Advanced Algorithmic Techniques (COMP523)

Recursion and Divide and Conquer Techniques #4

Recap and plan

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- **Last lecture:**
  - Finding the closest pair of points.
  - Integer Multiplication.

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- **Last lecture:**
  - Finding the closest pair of points.
  - Integer Multiplication.
- **This lecture:**
  - The Selection problem.
    - i.e., finding the i^{th}-order statistic in an array.

The selection problem

- **Definition:** The i^{th}-order statistic of a set of n (distinct) elements is the i^{th} smallest element.
  - i.e., the element which is larger than exactly *i-1* other elements.

The Selection Problem: Selection(**A**[*1,…,n*],*i*)

Input: A set of n (distinct) numbers (in an array **A**) and a number *i*, with $1 \leq i \leq n$.
Output: The i^{th}-order statistic of the set.

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- Return the *i*-th element of the sorted array.
- Is sorting an overkill?

Divide and conquer

- Split the input into smaller inputs.
- Solve the problem for the smaller inputs recursively.
- Combine the solutions into a solution for the original problem.

A glance back at the QuickSort algorithm

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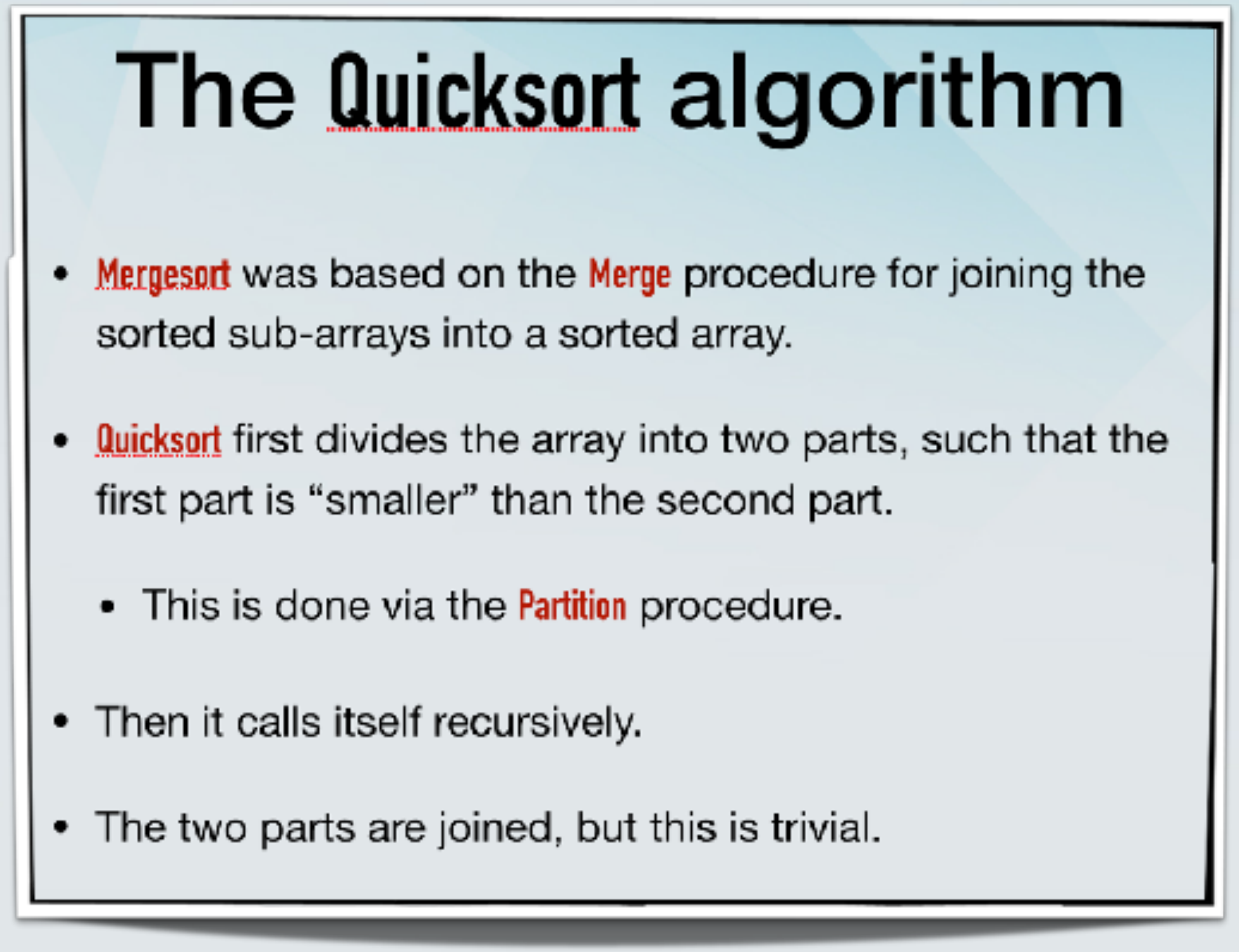

The Quicksort algorithm

- Mergesort was based on the Merge procedure for joining the sorted sub-arrays into a sorted array.
- Quicksort first divides the array into two parts, such that the first part is "smaller" than the second part.
  - This is done via the Partition procedure.
- Then it calls itself recursively.
- The two parts are joined, but this is trivial.

