Advanced Algorithmic Techniques (COMP523)

Graph Algorithms #2

Recap and plan

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- **Last lecture:**
  - Graph definitions
  - Graph representations
  - Depth-First Search, Breadth-First Search

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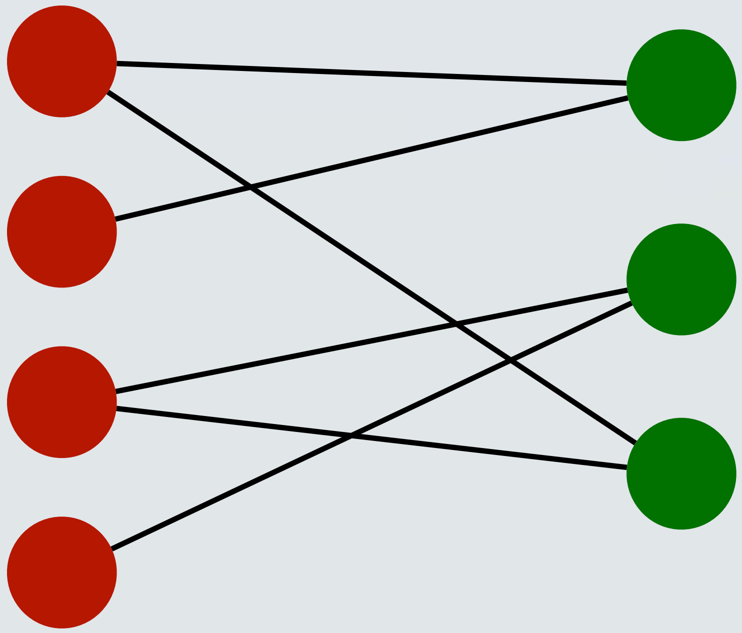
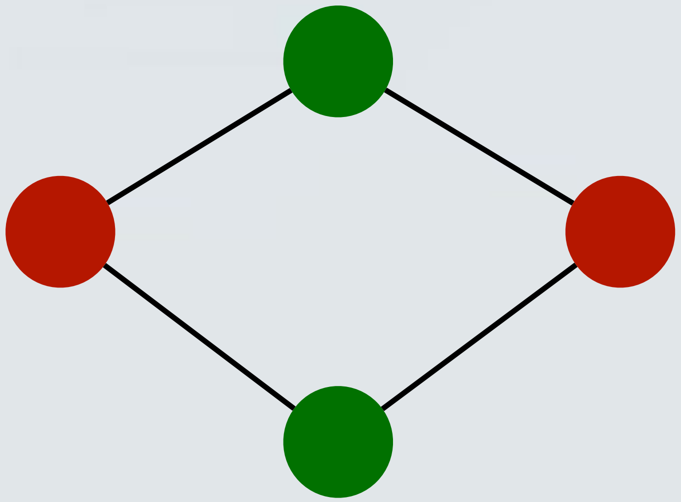
  - Graph definitions
  - Graph representations
  - Depth-First Search, Breadth-First Search
- **This lecture:**
  - Testing bipartiteness
  - DFS and BFS on directed graphs
  - Testing connectivity

Bipartite graphs

- A graph G=(V,E) is bipartite *if any only if* it can be partitioned into sets A and B such that each edge has one endpoint in A and one endpoint in B.
  - Often, we write G=(A U B,E).





Alternative definitions

- A graph G=(V,E) is bipartite *if any only if* its nodes can be coloured with 2 colours (say red and green), such that every vertex has one red endpoint and one green endpoint.
- A graph G=(V,E) is bipartite *if any only if* it does not contain any cycles of odd length.

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  - Generally, we observe that for all k in {1,2, … ,n}, u_k is red if k is odd and green if k is even.

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  - Generally, we observe that for all k in {1,2, … ,n}, u_k is red if k is odd and green if k is even.
  - By assumption, n is odd, so it must be red. But then u cannot be red, because G is bipartite.

