Advanced Algorithmic Techniques (COMP523)

Recursion and Divide and Conquer Techniques #3

Recap and plan

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- **Last lecture:**
  - Sorting with the **MergeSort** algorithm.
  - Sorting with the **QuickSort** algorithm.
  - The limitations of comparison-based sorting.

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- **Last lecture:**
  - Sorting with the **MergeSort** algorithm.
  - Sorting with the **QuickSort** algorithm.
  - The limitations of comparison-based sorting.
- **This lecture:**
  - Finding the closest pair of points.
  - Integer Multiplication.

Quick Recap

- **Searching:**
  - LinearSearch: Time O(n), (Aux.) Memory O(1)
  - BinarySearch: Time O(log n), (Aux.) Memory O(log n)
- **Sorting:**
  - InsertionSort: Time $O(n^2)$, (Aux.) Memory O(1)
  - MergeSort: Time O(n) log n, (Aux.) Memory **?**
  - QuickSort: Time $O(n^2)$, (Aux.) Memory **?**
- **Majority:**
  - General array: Time O(n), (Aux.) Memory **?**
  - Sorted array: Time O(log n), (Aux.) Memory **?**

Finding the closest pair of points

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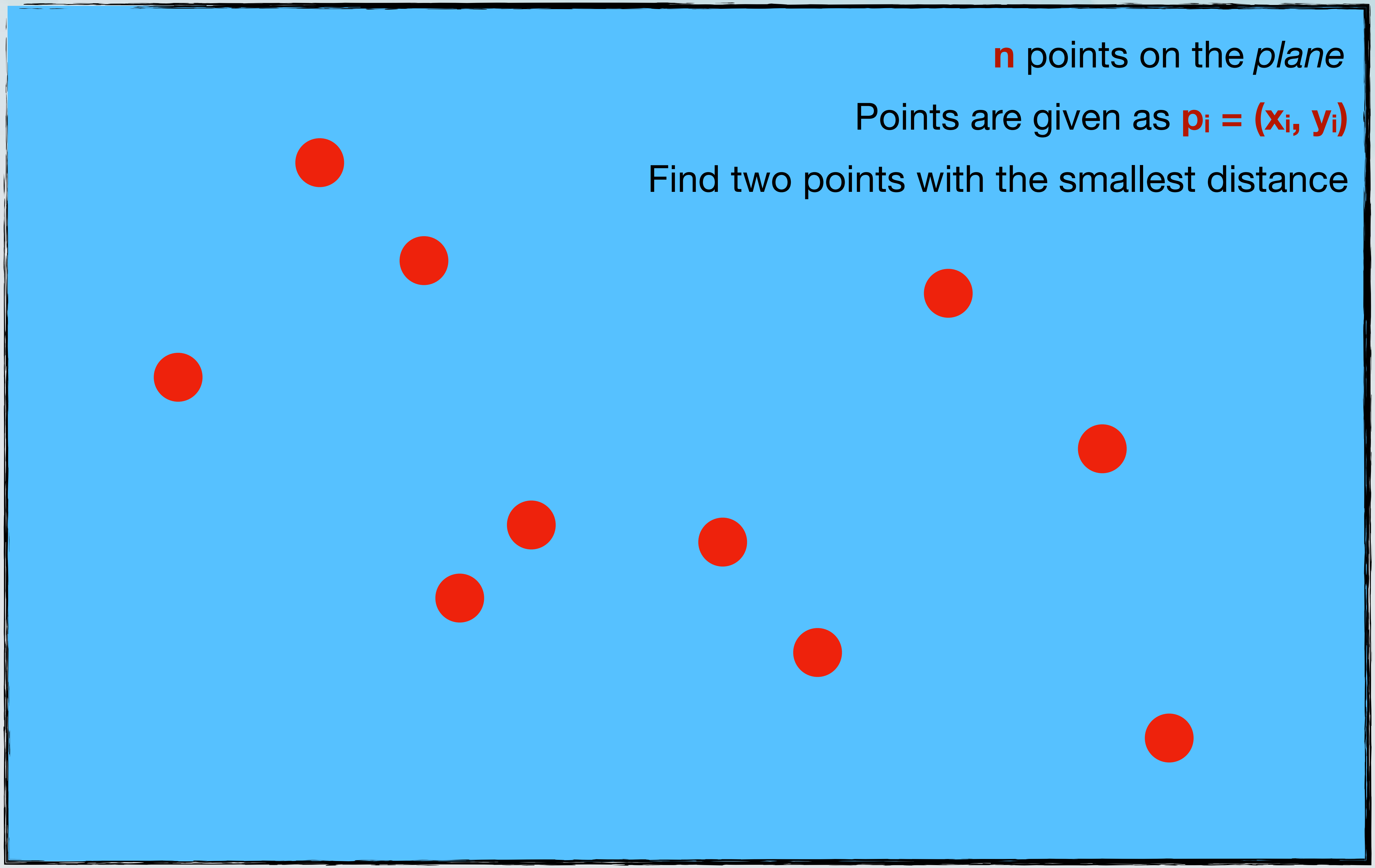
n points on the *plane*

Points are given as $p_i = (x_i, y_i)$

Find two points with the smallest distance



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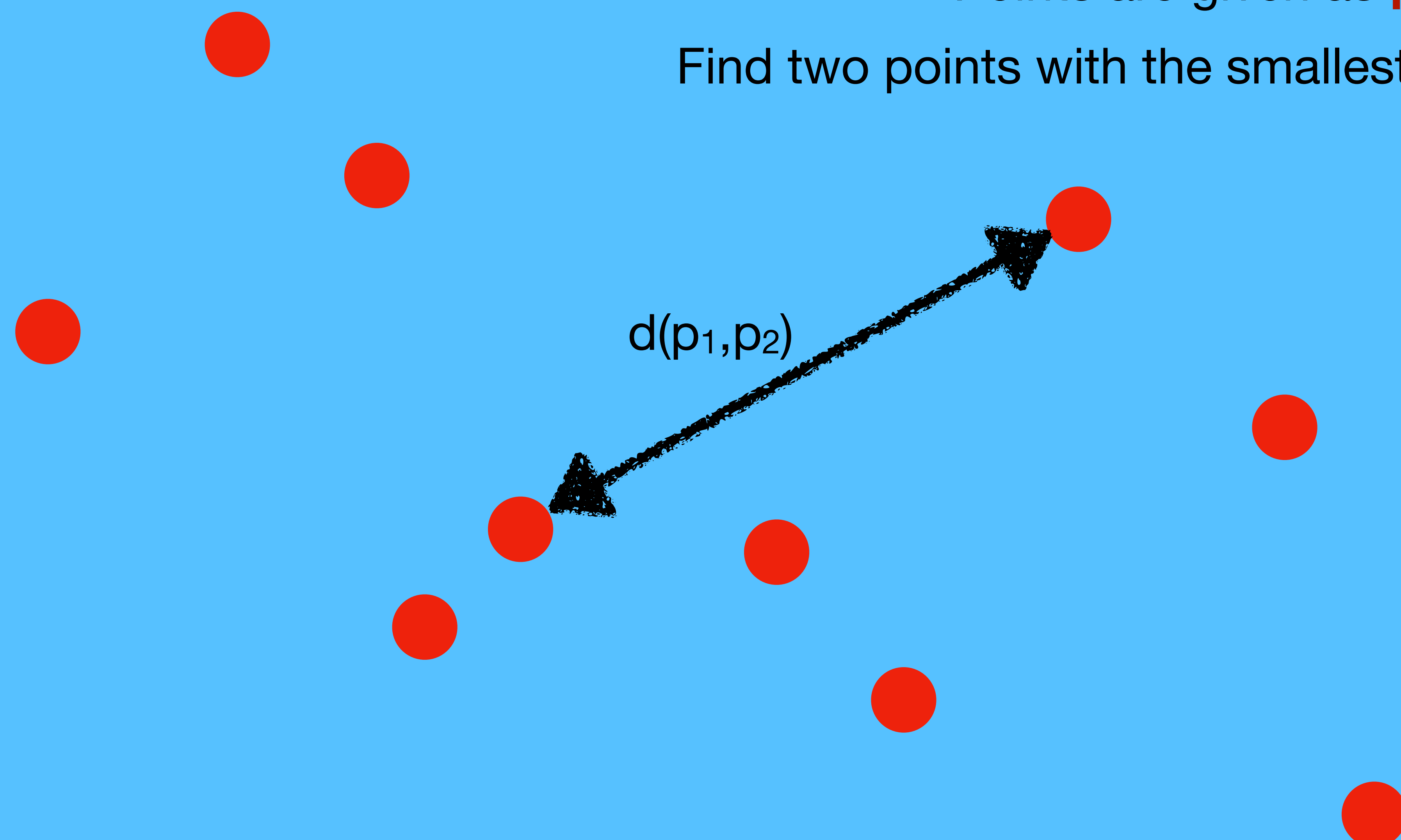

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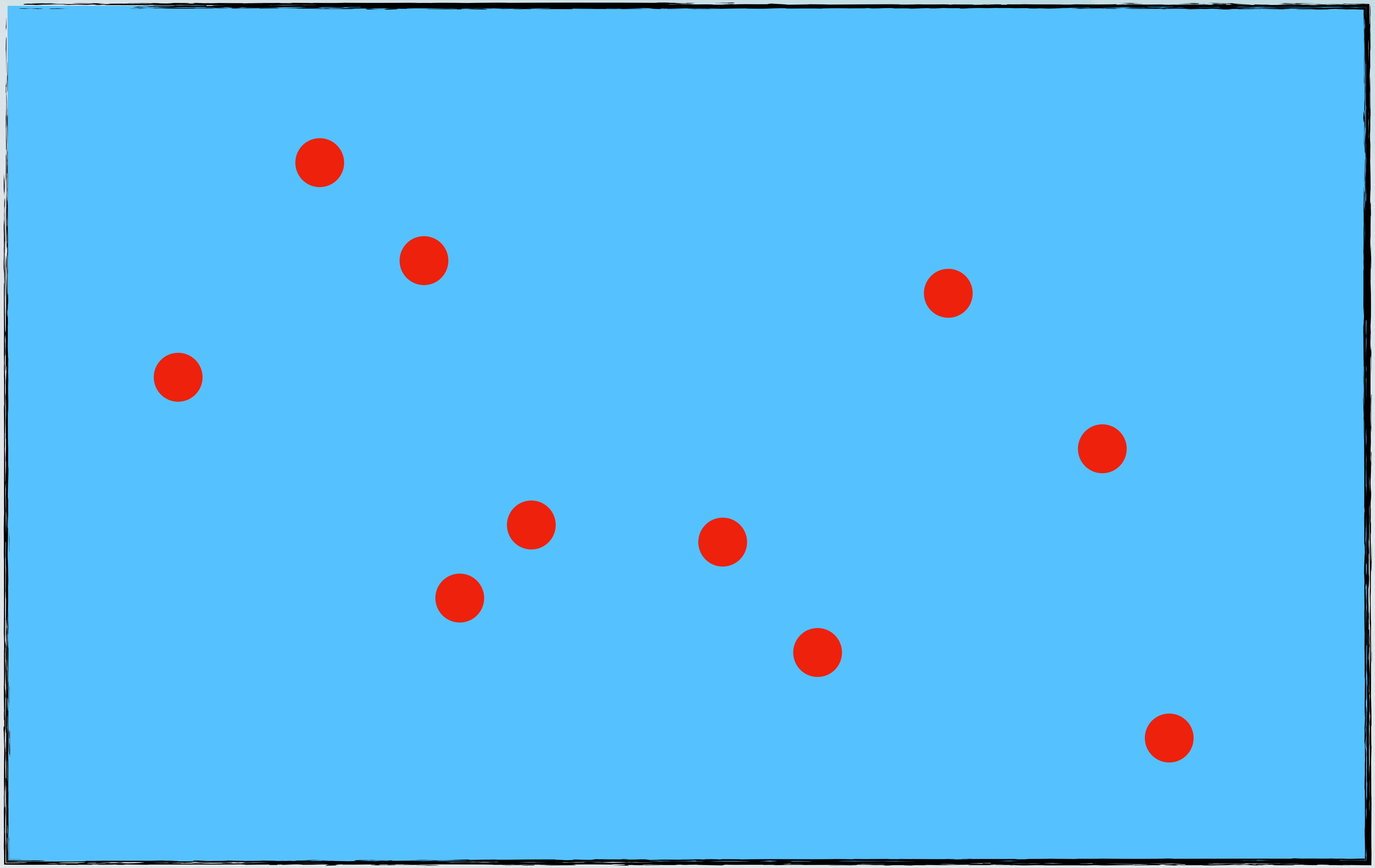
- Output two points that have the smallest distance.
- Running time?
  - **$\Omega(n^2)$**
- Can we do better?

Warmup: Points on the line



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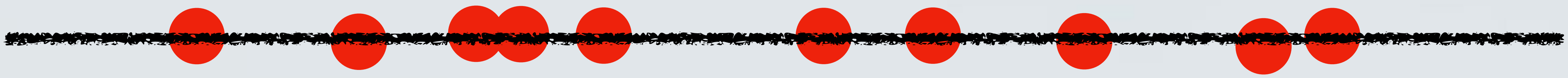

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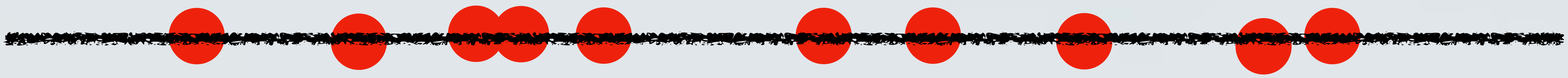

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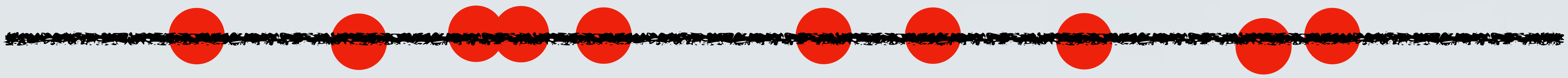

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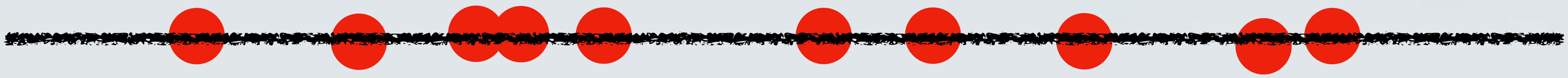

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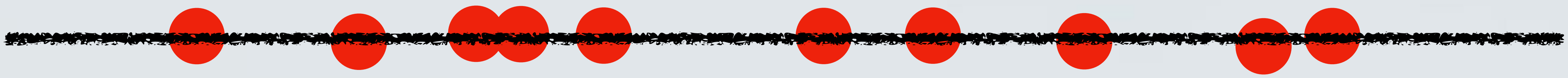

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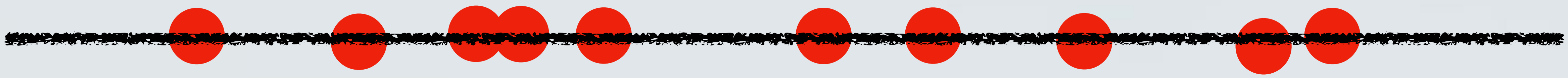
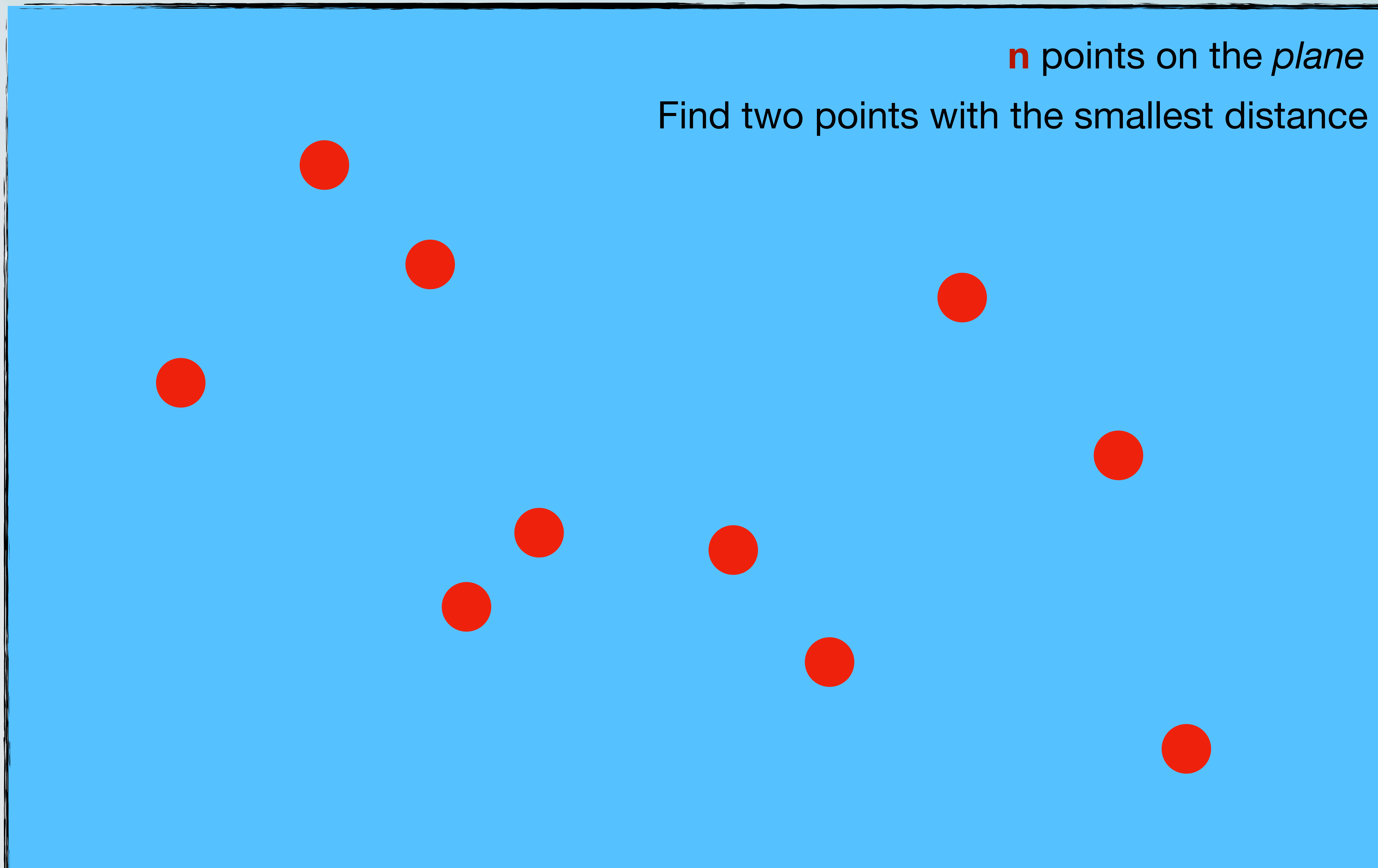


What is the worst-case running time?