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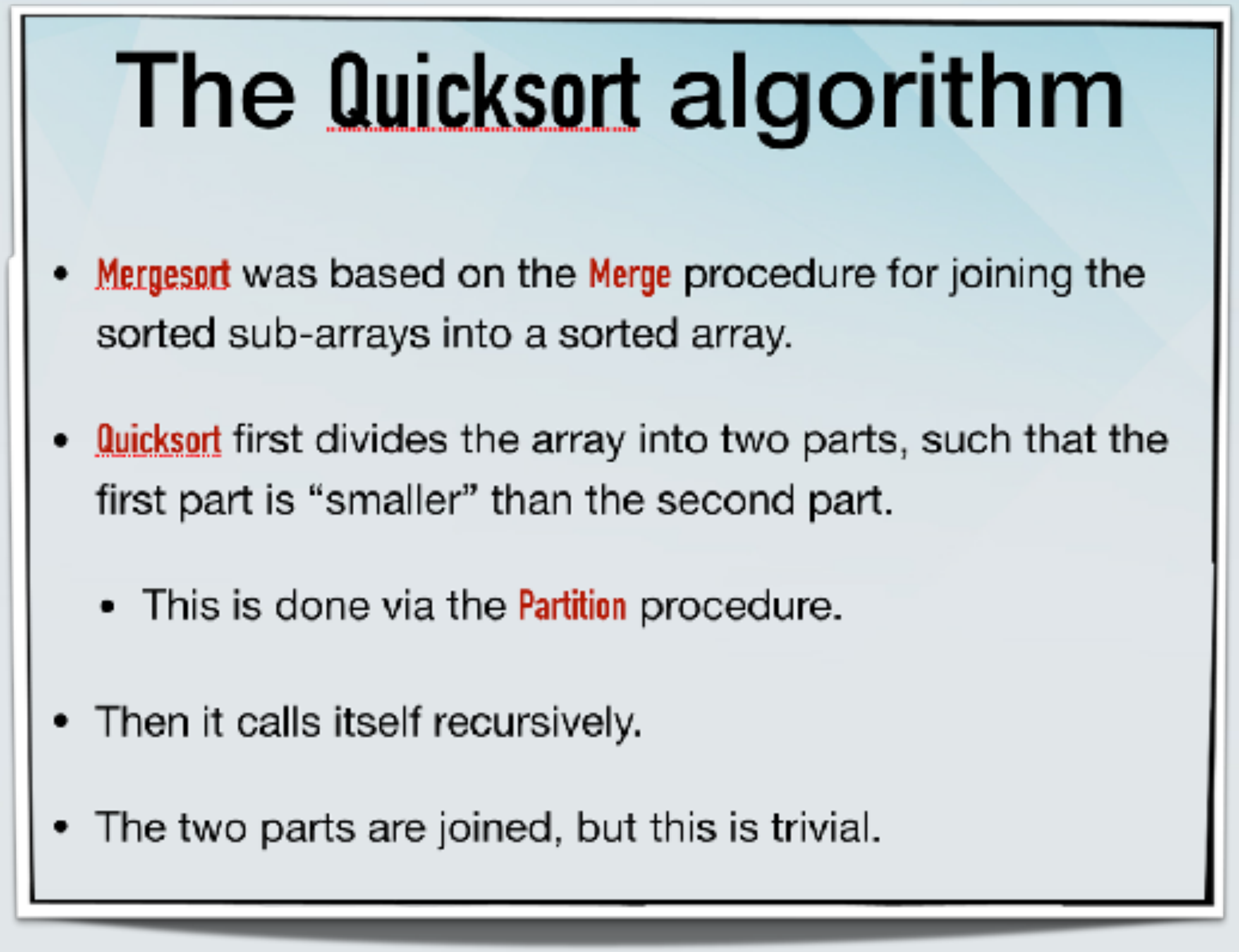

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Recall the Partition procedure

Revisiting the Partition procedure

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Procedure Partition(A[$i,\ldots,j$])

    Choose a pivot element x of A

    $k = i-1$

    For $h = i$ to $j-1$ do

        If A[$h$] $\leq$ x
            $k = k + 1$
            Swap A[$k$] with A[$h$]

    Swap A[$k+1$] with A[$j$]

Return $k+1$
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Running time **O(n)**

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- Then, we can reduce the search for the *i*-th element to one of the three subarrays.
- How can we choose the element **x** *appropriately*, such that the subarrays **A**[*1,…x-1*] and **A**[*x+1, … , n*] are of (approximately) equal size?