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- Sometimes, these alternatives definitions are also called "characterisations".

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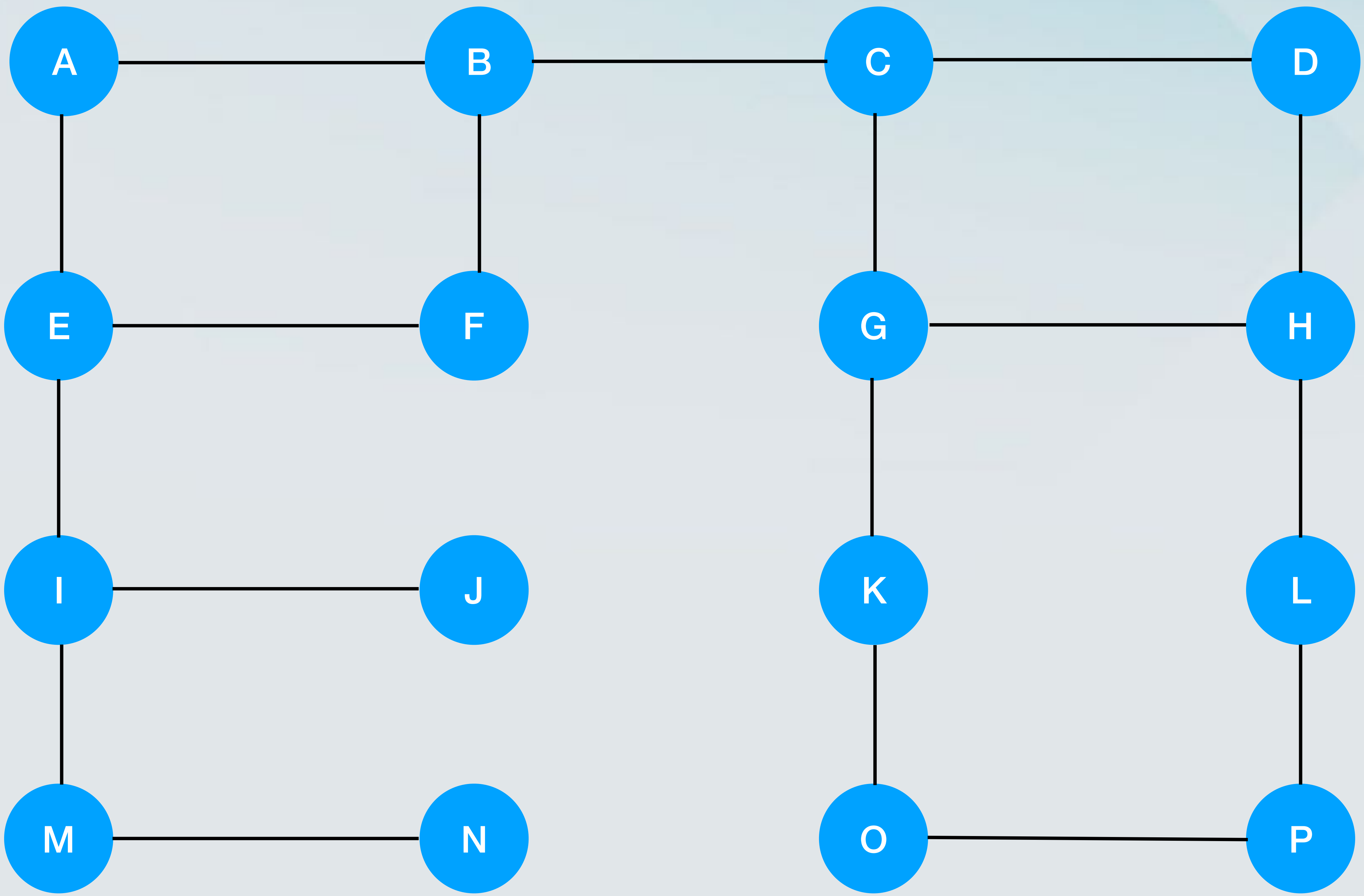