Can be done in **O(n log n)**

Finding the closest pair of points



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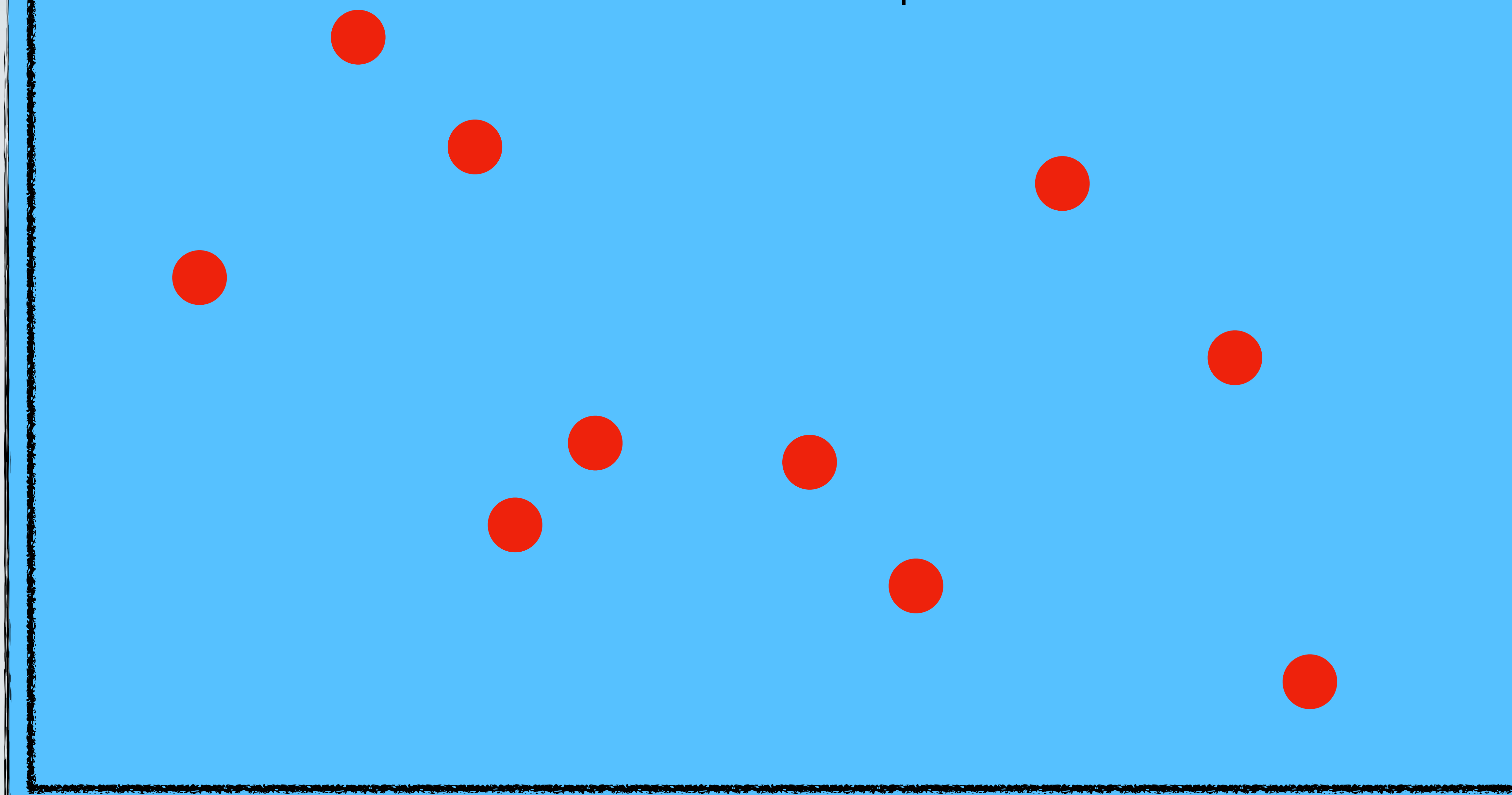

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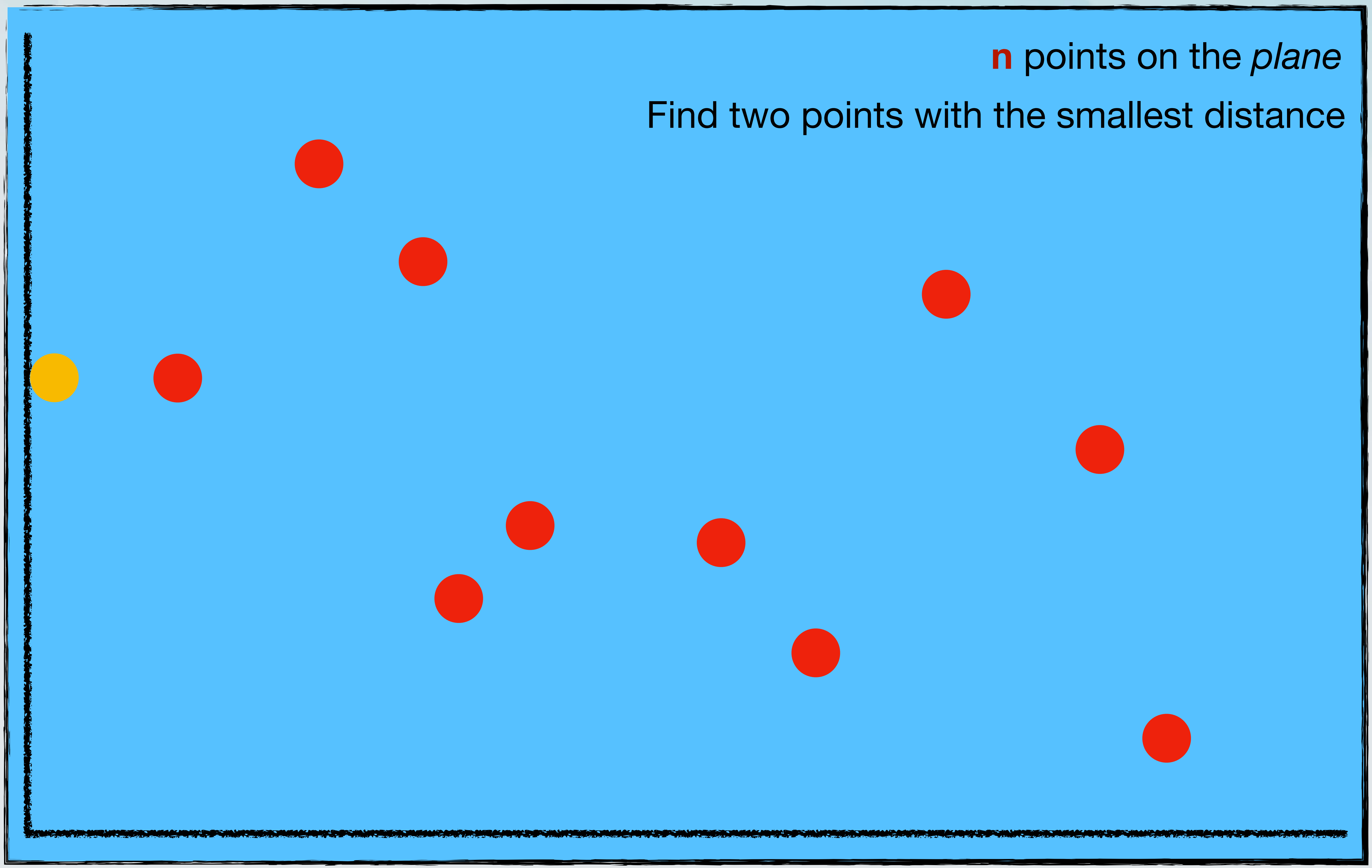

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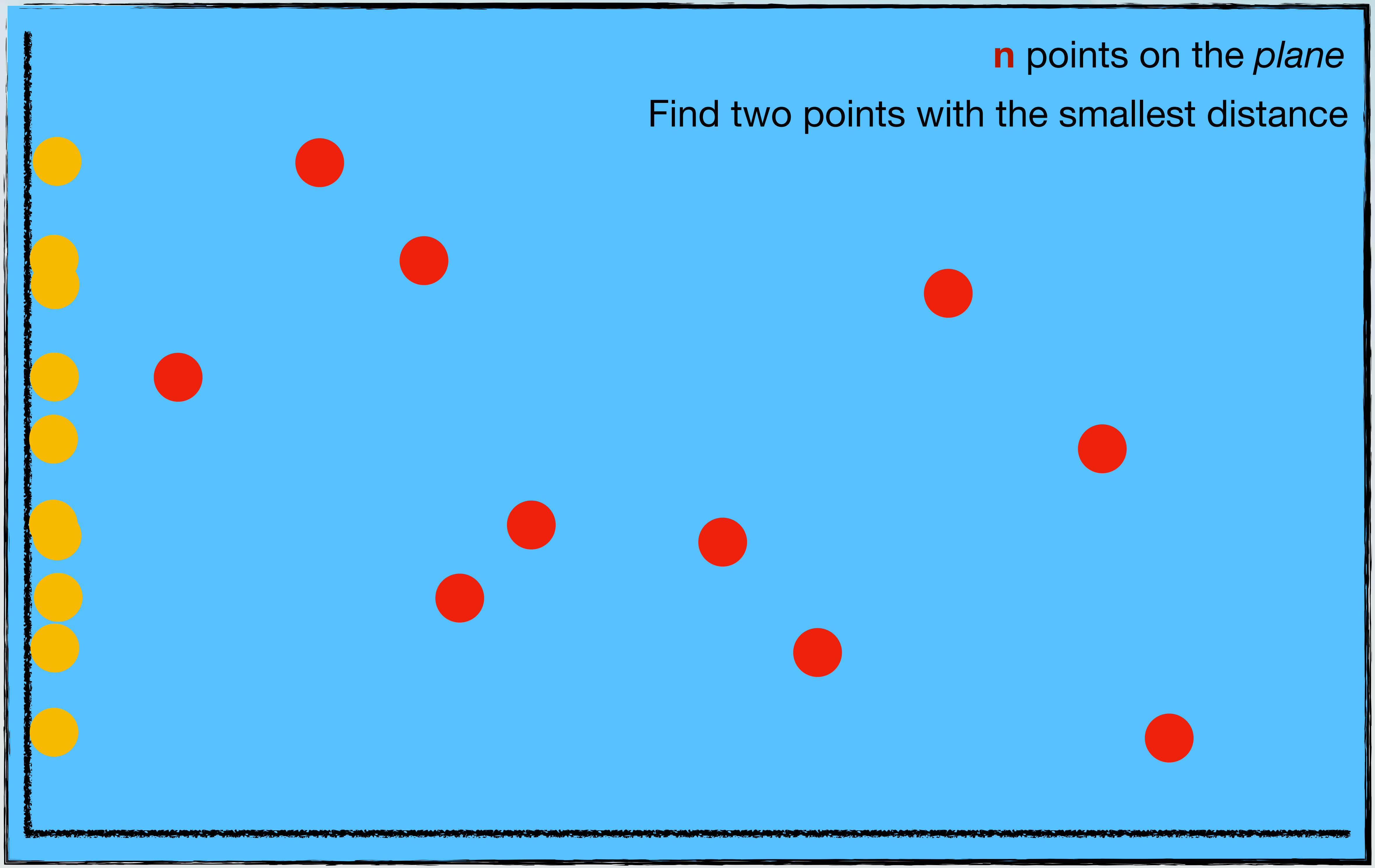

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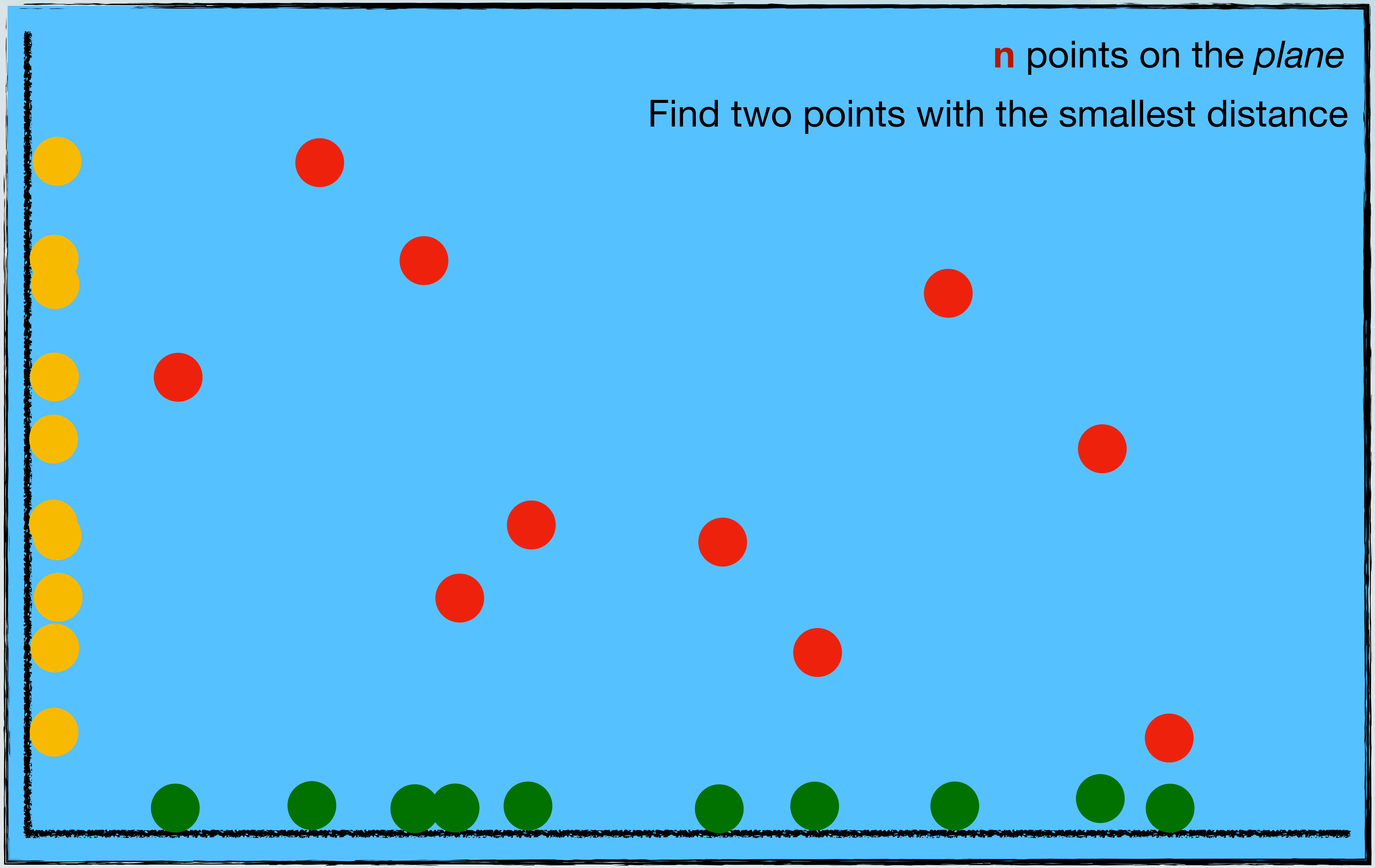

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- Maintain two arrays **H** and **V** for the horizontal and vertical coordinates of the points respectively.

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- Maintain two arrays **H** and **V** for the horizontal and vertical coordinates of the points respectively.
- To populate **H** and **V**, simply run through the given points $p_1=(x_1,y_1)$, $p_2=(x_2,y_2)$, … , $p_n=(x_n,y_n)$ and put $x_1, x_2, \ldots, x_n$ into **H** and $y_1, y_2, \ldots, y_n$ into **V**.

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- Sort the points in **H** and in **V**, using some sorting algorithm.

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Main algorithm:

Finding the closest pair of points

Main algorithm:

- Partition array **H** into two halves H_1 and H_2, according to the sorted order.