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- What is an algorithm for finding the median?

    - Selection(**A**[*1,…,n*],$(n+1)/2$)

Let’s try to do that…

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Algorithm Selection(A[$1,\ldots,n$],$i$)

    x = Selection(A[$1,\ldots,n$],$(n+1)/2$)
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Are we stuck?

- We need to partition the array into two using a good pivot element (the median).
    - Or otherwise the running time of the recursion will be bad!
- But to find the median, we need an algorithm for selection!

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- We need to partition the array into two using a good pivot element (something “close” to the median).
  - Or otherwise the running time of the recursion will be bad!
- But to find the median, we need an algorithm for selection!

A good pivot element

- Split the array **A** into sub-arrays with **5** elements each.
  - The last one might have fewer elements.

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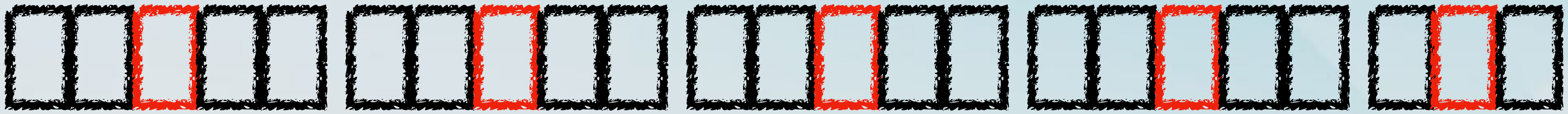

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Run InsertionSort

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- Find the **median-of-medians**.

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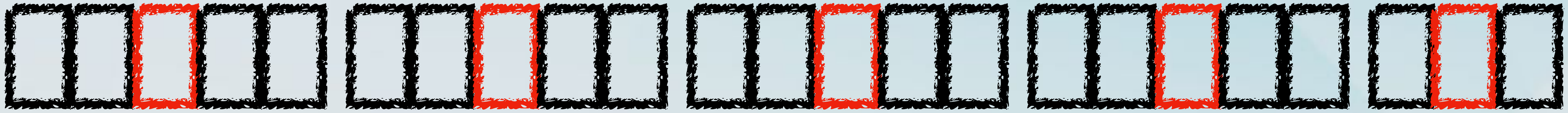

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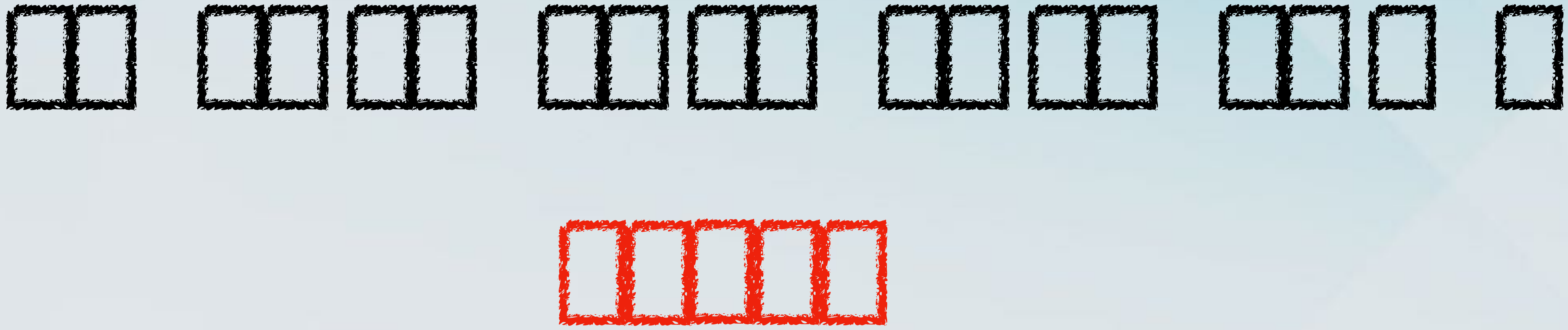