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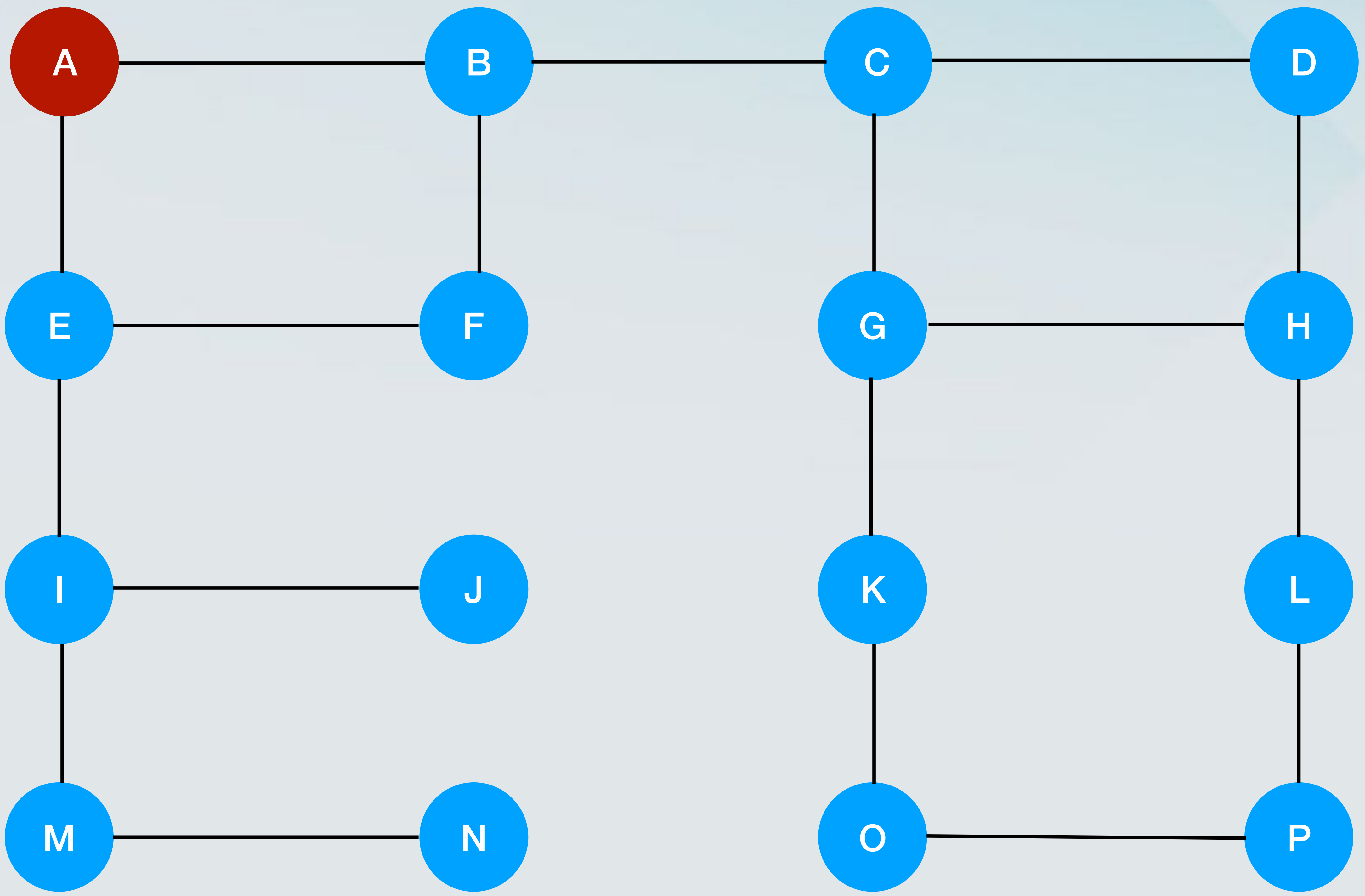