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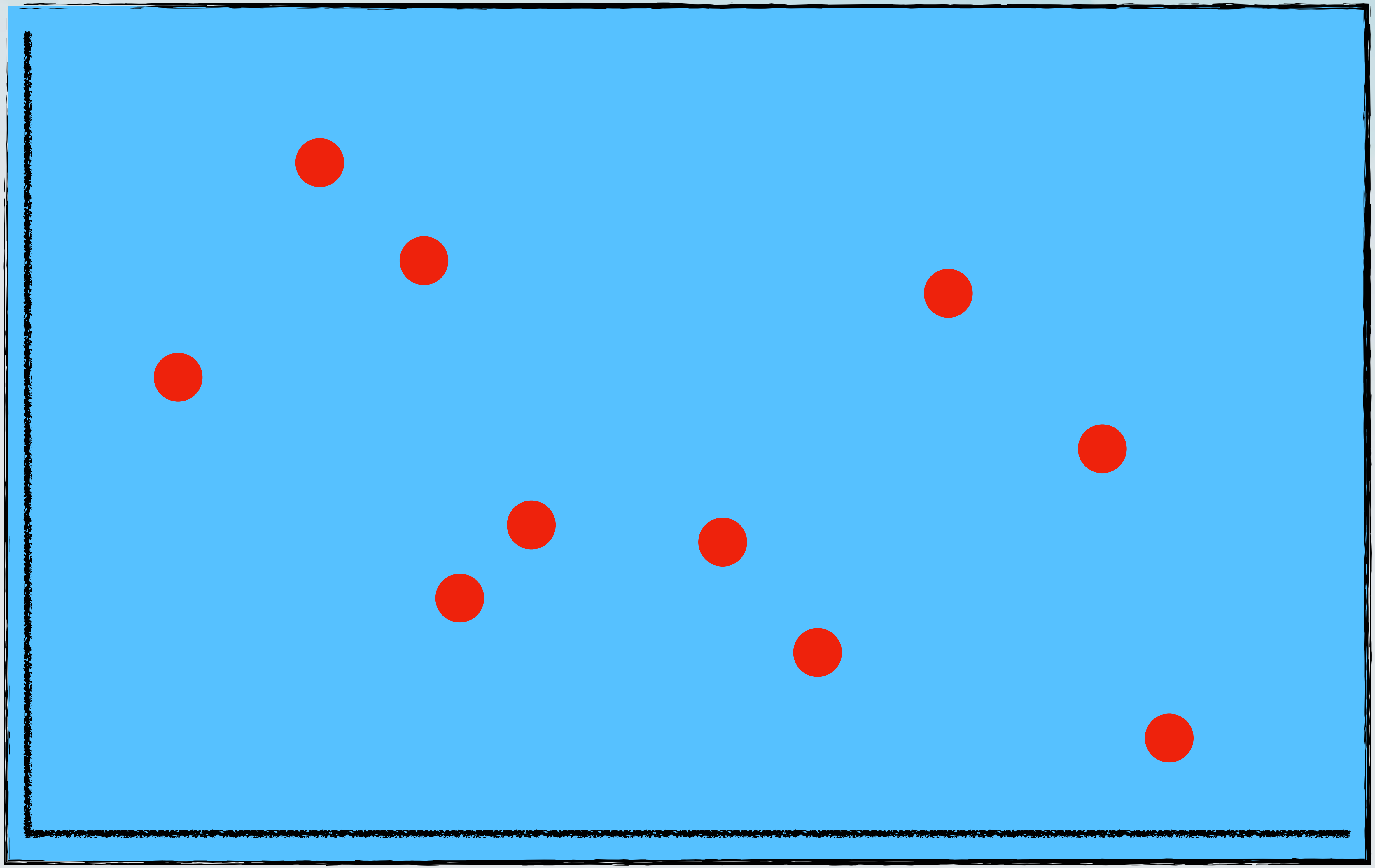

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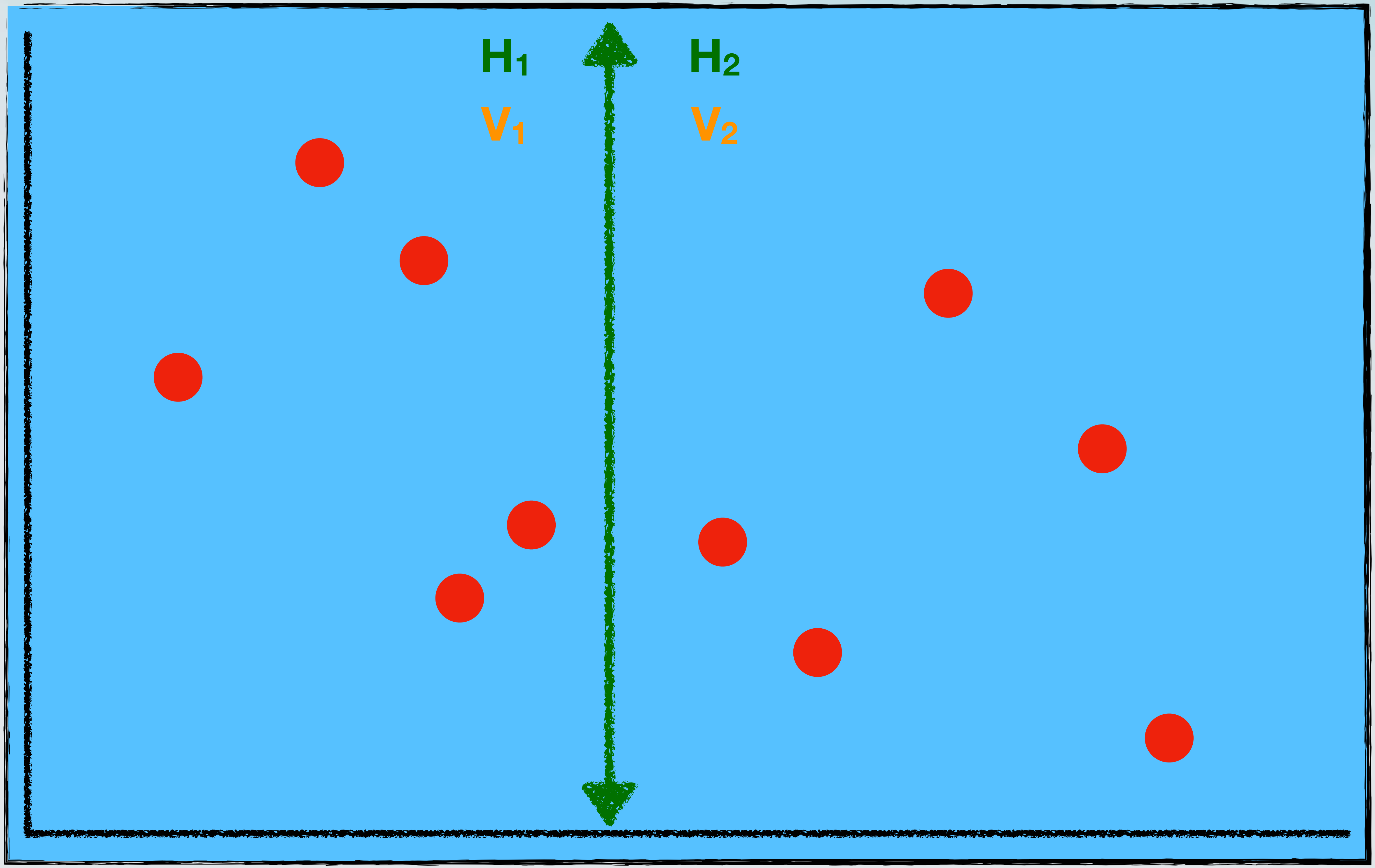

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- Let (l_1,l_2) and $(r1,r2)$ be the set of points returned by the runs of the algorithm on the two halves.
    - We haven't really developed that part yet!

ClosestPair Pseudocode

Algorithm ClosestPair($p_1,\ldots,p_n$)

Construct arrays **H** and **V**

(p_1,p_2) = ClosestPairRec(**H**,**V**)

Procedure ClosestPairRec(**H**,**V**)

Construct **H**$_1$, **H**$_2$, **V**$_1$, **V**$_2$

(l_1,l_2) = ClosestPairRec(**H**$_1$,**V**$_1$)

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Procedure **ClosestPairRec**(**H**,**V**)

If $|H| = |V| \leq 3$

Check all pairwise distances

Construct H_1, H_2, V_1, V_2

(l_1,l_2) = **ClosestPairRec**(H_1,V_1)

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Divide and Conquer

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- We have successfully divided the problem into smaller parts.

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Divide and Conquer

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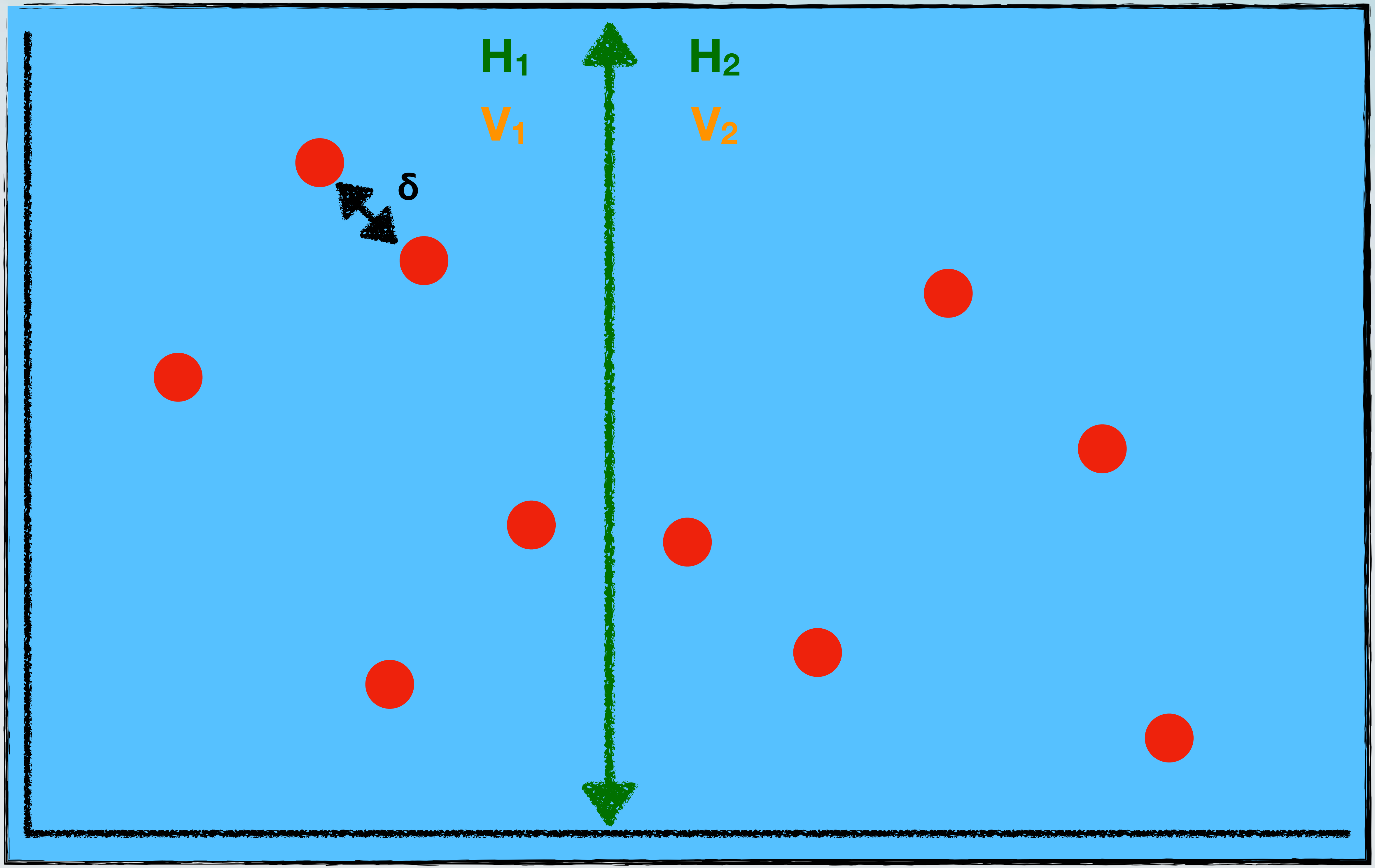
- How do we combine these parts to get a solution to the original problem?
- What might be the problem here?
- What if the smallest distance is between points in (**H_1**,**V_1**) and (**H_2**,**V_2**)?

Finding the closest pair of points



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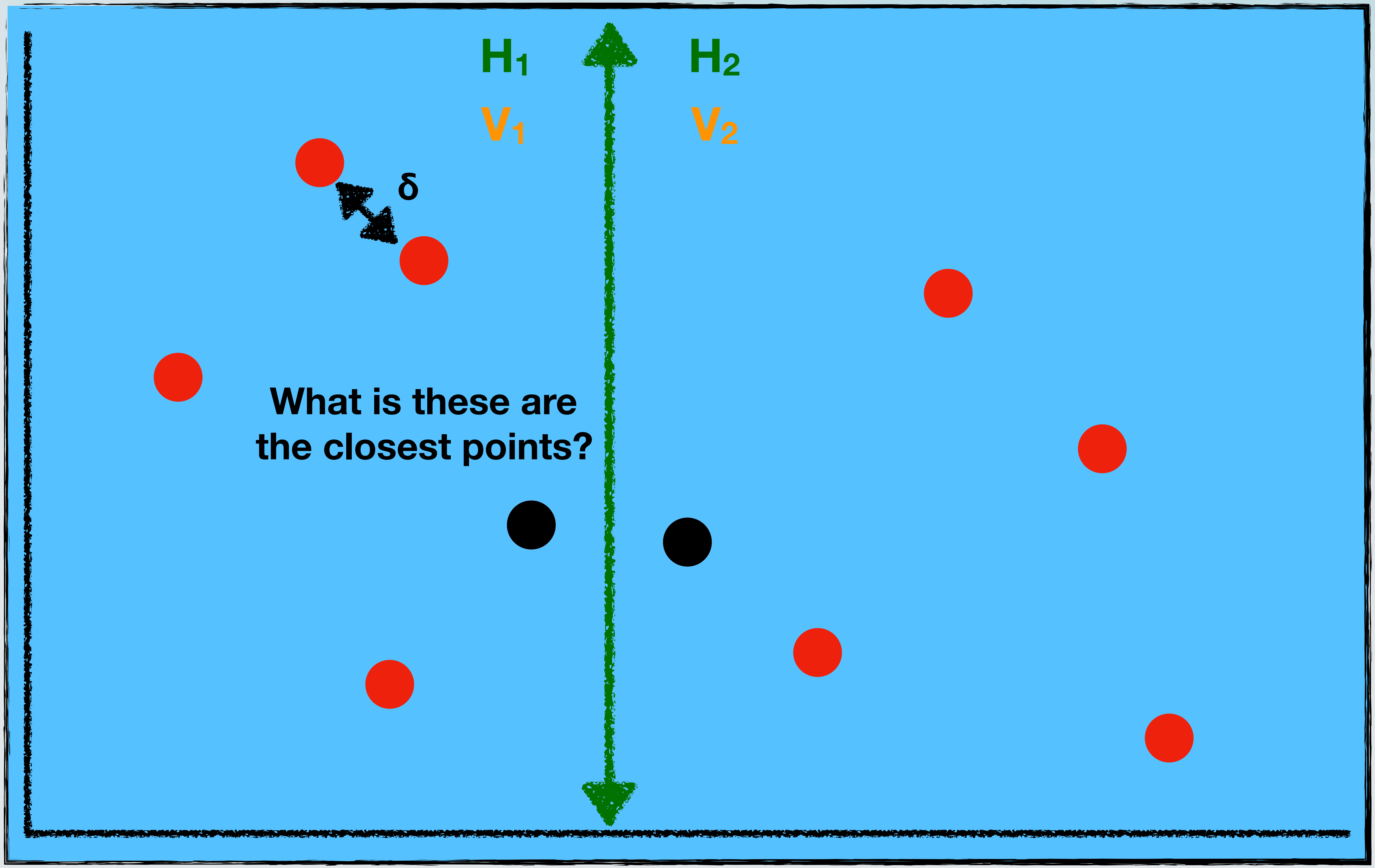

Combining the solutions

- Let δ be the $\min(d(l_1,l_2), d(r_1,r_2))$ be the minimum distance among the two solutions provided.

- Draw a vertical line **L** over the rightmost point of the set **H_1**.

