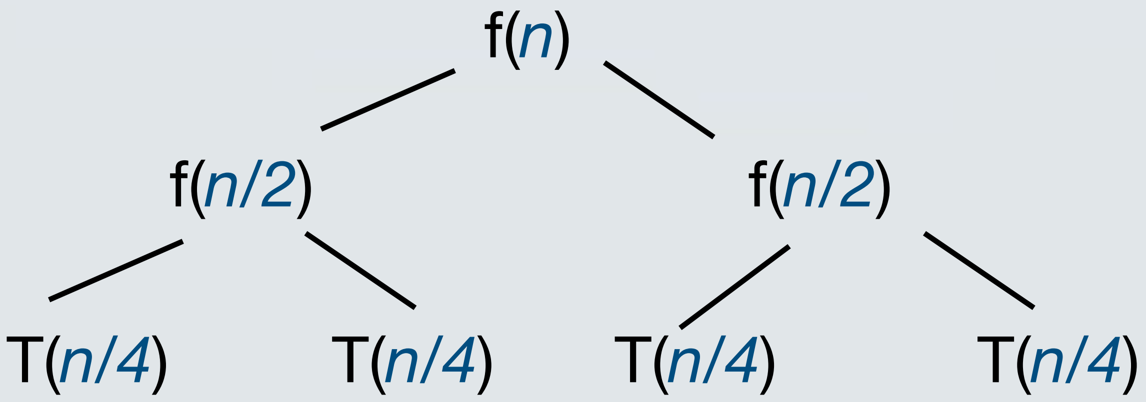


$f(n)$

$T(n/2)$ $T(n/2)$

Second iteration: Price of $f(n/2)$ for each subproblem, plus the cost of two subproblems of size n/4



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$T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$

Running time of Mergesort

- In total, there will be log n + 1 levels (input halved every time).
  - Level 0 has cost $C_0(n) = f(n)$
  - Level 1 has cost $C_1(n) = 2f(n/2)$
  - Level 2 has cost $C_2(n) = 4f(n/4)$
  - Level j has cost $C_j(n) = 2^j f(n/2^j)$
  - The last level has cost $f(n)$

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- It also holds that $C_j(n) = 2^j f(n/2^j)$ is at most dn for some large enough d.

- The overall running time is **O(n log n)**.

Sorting with Quicksort

The Quicksort algorithm

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  - This is done via the **Partition** procedure.

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    - This is done via the **Partition** procedure.

- Then it calls itself recursively.

- The two parts are joined, but this is trivial.

The Partition procedure

```
Procedure Partition(A[$i,\dots,j$])

    Choose a pivot element x of A

    $k = i-1$

    For $h = i$ to $j-1$ do

        If A[$h$] $\leq$ x
            $k = k + 1$
            Swap A[$k$] with A[$h$]

    Swap A[$k+1$] with A[$j$]

Return $k+1$
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Correctness of **Partition**:

(CLRS p. 171-173)

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    Swap A[$k+1$] with A[$j$]

Return $k+1$
```

Correctness of **Partition**:

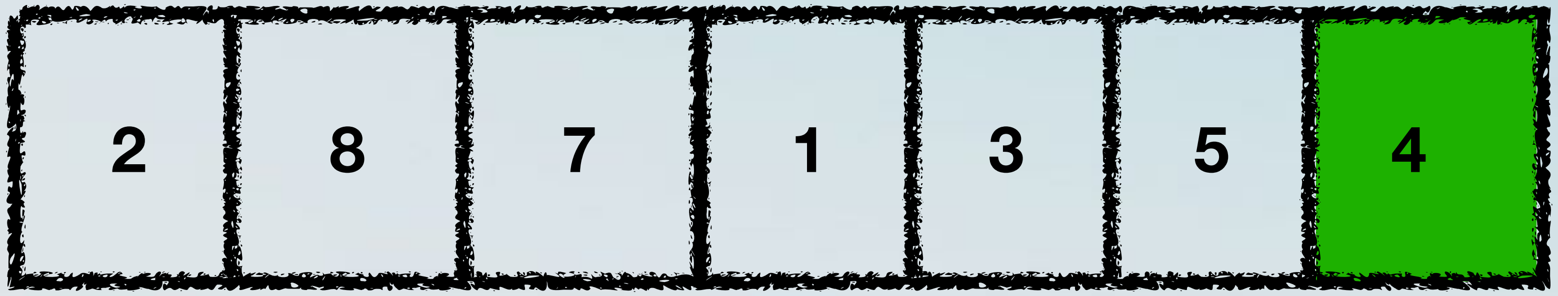
(CLRS p. 171-173)