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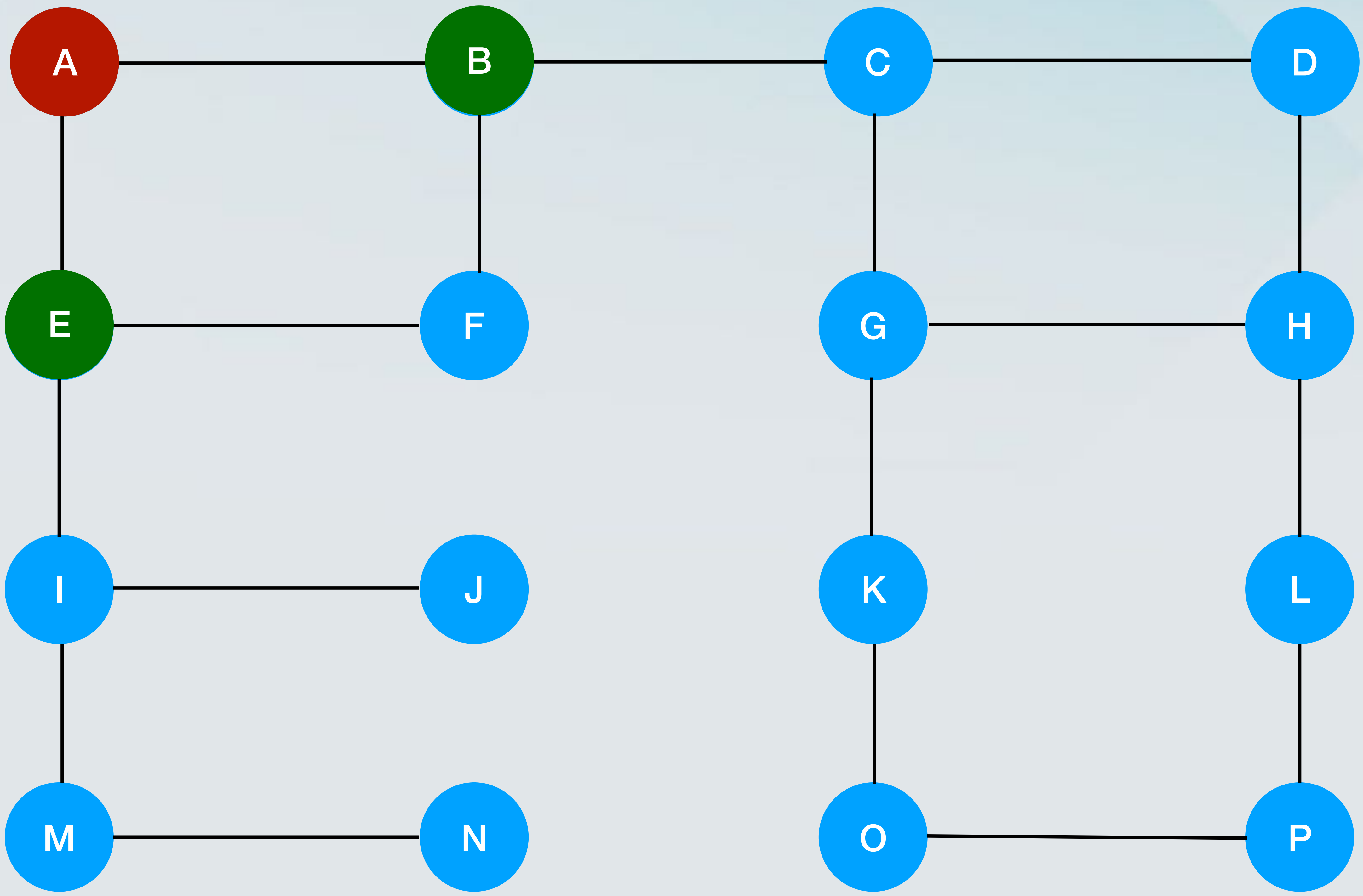