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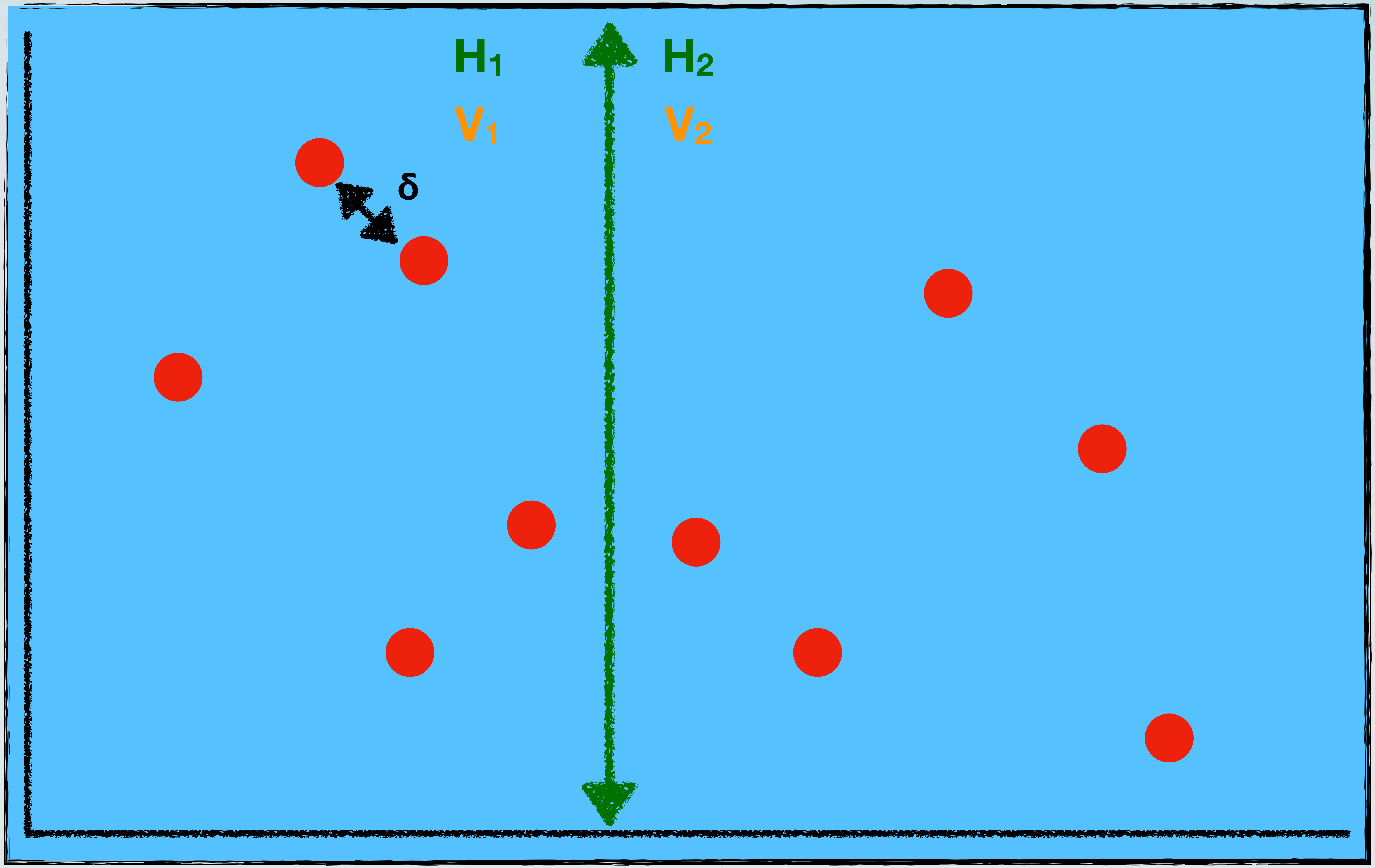

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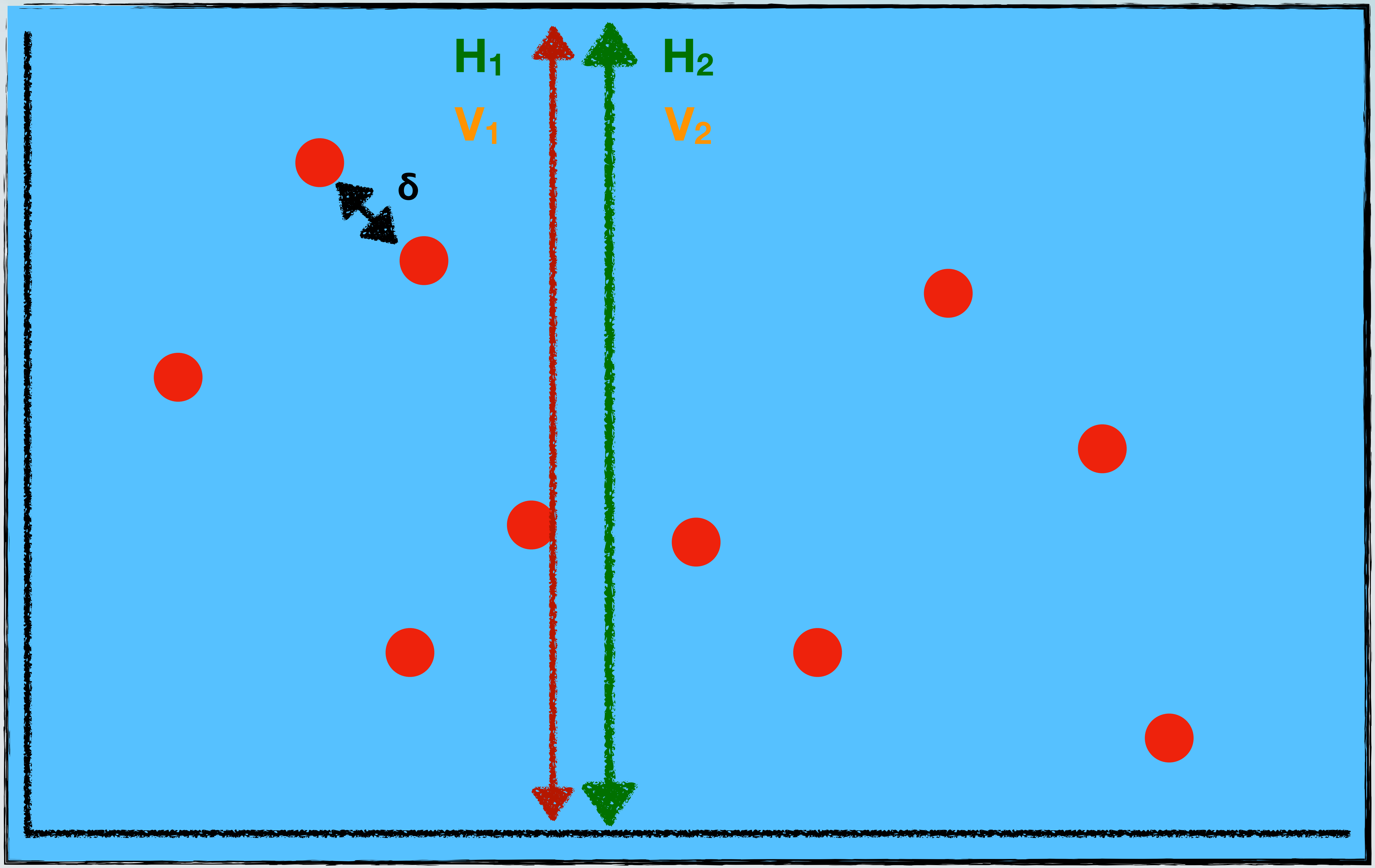

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    - This basically means that the separating line is a “tight” as possible.
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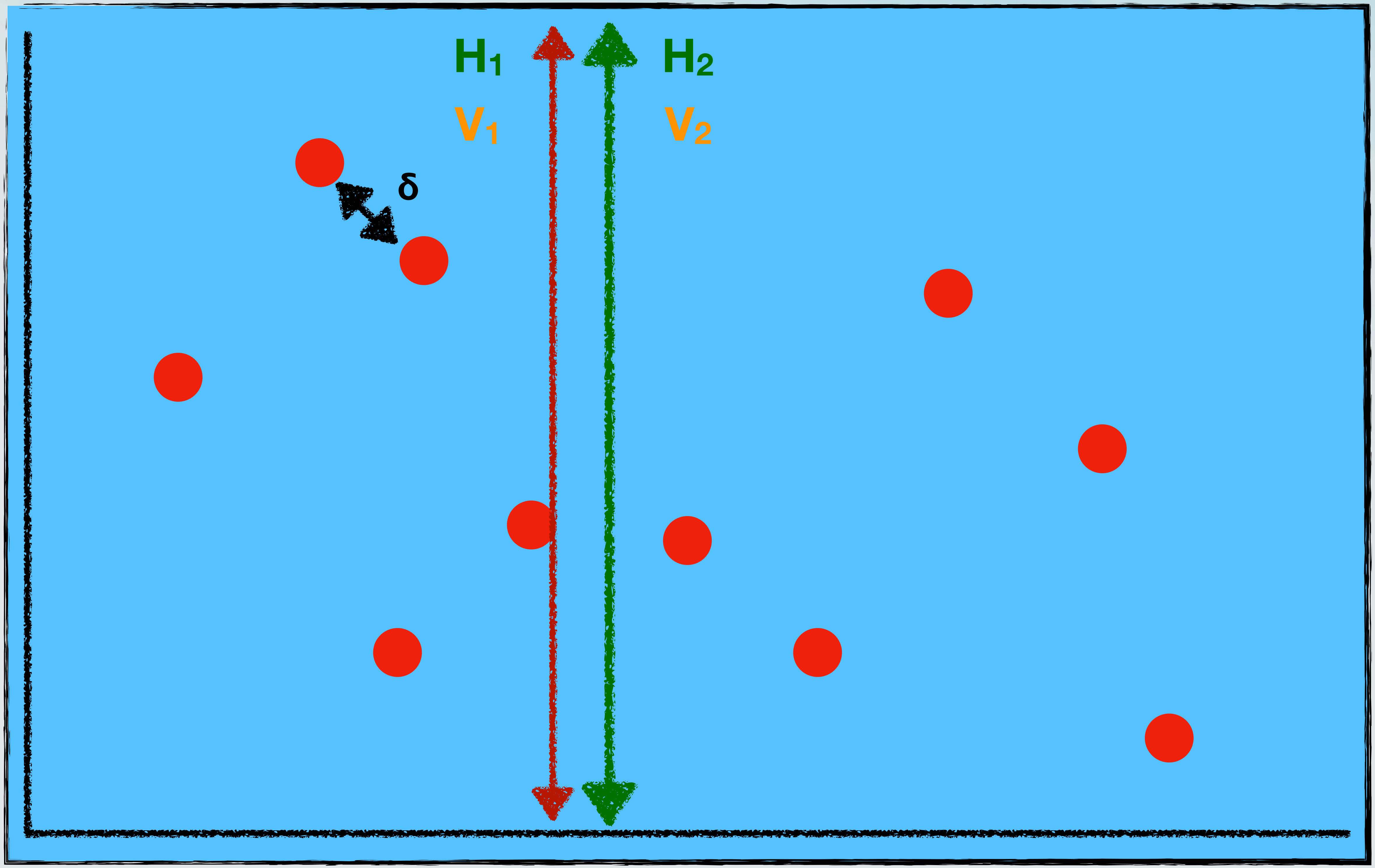

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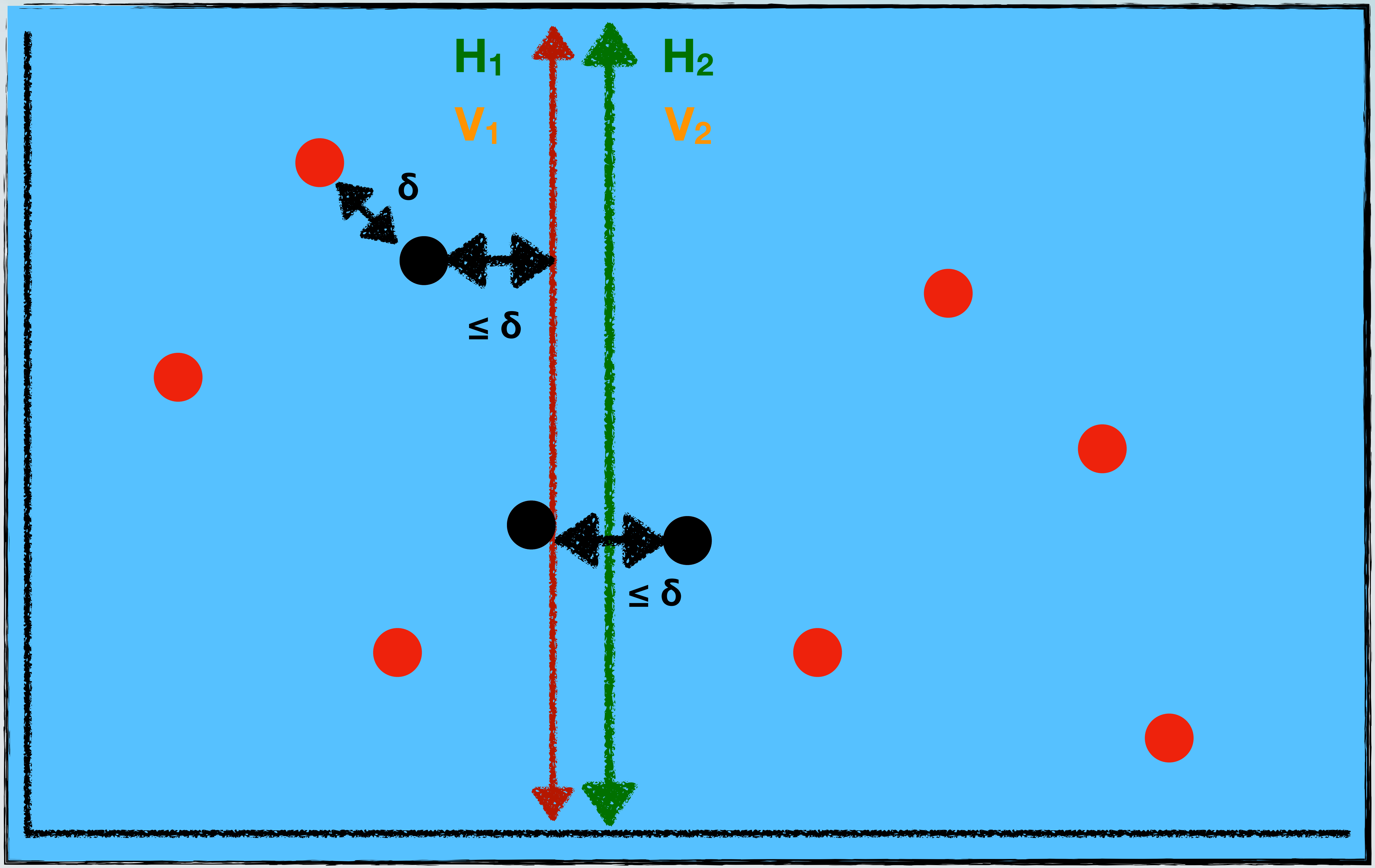

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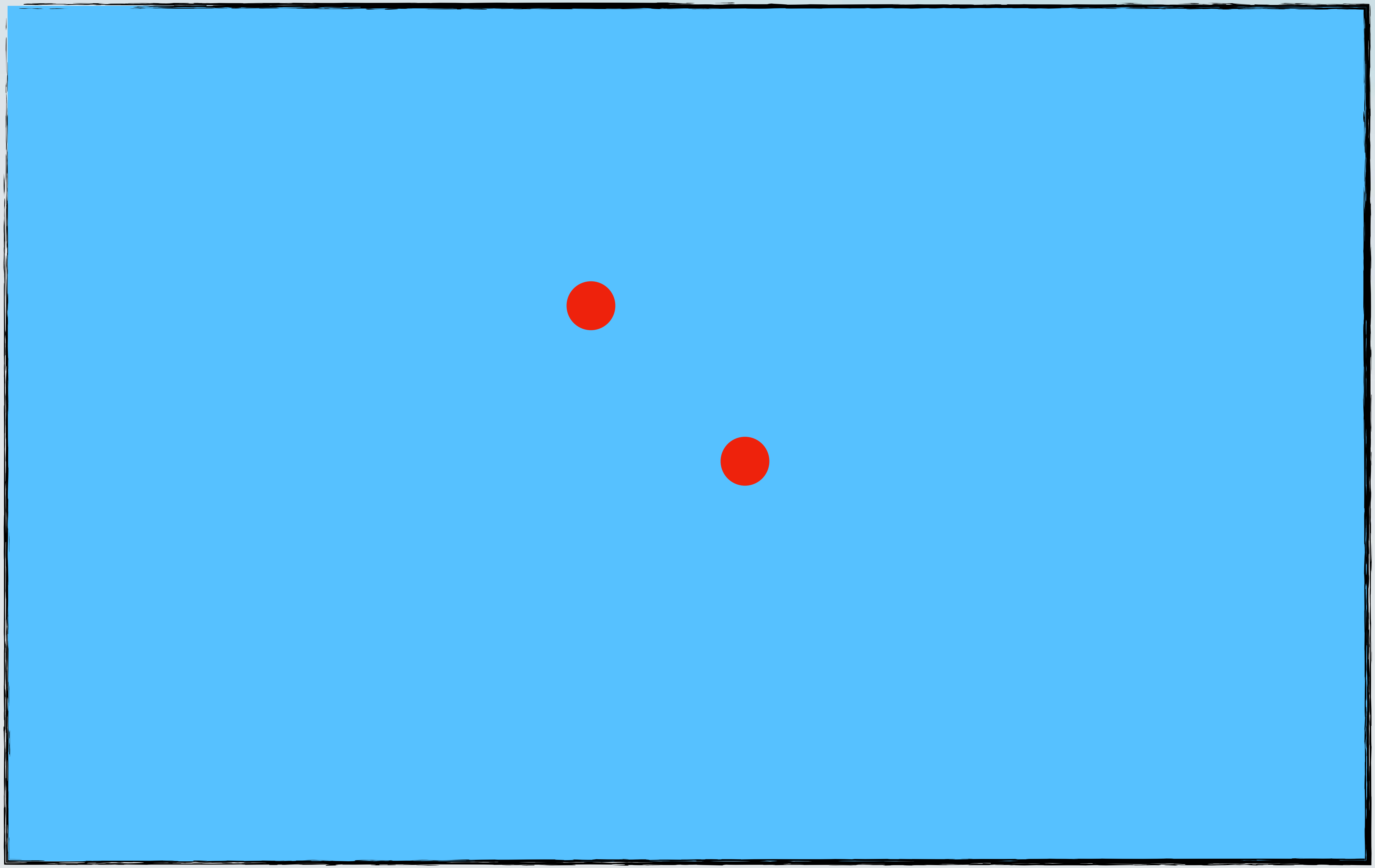
Is it safe to only consider the points in S?