Running time **O(n)**

The Quicksort algorithm

2	8	7	1	3	5	4

The Quicksort algorithm



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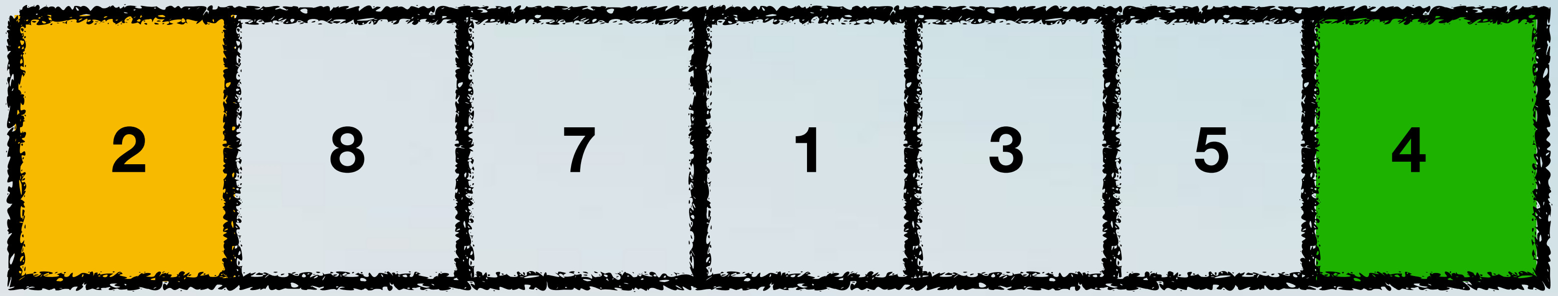

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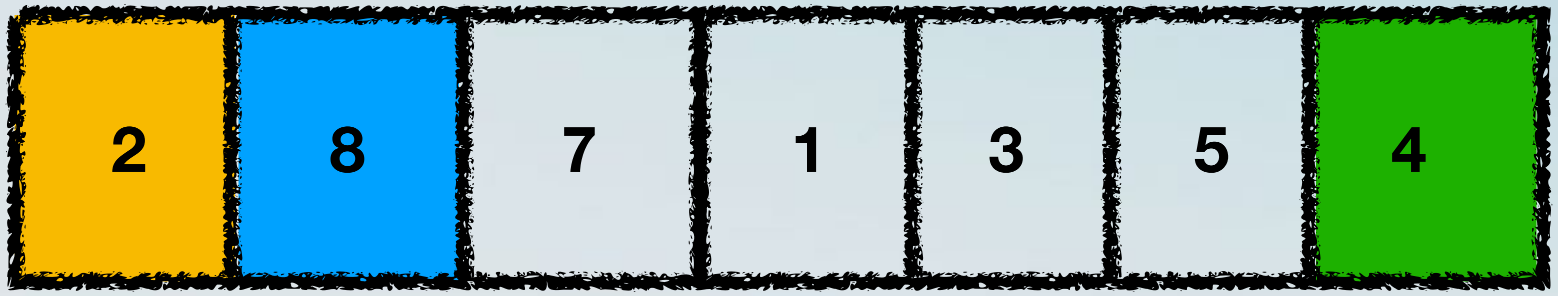

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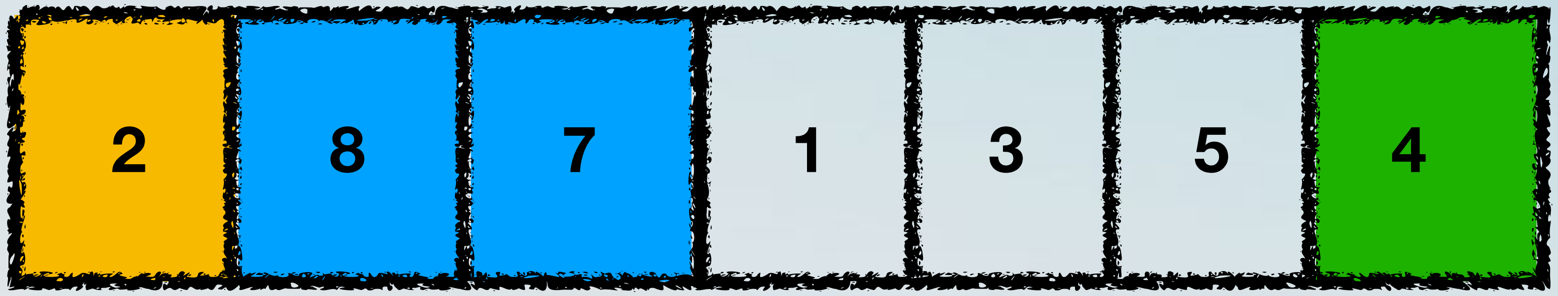