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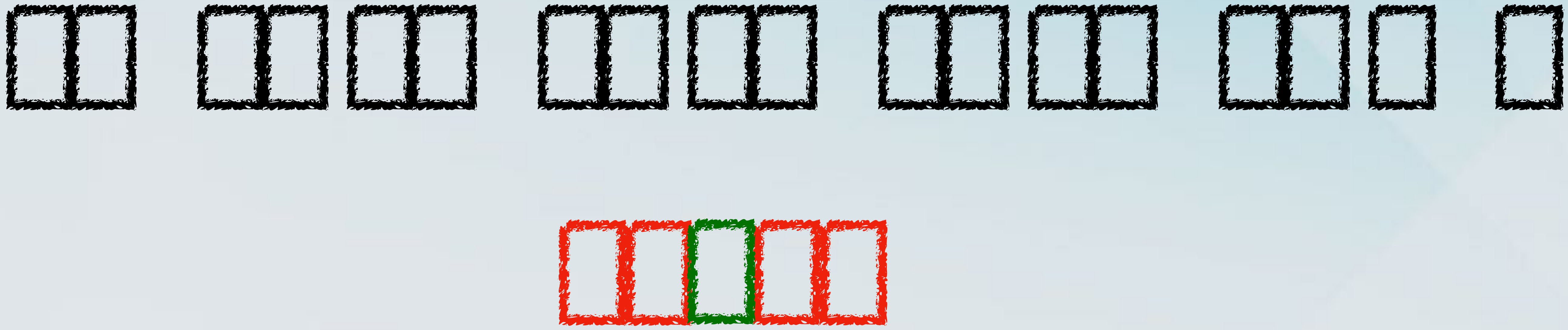

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    Run Selection

This failed...

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Algorithm Selection(A[$1,\ldots,n$],$i$)

    x = Selection(A[$1,\ldots,n$],$(n+1)/2$)
    y = Partition(A[$1,\ldots,n$],x)
```

...but this won't.

```
Algorithm Selection(A[$1,\ldots,n$],$i$)

    Split the array A into n/5 arrays of size 5
    For each subarray $A_i$, find the median
    Let $m_1, m_2, \ldots, m_{n/5}$ be those medians

    x = Selection(A[$1,\ldots,n$],$(n+1)/2$)   /*Find the median of medians */
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The Selection algorithm (not exactly pseudocode)

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     k-1 is the number of elements in the lower subarray.

     If $i$ = k, return x
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     If $i$ > k, return Selection(A[$k+1,\ldots,n$],$i-k$)
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Zooming in

```
If $i$ = k, return **x**
If $i$ < k, return Selection(A[$1,\ldots,k-1$],$i$)
If $i$ > k, return Selection(A[$k+1,\ldots,n$],$i-k$)
```

- We are looking for the i^{th}-order statistic.
- If i=k, then x is the answer - it is larger than k-1 elements.
- If $i \leq$ k, the answer cannot be in the second part, as then i would be larger than at least k-1 elements.
- For the same reason, if $i \geq$ k, the answer cannot be in the first part.

Running time

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$|S_{max}| = \max(k-1, n-k)$

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Before we proceed, we have to bound $|S_{max}|$.

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- Each one of these groups has at least 3 elements > **x.**
  - Because **x** ≤ their “baby median”.
  - Except possibly the group containing **x** and the group that has fewer than 5 elements.

Bounding the size of the subarrays

What is the total number of elements larger than **x**?

$$3\left(\left\lceil \frac{1}{2}\cdot\left\lceil\frac{n}{5}\right\rceil\right\rceil - 2\right) \geq \frac{3n}{10} - 6$$

# elements > x in each of those groups

# groups

groups who could be exceptions

# groups with “baby medians” > x

This means that the size of the lower subarray is at most **7n/10 + 6**

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- Back to the recurrence:

  $T(n) \leq T(n/5) + T\,(|S_{max}|) + cn = T(n/5) + T\,(7n/10+6) + bn$

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- This is at most cn whenever $-cn/10 + 7c + bn \le 0$, or equivalently, when $c \ge 10bn/(n-70)$.
- If $n \ge 140$, then $n/(n-70) \le 2$ and then, it suffices to have $c \ge 20b$.

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- We will prove the statement by induction.
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    - **Inductive Step:** Suppose that it holds for all n up to k=140. Then for $n=k+1$, we have $T(n) \leq cn + (-cn/10 + 7c + bn)$

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  - **Inductive Step:** Suppose that it holds for all n up to k=140. Then for $n=k+1$, we have $T(n) \leq cn + (-cn/10 + 7c + bn)$

    - This follows from the fact that $n > 140$ and $c \geq 20b$.

Bonus: The Master Theorem

Suppose $T(n) \leq \alpha T\left(\lceil n/b \rceil\right) + O(n^d)$

for some constants $\alpha > 0,\ b > 1$ and $d \geq 0$.

$$\text{Then,}\ \ T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

Bonus: The Master Theorem

Suppose $T(n) \leq \alpha T\left(\lceil n/b \rceil\right) + O(n^d)$
for some constants $\alpha > 0,\ b > 1$ and $d \geq 0$.

$$\text{Then,}\quad T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b \alpha \\ O(n^d \log_b n), & \text{if } d = \log_b \alpha \\ O(n^{\log_b \alpha}), & \text{if } d < \log_b \alpha \end{cases}$$

Example: For **MergeSort**, α=b=2 and d=1, we get **O(n log n).**