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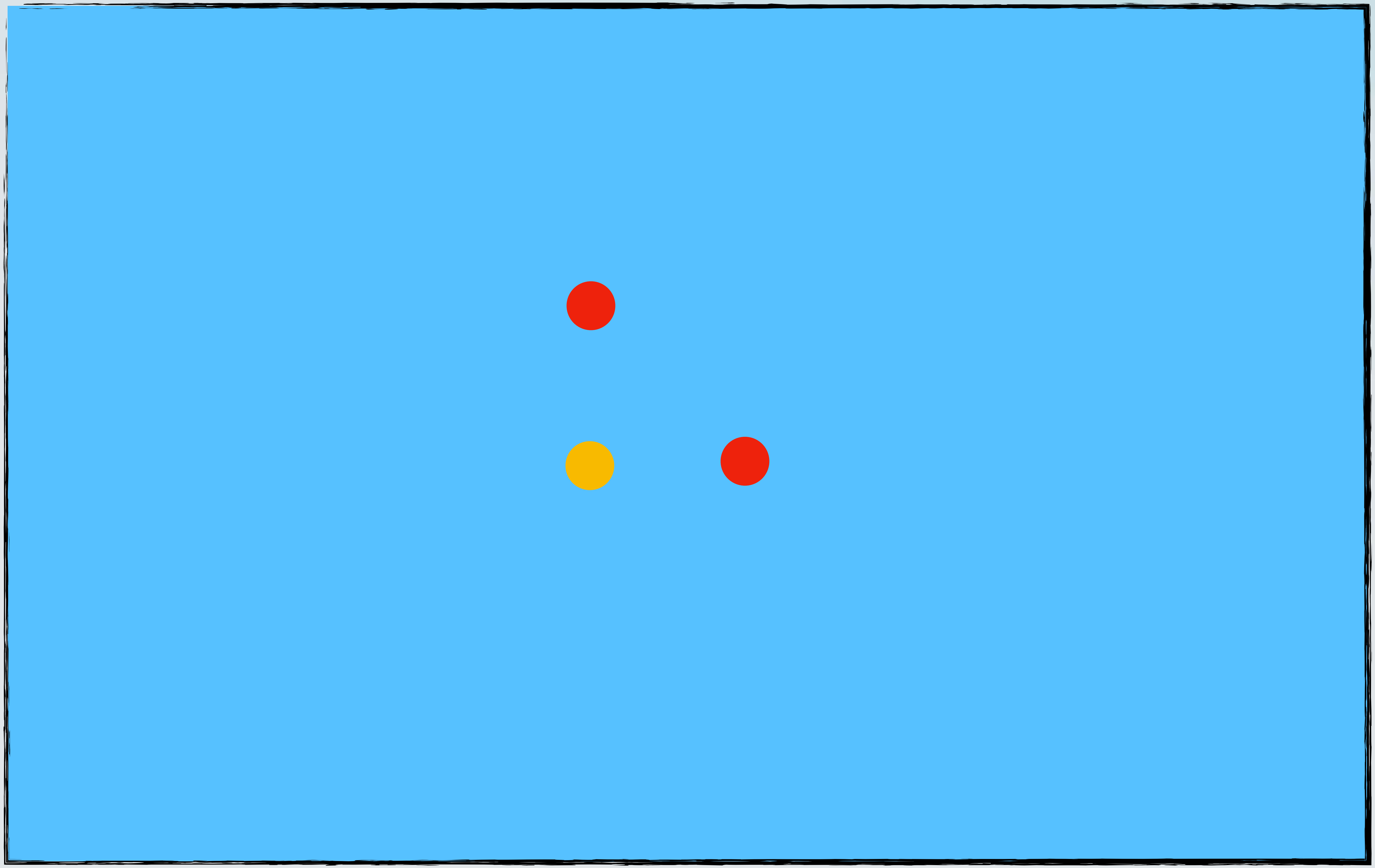

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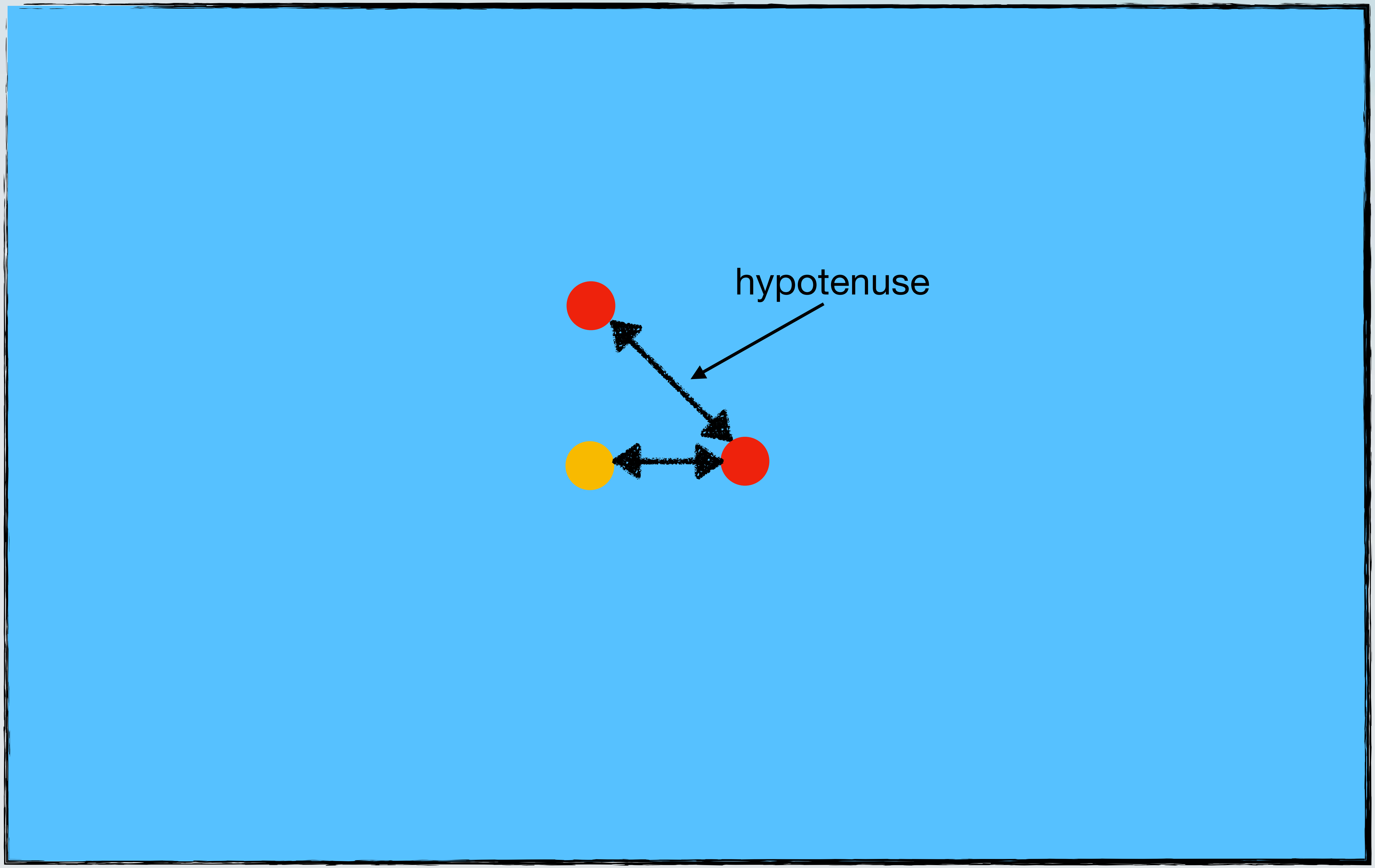

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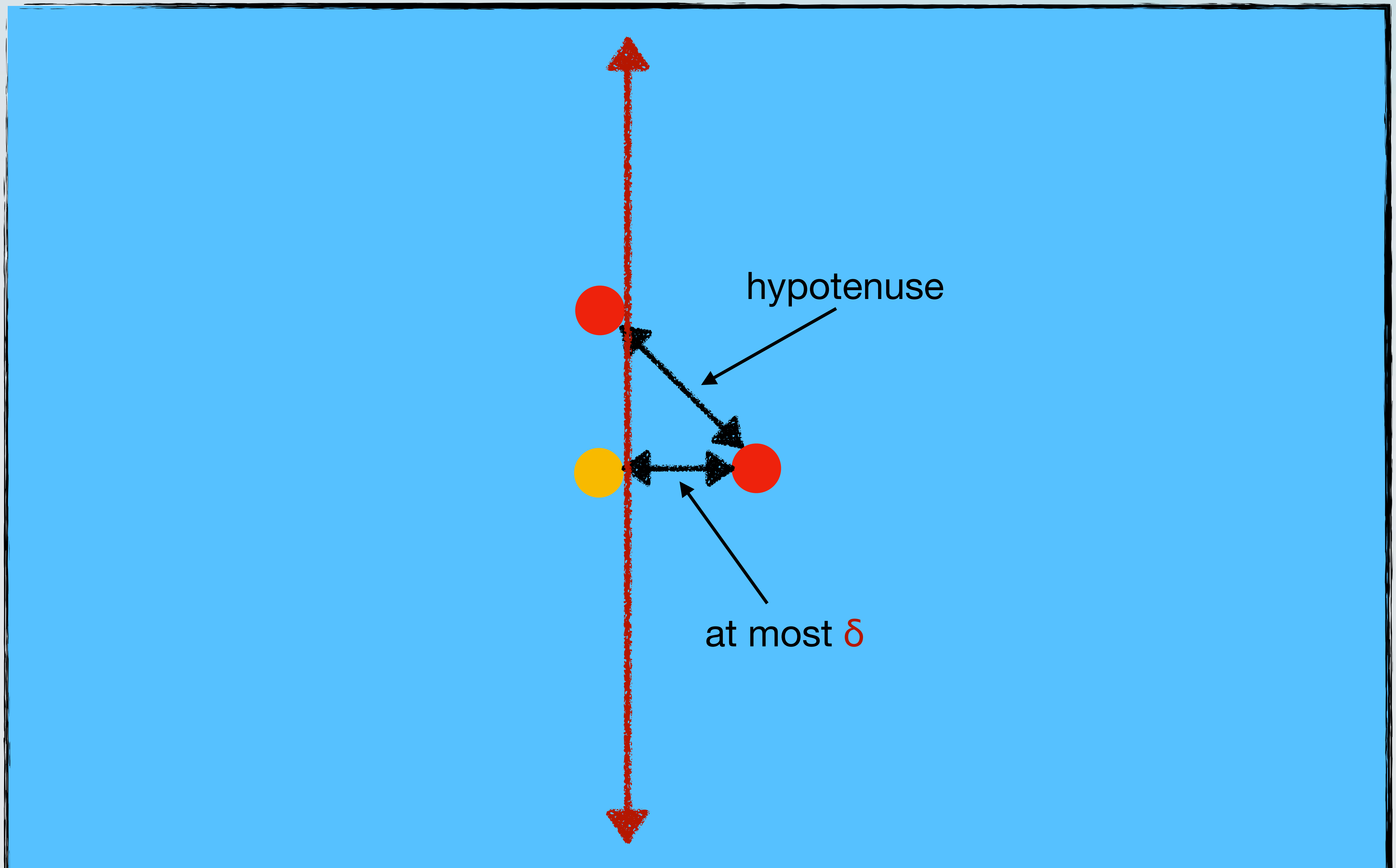

Constructing the set S

Array V

1	2	4	6	10	14	17	19	21	24

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Do we get more than a set?

Constructing the set **S**

Array **V**

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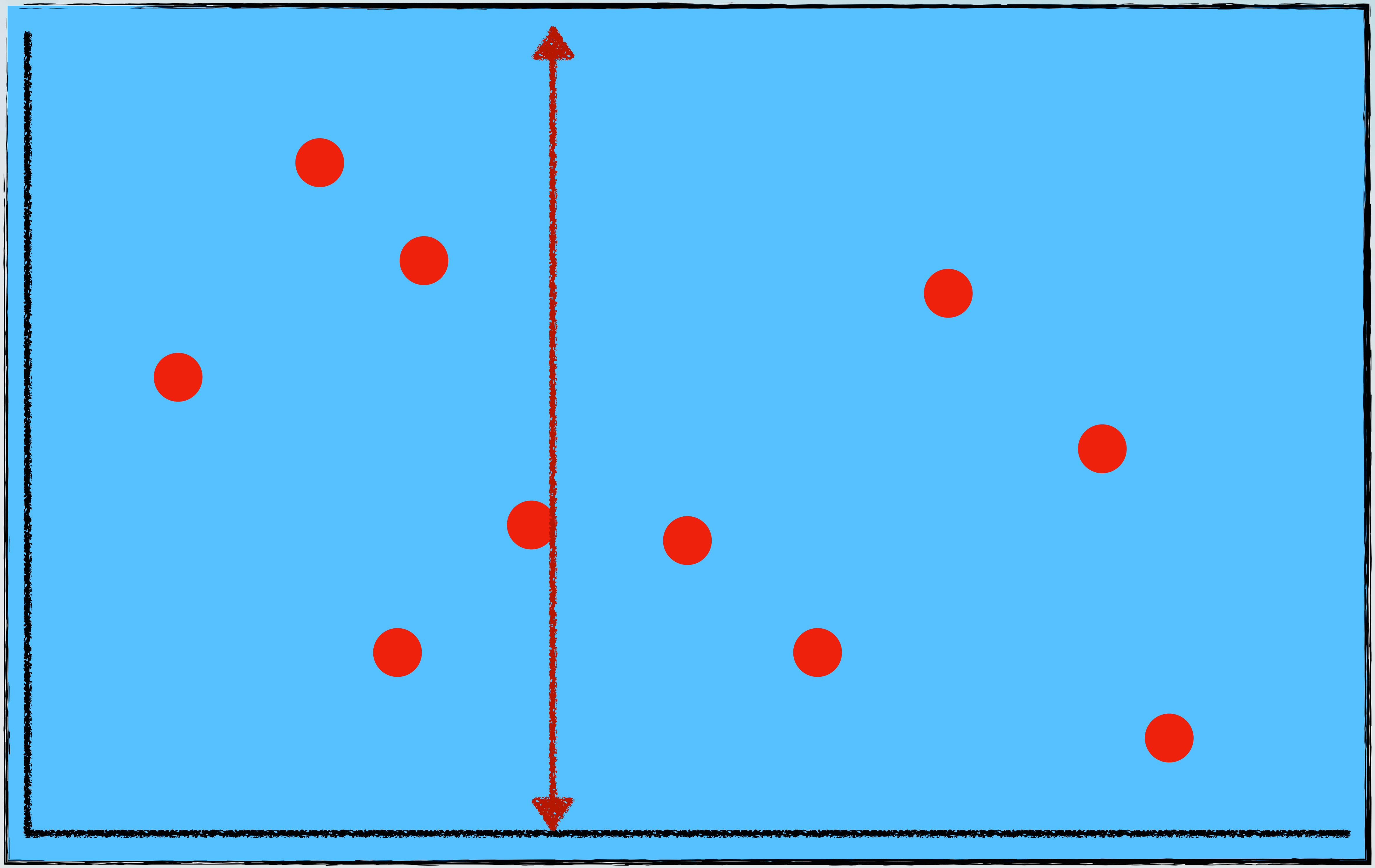
Do we get more than a set?

We actually get a **sorted list!**
(sorted in the y-coordinate)
Call this S_v.