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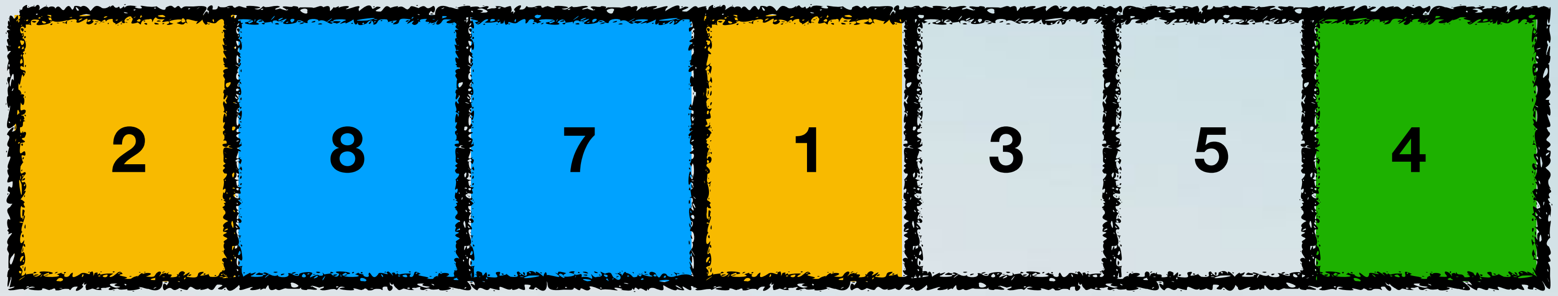

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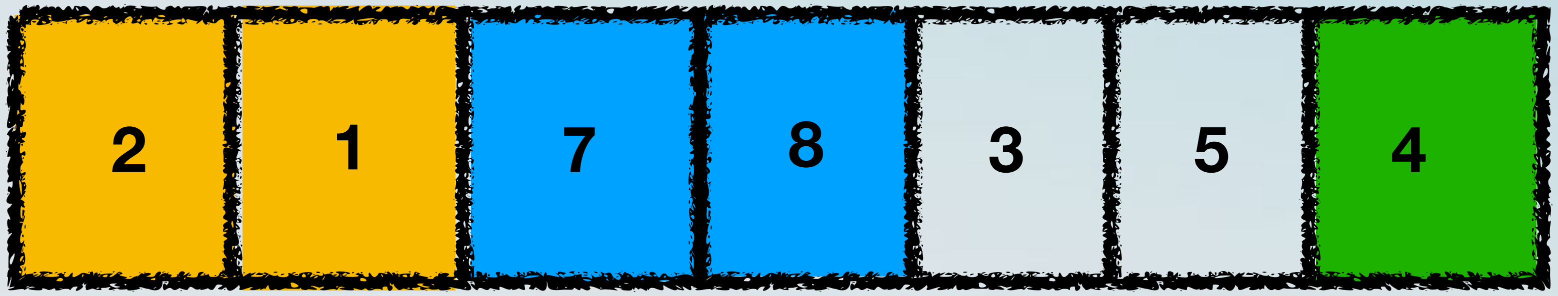

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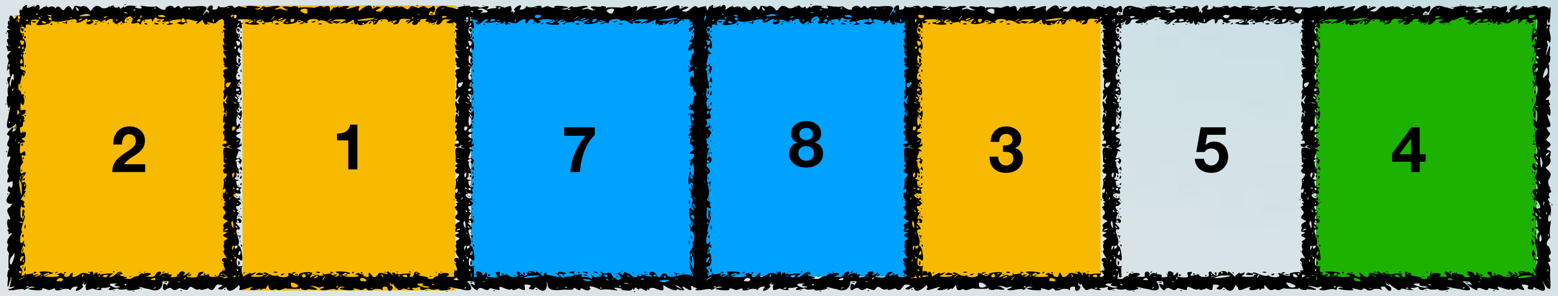