Finding the closest pair of points



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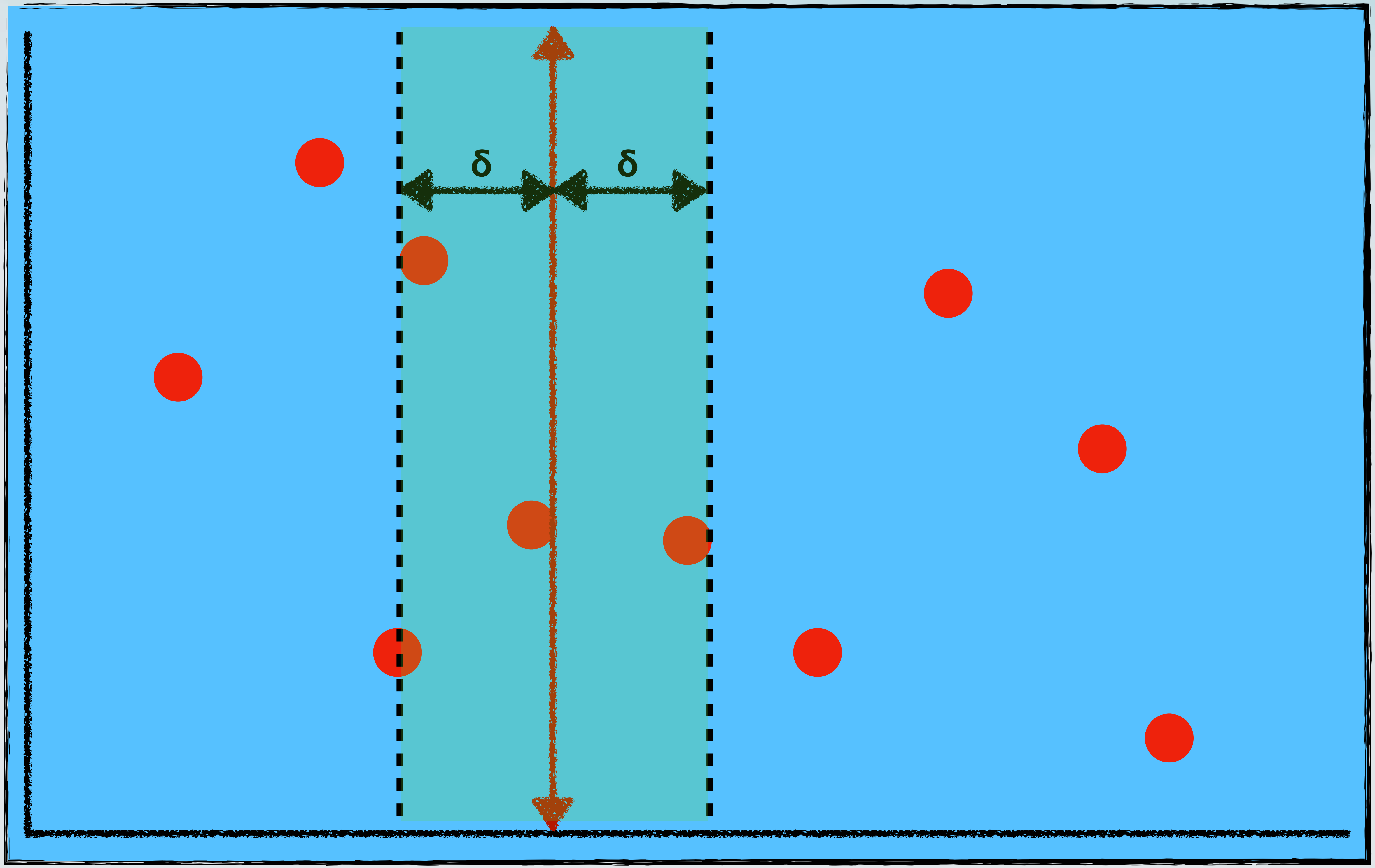
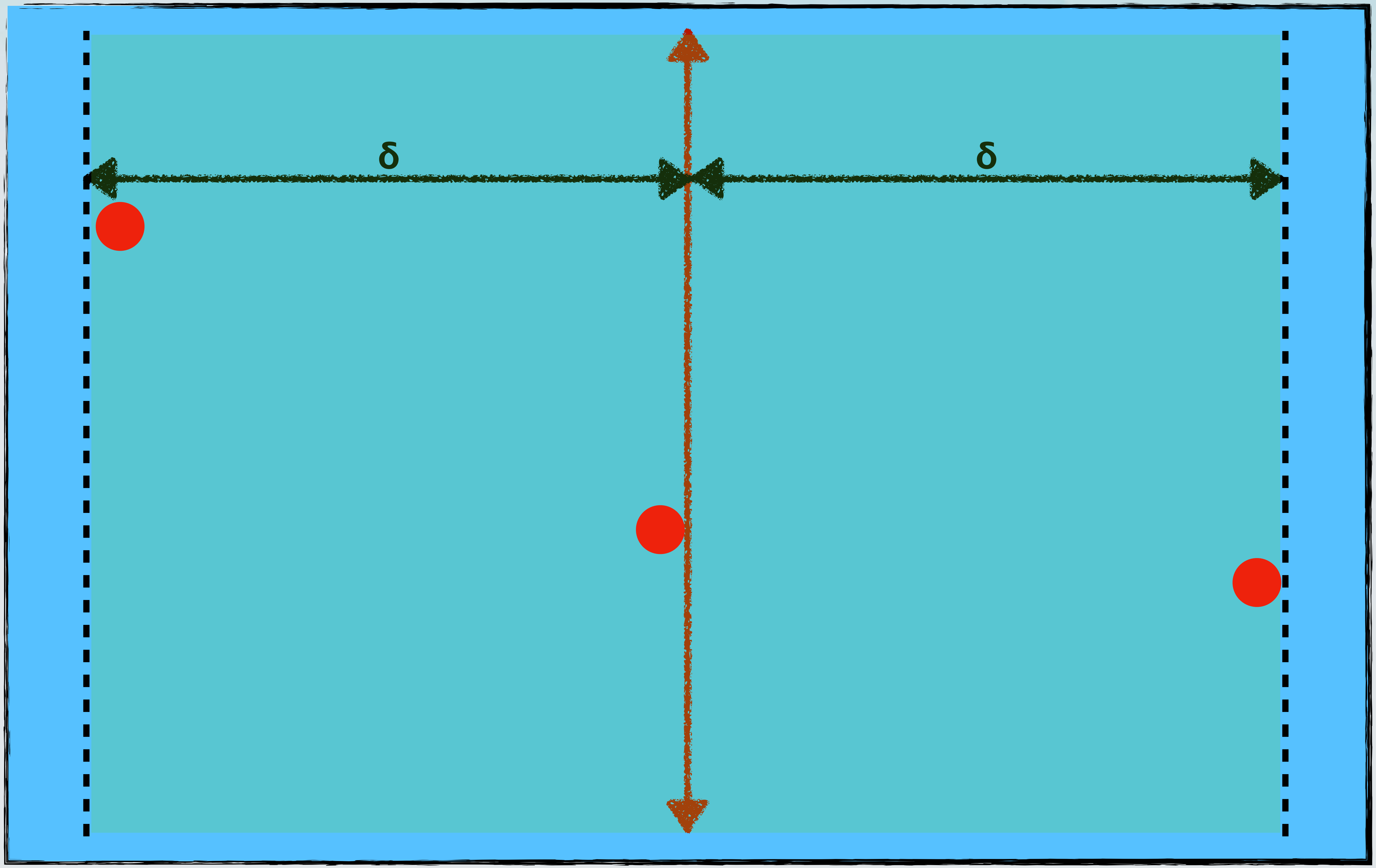
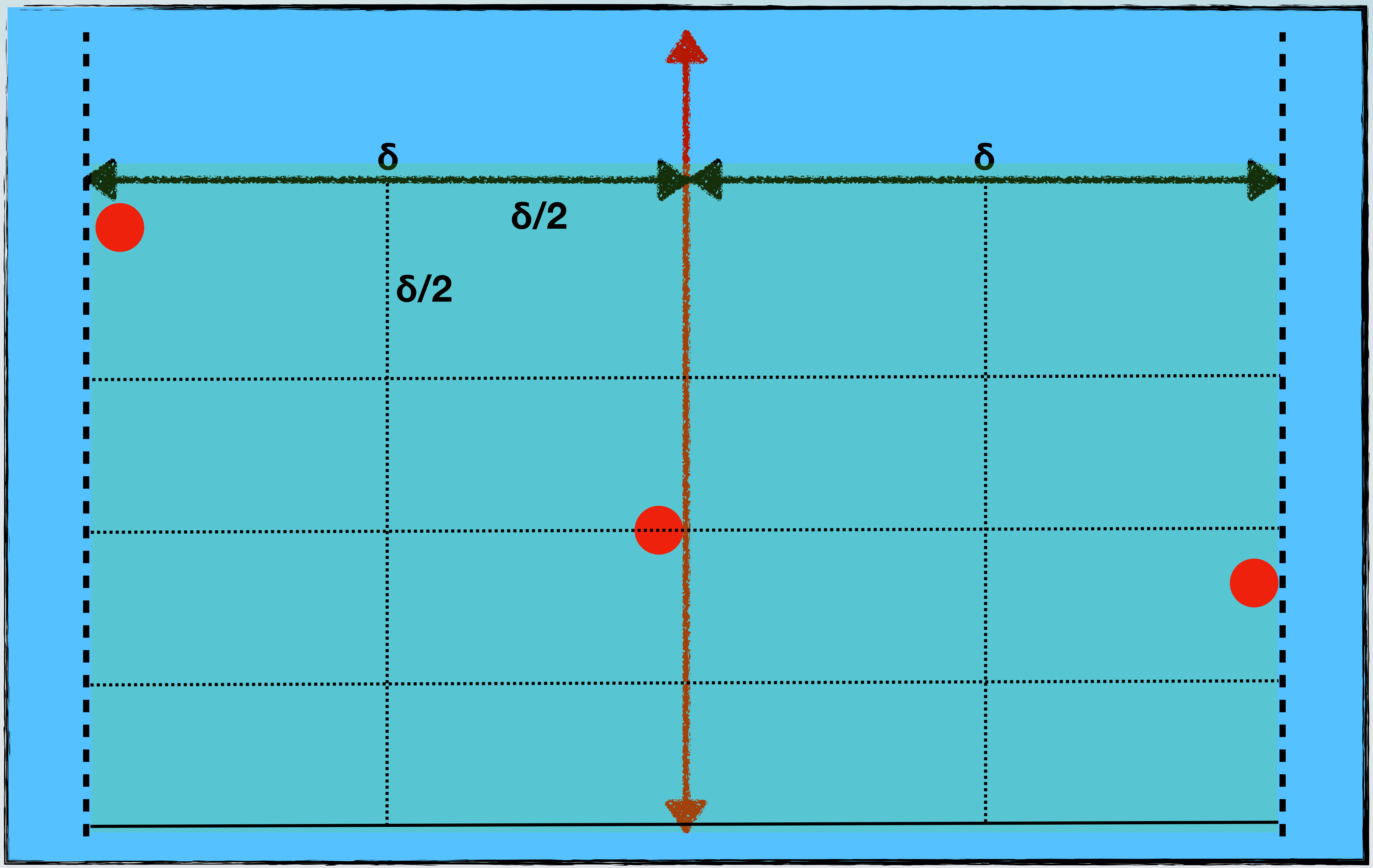


Zooming in



Partitioning the square



Claims

- **Claim 1:** In each box, there can only be a single point.

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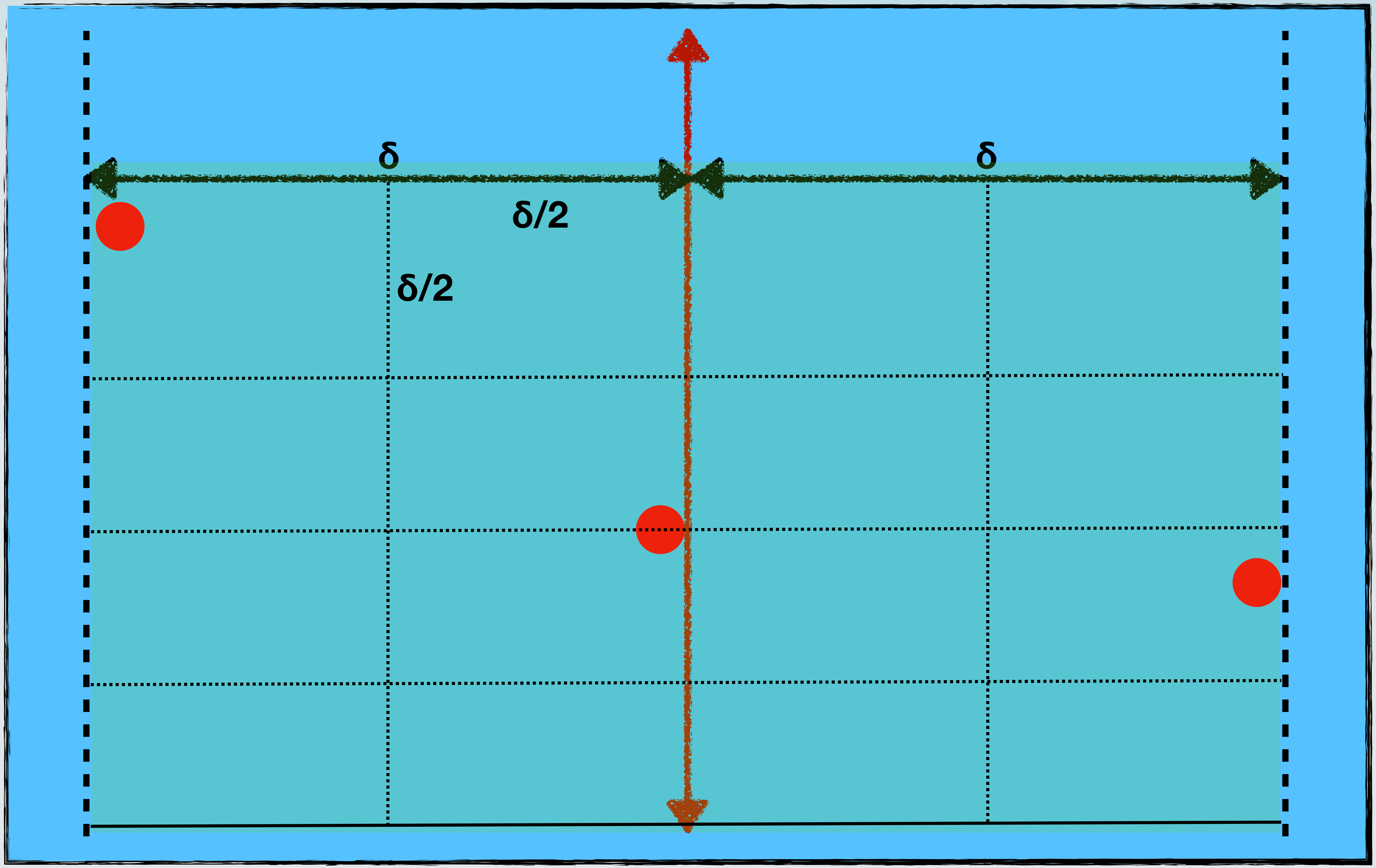

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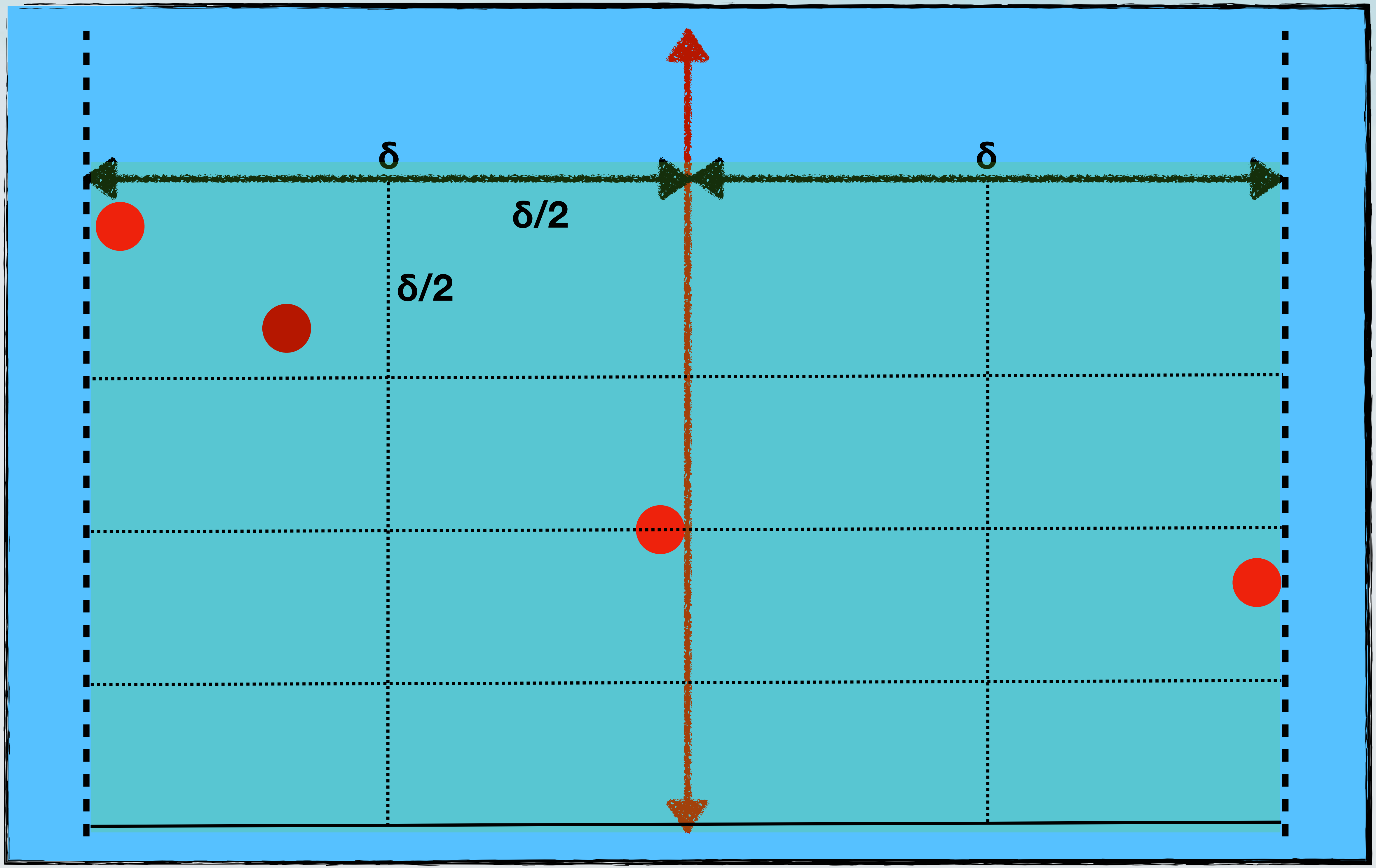