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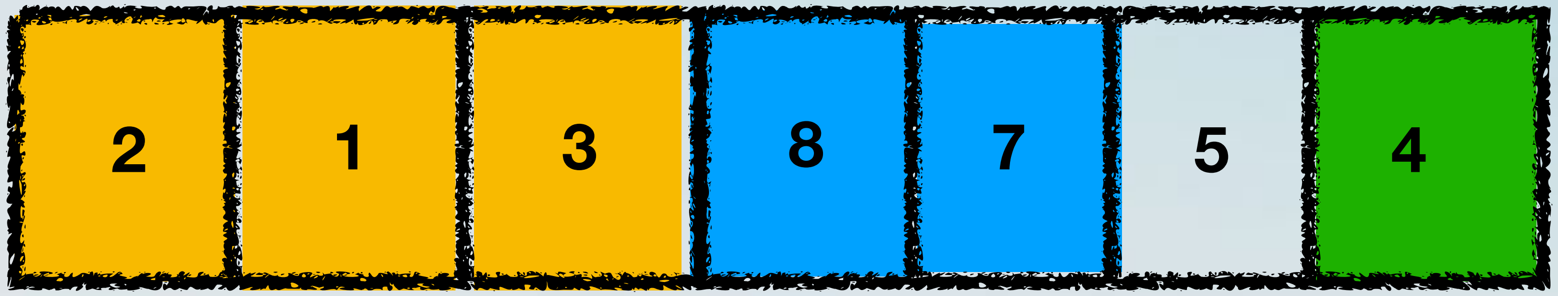

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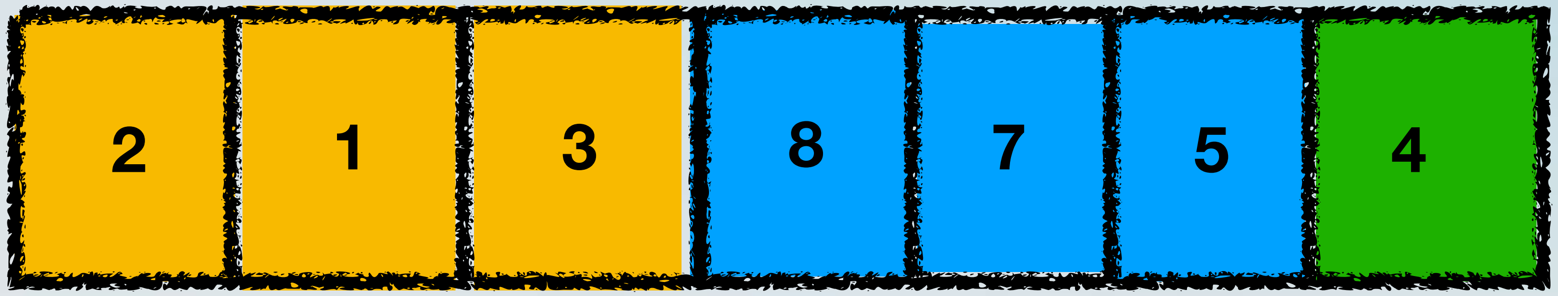

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The **Quicksort** algorithm

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Sort this using
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The Quicksort algorithm

Sort this using **Quicksort**

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```
Algorithm Quicksort(A[$i,\ldots,j$])