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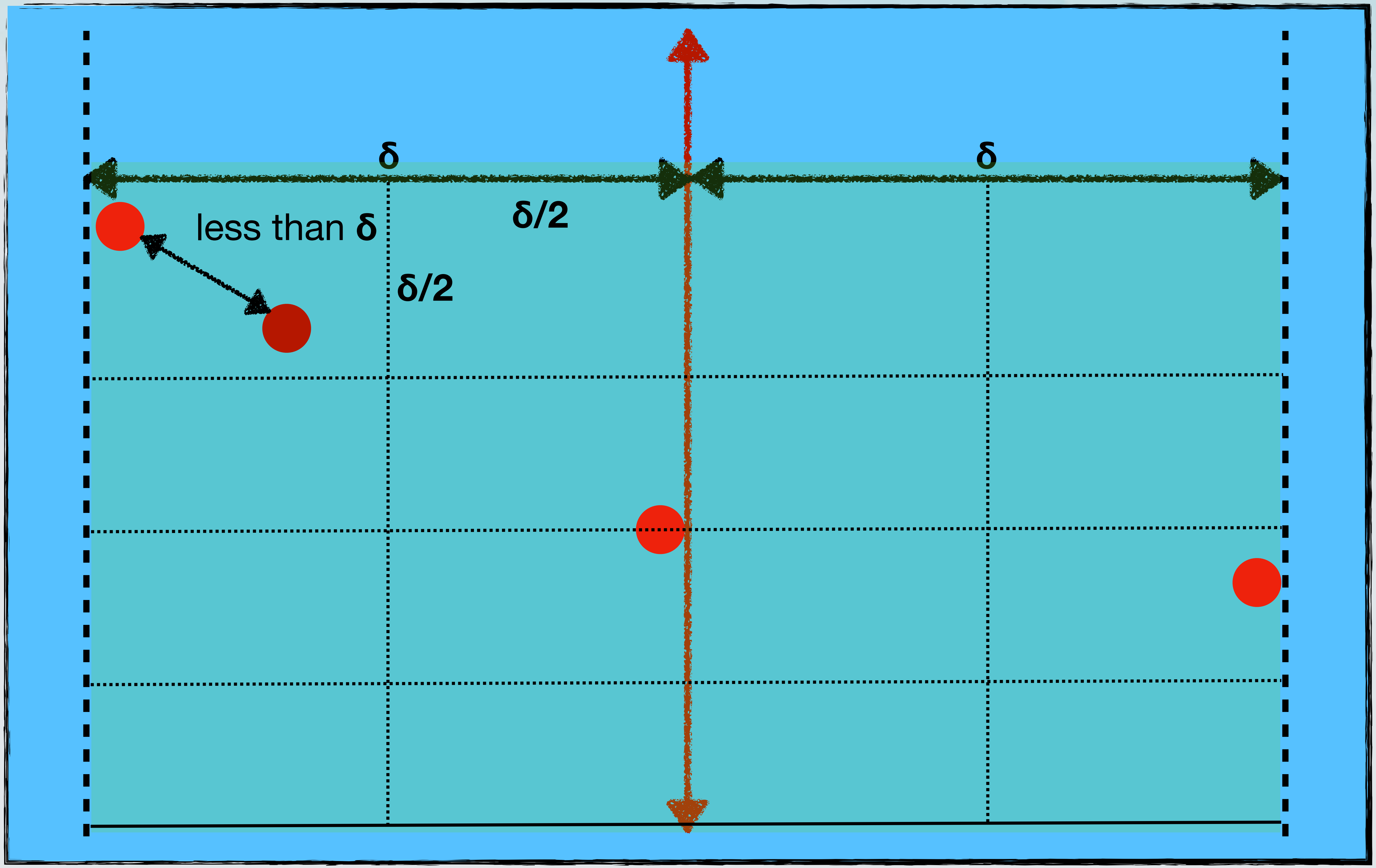

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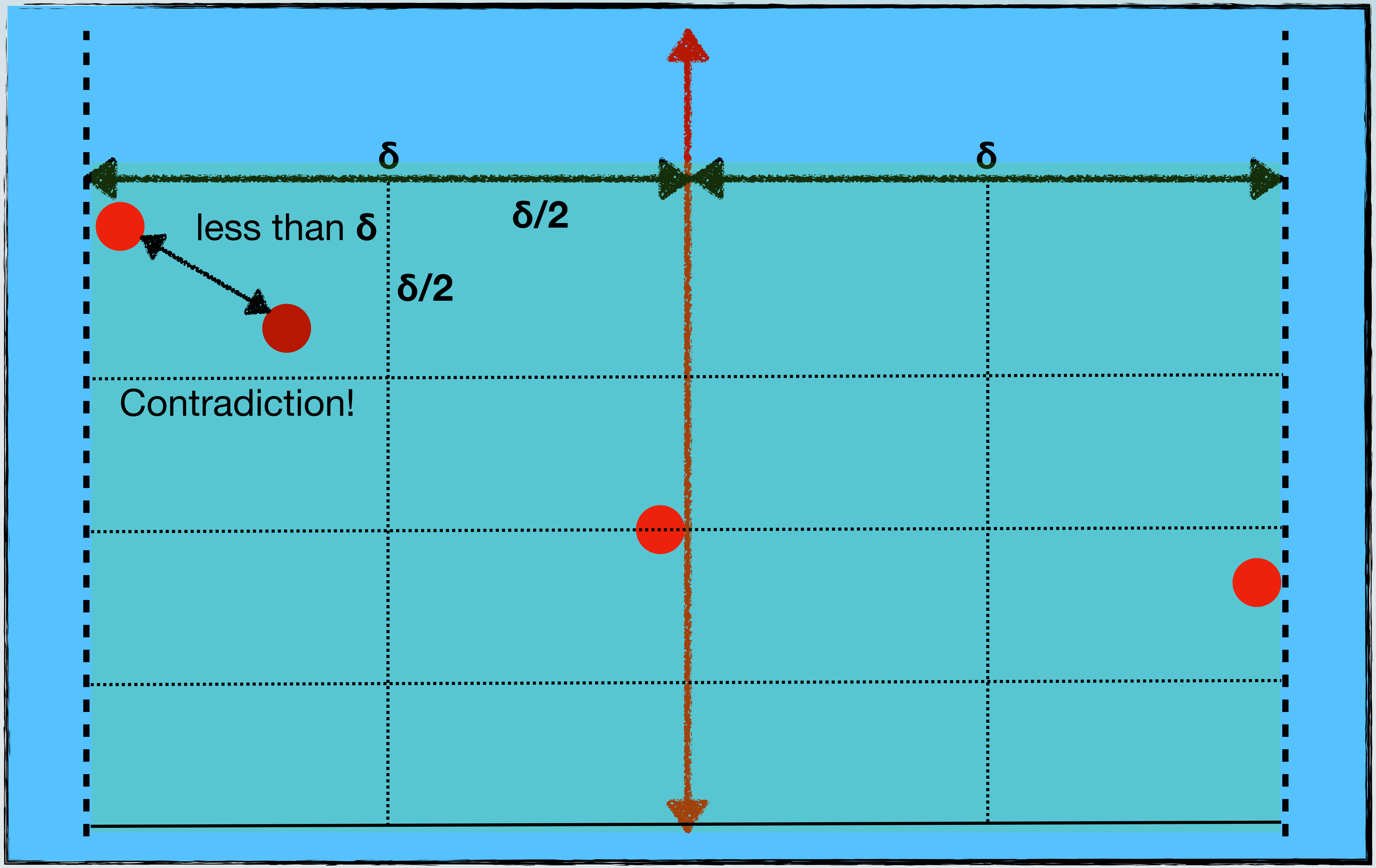

Claims

- **Claim 1:** In each box, there can only be a single point.
- **Claim 2:** If two points p_1 and p_2 are such that $d(p_1,p_2) < \delta$, then $\mathbf{S_v}[p_{1y}] - \mathbf{S_v}[p_{2y}] \leq 15$
  - In other words, the two points are within 15 positions of each other in the sorted array $\mathbf{S_v}$.

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- Assume by contradiction that this is not the case, and p_1 and p_2 are at least 16 positions apart in $\mathbf{S_v}$.

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- By **Claim 1**, there can be at most one point in each box.

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    - To be 16 positions apart, there must be at least **3 rows** of boxes separating the points.

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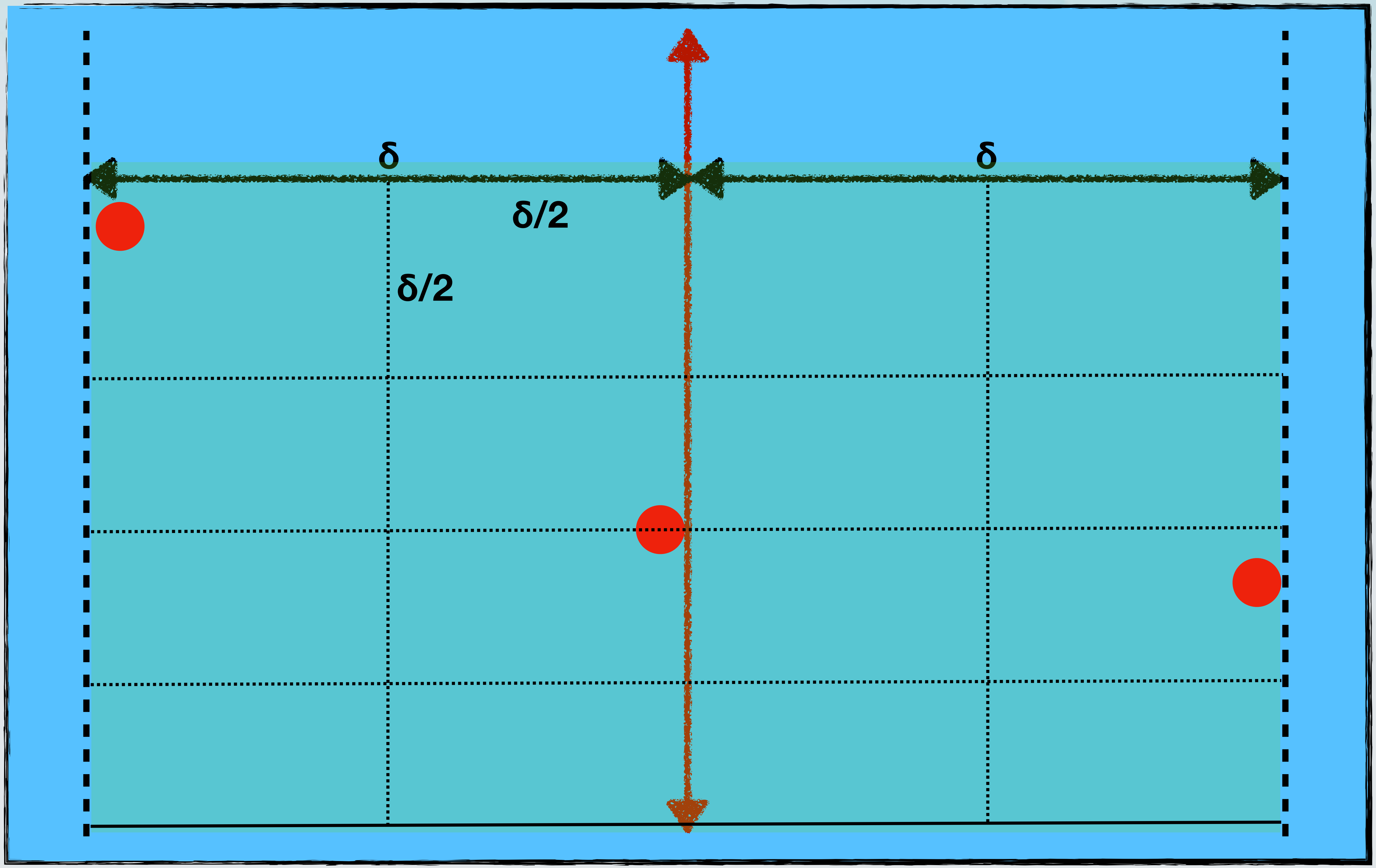
- Assume by contradiction that this is not the case, and p_1 and p_2 are at least 16 positions apart in $\mathbf{S_v}$.
- By **Claim 1**, there can be at most one point in each box.
    - To be 16 positions apart, there must be at least **3 rows** of boxes separating the points.
    - But then the distance is at least $3\delta/2$, a contradiction.

Partitioning the square



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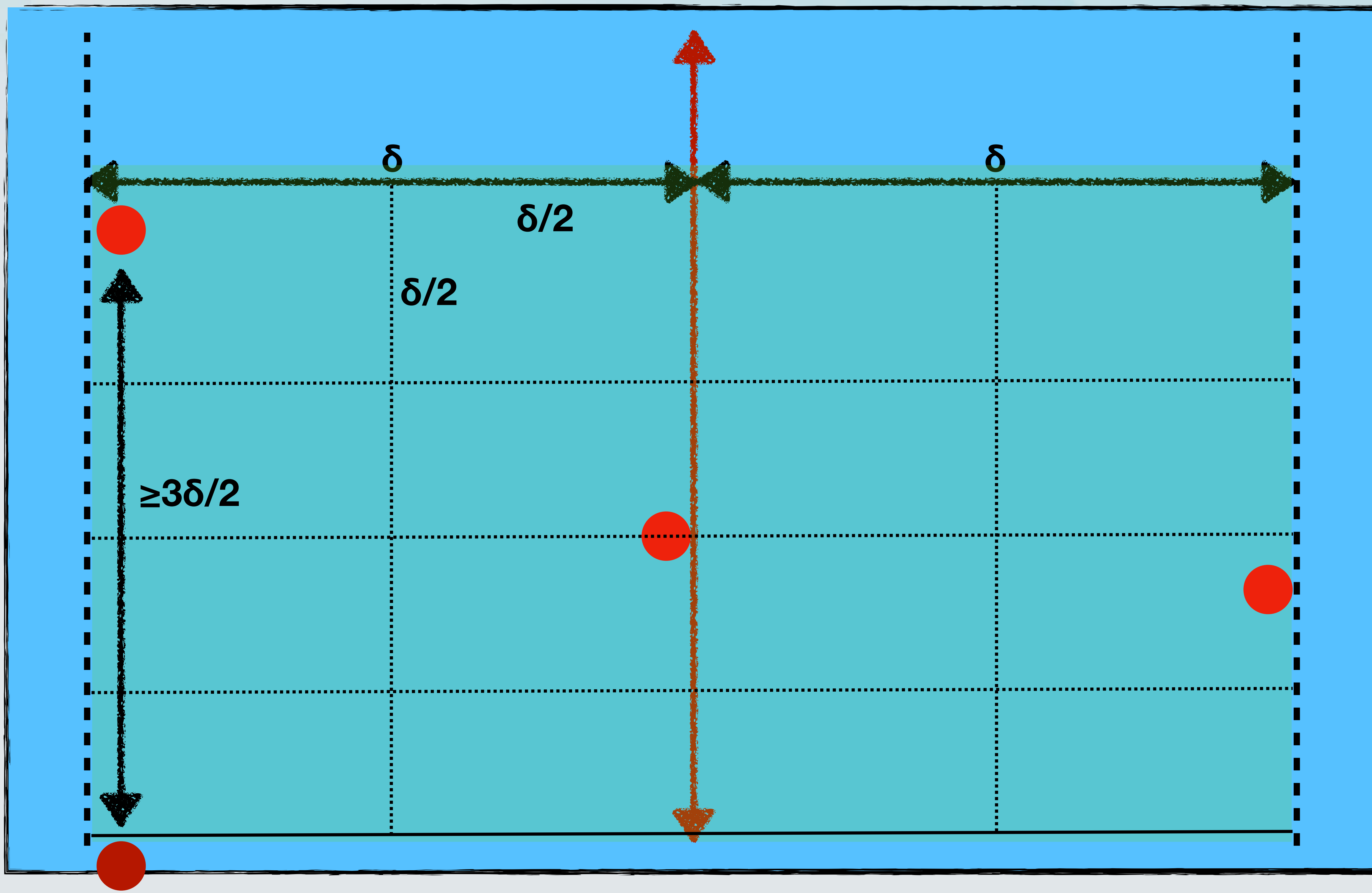

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- Find the pair of points p_1 and p_2 for which the minimum of all these distances is achieved.
- Compare this to δ and output the pair with the minimum distance.

ClosestPair Pseudocode

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Procedure ClosestPairRec(**H**,**V**)

If |**H**| = |**V**| ≤ 3

Check all pairwise distances

Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = ClosestPairRec($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = ClosestPairRec($\mathbf{H_2}$,$\mathbf{V_2}$)

$\delta = \min(d(l_1,l_2), d(r_1,r_2))$

$x^* = \max_i x_i$ for $i=1,\ldots,n$

$\mathbf{L} = \{(x,y) : x=x^*\}$

S = set of points within distance δ of **L.**

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

Compute distance between s

and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2)

that minimises this distance.