        $y$ = Partition(A[$i,\ldots,j$])
              Quicksort(A[$i,\ldots,y-1$])
              Quicksort(A[$y+1,\ldots,j$])
```

Running time of **Quicksort**

- Can it be as fast as **Mergesort**?
- Can it be slower than **Mergesort**?
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- This will depend on the pivot element!

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What is the worst possible running time?

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  $n_1 = n-1$
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  - $T(n) = \Theta(n^2)$

Running time of Quicksort

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Running time of **Quicksort**

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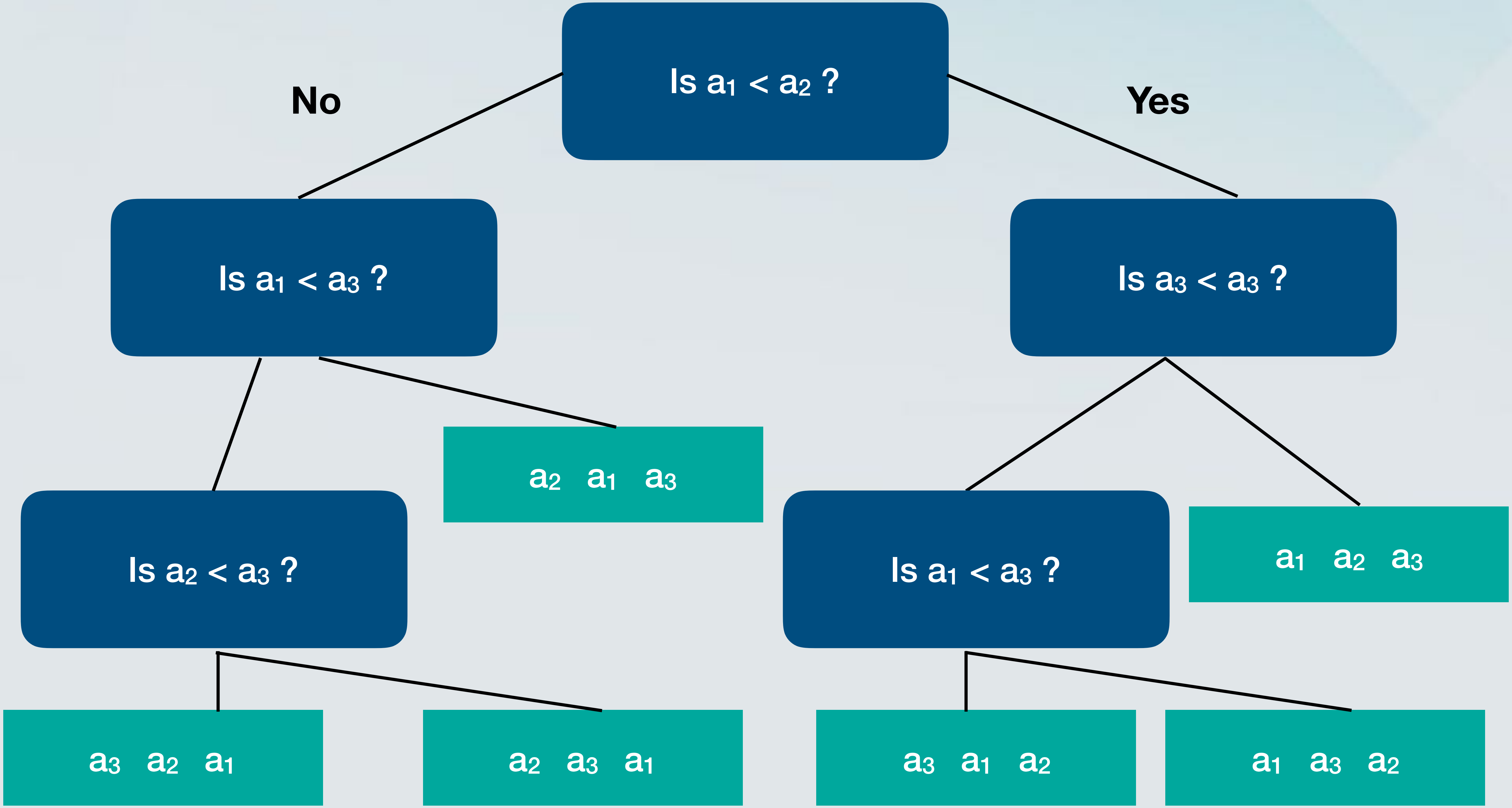
- In other words, we will prove that for any such algorithm, the running time is **Ω(n log n).**

Lower bound for (comparison-based) sorting



Is $a_1 < a_2$?

No

Yes

Is $a_1 < a_3$?

Is $a_3 < a_3$?

a_2 a_1 a_3

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a_1 a_2 a_3

a_3 a_2 a_1

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  - A leaf is a permutation of $\{a_1, a_2, \dots, a_n\}$
  - Every possible permutation can appear as a leaf, since every possible permutation is a valid output.

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$$
\begin{aligned}
\log(n!) = \log(1 * 2 * \ldots * n) &= \log(1) + \log(2) + \ldots + \log(n) \\
&\geq \log(n/2) + \ldots + \log(n) \text{ (half)} \\
&\geq \log(n/2) + \ldots + \log(n/2) \\
&= (n/2) * \log(n/2)
\end{aligned}
$$

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- **Upper bound:** We construct an algorithm that has performance $O(g(n))$ for criterion A.
- **Lower bound:** We show that for any algorithm, the performance for criterion A is $\Omega(g(n))$.

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- How do we find this function?
  - No easy answer!
  - We try to design algorithms which are as good as possible and when we feel that we can not improve more, we try to prove the matching lower bound.

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    - Stay tuned!