If $d(s_1,s_2) \leq \delta$

Return (s_1,s_2)

Else if $d(l_1,l_2) < d(r_1,r_2)$

Return (l_1,l_2)

Else return (r_1,r_2)

Running Time

Algorithm ClosestPair($p_1,\ldots,p_n$)

Construct arrays **H** and **V**

(p_1,p_2) = ClosestPairRec(**H**,**V**)

Procedure ClosestPairRec(**H**,**V**)

If |**H**| = |**V**| ≤ 3

Check all pairwise distances

Construct **H₁**, **H₂**, **V₁**, **V₂**

(l_1,l_2) = ClosestPairRec($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = ClosestPairRec($\mathbf{H_2}$,$\mathbf{V_2}$)

$\delta = \min(d(l_1,l_2), d(r_1,r_2))$

$x^* = \max_i x_i$ for $i=1,\ldots,n$

L = $\{(x,y) : x=x^*\}$

S = set of points within distance δ of **L.**

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

Compute distance between s

and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2)

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Running Time

Algorithm **ClosestPair**($p_1,\dots,p_n$)

    Construct arrays **H** and **V**

    (p_1,p_2) = **ClosestPairRec**(**H**,**V**)

Procedure **ClosestPairRec**(**H**,**V**)

    If $|\mathbf{H}| = |\mathbf{V}| \leq 3$

        Check all pairwise distances

    Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

    (l_1,l_2) = **ClosestPairRec**($\mathbf{H_1}$,$\mathbf{V_1}$)

    (r_1,r_2) = **ClosestPairRec**($\mathbf{H_2}$,$\mathbf{V_2}$)

    $\delta = \min(d(l_1,l_2), d(r_1,r_2))$

    $x^* = \max_i x_i$ for $i=1,\dots,n$

    $\mathbf{L} = \{(x,y) : x=x^*\}$

    S = set of points within distance δ of **L**.

?

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

    Compute distance between s

    and the next 15 points of $\mathbf{S_v}$.

    Return the pair of points (s_1,s_2)

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If $d(s_1,s_2) \leq \delta$

    Return (s_1,s_2)

Else if $d(l_1,l_2) < d(r_1,r_2)$

    Return (l_1,l_2)

Else return (r_1,r_2)

Finding the closest pair of points

Finding the closest pair of points

Preprocessing:

Finding the closest pair of points

Preprocessing:

- Maintain two arrays **H** and **V** for the horizontal and vertical coordinates of the points respectively.

Finding the closest pair of points

Preprocessing:

- Maintain two arrays **H** and **V** for the horizontal and vertical coordinates of the points respectively.
- To populate **H** and **V**, simply run through the given points $p_1=(x_1,y_1)$, $p_2=(x_2,y_2)$, … , $p_n=(x_n,y_n)$ and put $x_1, x_2, \dots, x_n$ into **H** and $y_1, y_2, \dots, y_n$ into **V**.

Finding the closest pair of points

Preprocessing:

- Maintain two arrays **H** and **V** for the horizontal and vertical coordinates of the points respectively.
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- Sort the points in **H** and in **V**, using some sorting algorithm.

Running Time

Algorithm ClosestPair($p_1,\ldots,p_n$)

?

Construct arrays **H** and **V**

(p_1,p_2) = ClosestPairRec(**H**,**V**)

Procedure ClosestPairRec(**H**,**V**)

If |**H**| = |**V**| ≤ 3

Check all pairwise distances

Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = ClosestPairRec($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = ClosestPairRec($\mathbf{H_2}$,$\mathbf{V_2}$)

δ = min(d(l_1,l_2), d(r_1,r_2))

x^* = $\max_i x_i$ for i=$1,\ldots,n$

L = {(x,y) : x=x^*}

S = set of points within distance δ of **L.**

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

Compute distance between s and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2) that minimises this distance.

If d(s_1,s_2) ≤ δ

Return (s_1,s_2)

Else if d(l_1,l_2) < d(r_1,r_2)

Return (l_1,l_2)

Else return (r_1,r_2)

Running Time

Algorithm **ClosestPair**($p_1,\ldots,p_n$)

Construct arrays **H** and **V**

(p_1,p_2) = **ClosestPairRec**(**H**,**V**)

O(n log n)

Procedure **ClosestPairRec**(**H**,**V**)

If |**H**| = |**V**| ≤ 3

Check all pairwise distances

Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = **ClosestPairRec**($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = **ClosestPairRec**($\mathbf{H_2}$,$\mathbf{V_2}$)

$\delta = \min(d(l_1,l_2), d(r_1,r_2))$

$x^* = \max_i x_i$ for $i=1,\ldots,n$

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Construct $\mathbf{S_v}$

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Procedure ClosestPairRec(**H**,**V**)

If |**H**| = |**V**| ≤ 3

Check all pairwise distances

Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = ClosestPairRec($\mathbf{H_1}$,$\mathbf{V_1}$)

?

(r_1,r_2) = ClosestPairRec($\mathbf{H_2}$,$\mathbf{V_2}$)

δ = min(d(l_1,l_2), d(r_1,r_2))

$x^* = \max_i x_i$ for i=$1,\ldots,n$

L = {(x,y) : x=x^*}

S = set of points within distance δ of **L.**

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

Compute distance between s

and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2)

that minimises this distance.

If d(s_1,s_2) ≤ δ

Return (s_1,s_2)

Else if d(l_1,l_2) < d(r_1,r_2)

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Running Time

Algorithm ClosestPair($p_1,\ldots,p_n$)

Construct arrays **H** and **V**

(p_1,p_2) = ClosestPairRec(**H**,**V**)

O(n log n)

Procedure ClosestPairRec(**H**,**V**)

If $|\mathbf{H}| = |\mathbf{V}| \leq 3$

Check all pairwise distances

Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = ClosestPairRec($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = ClosestPairRec($\mathbf{H_2}$,$\mathbf{V_2}$)

O(n log n)
known recurrence

$\delta = \min(d(l_1,l_2), d(r_1,r_2))$

$x^* = \max_i x_i$ for $i=1,\ldots,n$

$\mathbf{L} = \{(x,y) : x=x^*\}$

S = set of points within distance δ of **L**.

Construct $\mathbf{S_v}$

For each point s in $\mathbf{S_v}$

Compute distance between s and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2) that minimises this distance.

If $d(s_1,s_2) \leq \delta$

Return (s_1,s_2)

Else if $d(l_1,l_2) < d(r_1,r_2)$

Return (l_1,l_2)

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Running Time

Algorithm **ClosestPair**($p_1,\dots,p_n$)

Construct arrays **H** and **V**

(p_1,p_2) = **ClosestPairRec**(**H**,**V**)

O(n log n)

Procedure **ClosestPairRec**(**H**,**V**)

If $|\mathbf{H}| = |\mathbf{V}| \leq 3$

Check all pairwise distances

O(n) Construct $\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$

(l_1,l_2) = **ClosestPairRec**($\mathbf{H_1}$,$\mathbf{V_1}$)

(r_1,r_2) = **ClosestPairRec**($\mathbf{H_2}$,$\mathbf{V_2}$)

O(n log n)
known recurrence

$\delta = \min(d(l_1,l_2), d(r_1,r_2))$

$x^* = \max_i x_i$ for $i=1,\dots,n$

O(n) **L** = $\{(x,y) : x=x^*\}$

S = set of points within distance δ of **L**.

Construct $\mathbf{S_v}$ O(n)

For each point s in $\mathbf{S_v}$ O(n)

Compute distance between s

and the next 15 points of $\mathbf{S_v}$.

Return the pair of points (s_1,s_2)

that minimises this distance.

If $d(s_1,s_2) \leq \delta$

Return (s_1,s_2) O(1)

Else if $d(l_1,l_2) < d(r_1,r_2)$

Return (l_1,l_2)

Else return (r_1,r_2)

Integer Multiplication

Integer Multiplication

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

```
    1100
x   1101
--------
```

Integer Multiplication

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

$$\begin{array}{r} 1100 \\ \times\ 1101 \\ \hline 1100 \end{array}$$

Integer Multiplication

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

```
   1100
x  1101
-------
   1100
  0000
```

Integer Multiplication

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

```
      1100
  x   1101
----------
      1100
     0000
    1100
```

Integer Multiplication

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

```
    1100
x   1101
--------
    1100
   0000
  1100
 1100
```

Integer Multiplication

```
     1100
 x   1101
---------
     1100
    0000
   1100
  1100
---------
```

Input: 2 integer numbers, in *binary*

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```
      1100
   x  1101
  ---------
      1100
     0000
    1100
   1100
  ---------
  10011100
```

Integer Multiplication

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      1100
   x  1101
  ---------
      1100
     0000
    1100
   1100
  ---------
  10011100
```

Input: 2 integer numbers, in *binary*
n is the number of *bits* of each number

What is the running time of this algorithm?

Integer Multiplication

```
     1100
 x   1101
---------
     1100
    0000
   1100
  1100
---------
 10011100
```

Input: 2 integer numbers, in *binary*

n is the number of *bits* of each number

What is the running time of this algorithm?

O(n) time to compute each partial product

n partial products

Time $O(n^2)$

Can we do it faster?

- We will use the Divide-and-Conquer approach.
- We will reduce the problem to solving some instances with $n/2$ bits.
- Then we will use this approach recursively to get a solution for the original problem.

Faster multiplication

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$$

$$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T}(n) \leq \mathbf{4T}(n/2) + \mathbf{c}n$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

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4 multiplications of **n/2** bit numbers

T(*n*) ≤ 4T(*n/2*) + c*n*

Solving the recurrence gets us to **T(n) = O($n^{\log 4}$) = O(*n*^2)**

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T(\mathit{n}) \leq 4T(\mathit{n/2}) + c\mathit{n}}$

Solving the recurrence gets us to $\mathbf{T(n) = O(\mathit{n}^{\log 4}) = O(\mathit{n}\hat{\ }2)}$

Generally, solving the recurrence $\mathbf{T(\mathit{n}) \leq \mathit{q}T(\mathit{n/2}) + c\mathit{n}}$ gets us to $\mathbf{T(n) = O(\mathit{n}^{\log \mathit{q}})}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T(}n\mathbf{) \leq 4T(}n/2\mathbf{) + c}n$

Solving the recurrence gets us to $\mathbf{T(n) = O(}n^{\log 4}\mathbf{) = O(}n^{\wedge 2}\mathbf{)}$

Generally, solving the recurrence $\mathbf{T(}n\mathbf{)} \leq q\mathbf{T(}n/2\mathbf{) + c}n$ gets us to $\mathbf{T(n) = O(}n^{\log q}\mathbf{)}$

$(\mathbf{x_1} + \mathbf{x_0})(\mathbf{y_1} + \mathbf{y_0}) = \mathbf{x_1} \cdot \mathbf{y_1} + \mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1} + \mathbf{x_0} \cdot \mathbf{y_0}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

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4 multiplications of **n/2** bit numbers

$\mathbf{T(\mathit{n}) \leq 4T(\mathit{n/2}) + c\mathit{n}}$

Solving the recurrence gets us to $\mathbf{T(n) = O(\mathit{n}^{\log 4}) = O(\mathit{n}^{\wedge 2})}$

Generally, solving the recurrence $\mathbf{T(\mathit{n}) \leq \mathit{q}T(\mathit{n/2}) + c\mathit{n}}$ gets us to $\mathbf{T(n) = O(\mathit{n}^{\log \mathit{q}})}$

$(\mathbf{x_1} + \mathbf{x_0})\,(\mathbf{y_1} + \mathbf{y_0}) = \mathbf{x_1} \cdot \mathbf{y_1} + \mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1} + \mathbf{x_0} \cdot \mathbf{y_0}$

$= \mathbf{p}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{p} - \mathbf{x_1} \cdot \mathbf{y_1} - \mathbf{x_0} \cdot \mathbf{y_0}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T(}n\mathbf{)} \leq \mathbf{4T(}n/2\mathbf{)} + \mathbf{c}n$

Solving the recurrence gets us to $\mathbf{T(n) = O(}n^{\log 4}\mathbf{) = O(}n\text{^}2\mathbf{)}$

Generally, solving the recurrence $\mathbf{T(}n\mathbf{)} \leq q\mathbf{T(}n/2\mathbf{)} + \mathbf{c}n$ gets us to $\mathbf{T(n) = O(}n^{\log q}\mathbf{)}$

$(\mathbf{x_1} + \mathbf{x_0})\,(\mathbf{y_1} + \mathbf{y_0}) = \mathbf{x_1} \cdot \mathbf{y_1} + \mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1} + \mathbf{x_0} \cdot \mathbf{y_0}$

$= \mathbf{p}$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{p} - \mathbf{x_1} \cdot \mathbf{y_1} - \mathbf{x_0} \cdot \mathbf{y_0}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T}(n) \leq \mathbf{4T}(n/2) + \mathbf{c}n$

Solving the recurrence gets us to $\mathbf{T}(n) = \mathbf{O}(n^{\log 4}) = \mathbf{O}(n\text{^}2)$

Generally, solving the recurrence $\mathbf{T}(n) \leq q\mathbf{T}(n/2) + \mathbf{c}n$ gets us to $\mathbf{T}(n) = \mathbf{O}(n^{\log q})$

$(\mathbf{x_1} + \mathbf{x_0})(\mathbf{y_1} + \mathbf{y_0}) = \mathbf{x_1} \cdot \mathbf{y_1} + \mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1} + \mathbf{x_0} \cdot \mathbf{y_0}$

$= \mathbf{p}$

We get the recurrence $\mathbf{T}(n) \leq \mathbf{3T}(n/2) + \mathbf{c}n$

Faster multiplication

$\mathbf{x} = \mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}$

$\mathbf{y} = \mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0}$

$\mathbf{x} \cdot \mathbf{y} = (\mathbf{x_1} \cdot 2^{n/2} + \mathbf{x_0}) \cdot (\mathbf{y_1} \cdot 2^{n/2} + \mathbf{y_0})$

$= \mathbf{x_1} \cdot \mathbf{y_1} \cdot 2^{n/2} + (\mathbf{p} - \mathbf{x_1} \cdot \mathbf{y_1} - \mathbf{x_0} \cdot \mathbf{y_0}) \cdot 2^{n/2} + \mathbf{x_0} \cdot \mathbf{y_0}$

4 multiplications of **n/2** bit numbers

$\mathbf{T}(n) \leq \mathbf{4T}(n/2) + \mathbf{c}n$

Solving the recurrence gets us to $\mathbf{T(n) = O(}n^{\log 4}\mathbf{) = O(}n\text{\^{}}2\mathbf{)}$

Generally, solving the recurrence $\mathbf{T}(n) \leq q\mathbf{T}(n/2) + \mathbf{c}n$ gets us to $\mathbf{T(n) = O(}n^{\log q}\mathbf{)}$

$(\mathbf{x_1} + \mathbf{x_0})(\mathbf{y_1} + \mathbf{y_0}) = \mathbf{x_1} \cdot \mathbf{y_1} + \mathbf{x_1} \cdot \mathbf{y_0} + \mathbf{x_0} \cdot \mathbf{y_1} + \mathbf{x_0} \cdot \mathbf{y_0}$

$= \mathbf{p}$

We get the recurrence $\mathbf{T}(n) \leq \mathbf{3T}(n/2) + \mathbf{c}n$

Faster multiplication

$\mathbf{x} = x_1 \cdot 2^{n/2} + x_0$

$\mathbf{y} = y_1 \cdot 2^{n/2} + y_0$

$\mathbf{x} \cdot \mathbf{y} = (x_1 \cdot 2^{n/2} + x_0) \cdot (y_1 \cdot 2^{n/2} + y_0)$

$= x_1 \cdot y_1 \cdot 2^{n/2} + (p - x_1 \cdot y_1 - x_0 \cdot y_0) \cdot 2^{n/2} + x_0 \cdot y_0$

4 multiplications of **n/2** bit numbers

$\mathbf{T(n) \leq 4T(n/2) + cn}$

Solving the recurrence gets us to $\mathbf{T(n) = O(n^{\log 4}) = O(n\text{^}2)}$

Generally, solving the recurrence $\mathbf{T(n) \leq qT(n/2) + cn}$ gets us to $\mathbf{T(n) = O(n^{\log q})}$

$\mathbf{\log_2 3 = 1.59}$

$(x_1 + x_0)(y_1 + y_0) = x_1 \cdot y_1 + x_1 \cdot y_0 + x_0 \cdot y_1 + x_0 \cdot y_0$

$= p$

We get the recurrence $\mathbf{T(n) \leq 3T(n/2) + cn}$

Faster multiplication

$\mathbf{x} = x_1 \cdot 2^{n/2} + x_0$

$\mathbf{y} = y_1 \cdot 2^{n/2} + y_0$

$\mathbf{x} \cdot \mathbf{y} = (x_1 \cdot 2^{n/2} + x_0) \cdot (y_1 \cdot 2^{n/2} + y_0)$

$= x_1 \cdot y_1 \cdot 2^{n/2} + (p - x_1 \cdot y_1 - x_0 \cdot y_0) \cdot 2^{n/2} + x_0 \cdot y_0$

4 multiplications of **n/2** bit numbers

$\mathbf{T(n) \leq 4T(n/2) + cn}$

Solving the recurrence gets us to $\mathbf{T(n) = O(n^{\log 4}) = O(n\textasciicircum 2)}$

Generally, solving the recurrence $\mathbf{T(n) \leq qT(n/2) + cn}$ gets us to $\mathbf{T(n) = O(n^{\log q})}$

$\mathbf{\log_2 3 = 1.59}$

$\mathbf{T(n) = O(n^{1.59})}$

$(x_1 + x_0)(y_1 + y_0) = x_1 \cdot y_1 + x_1 \cdot y_0 + x_0 \cdot y_1 + x_0 \cdot y_0$

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We get the recurrence $\mathbf{T(n) \leq 3T(n/2) + cn}$