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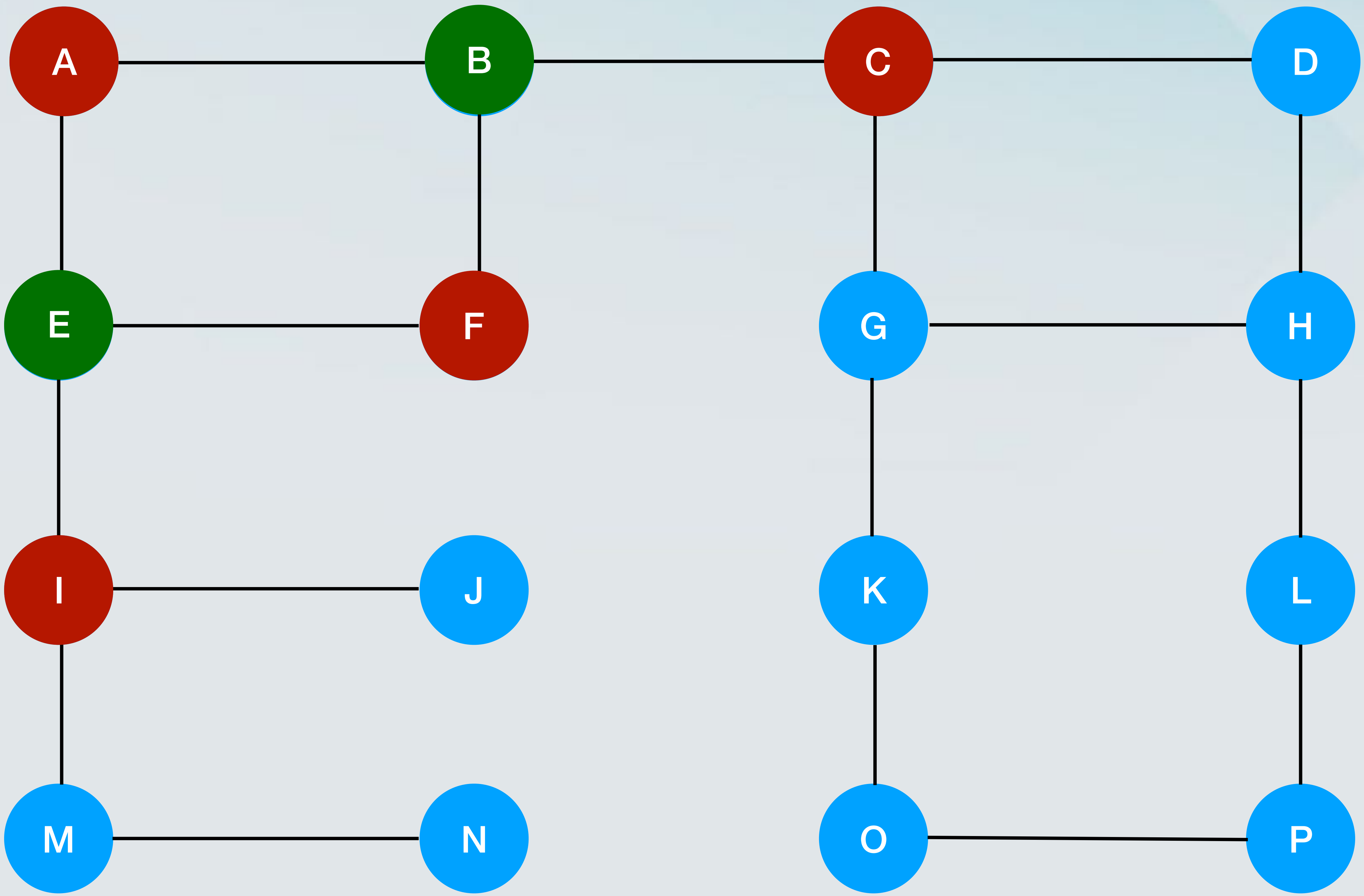

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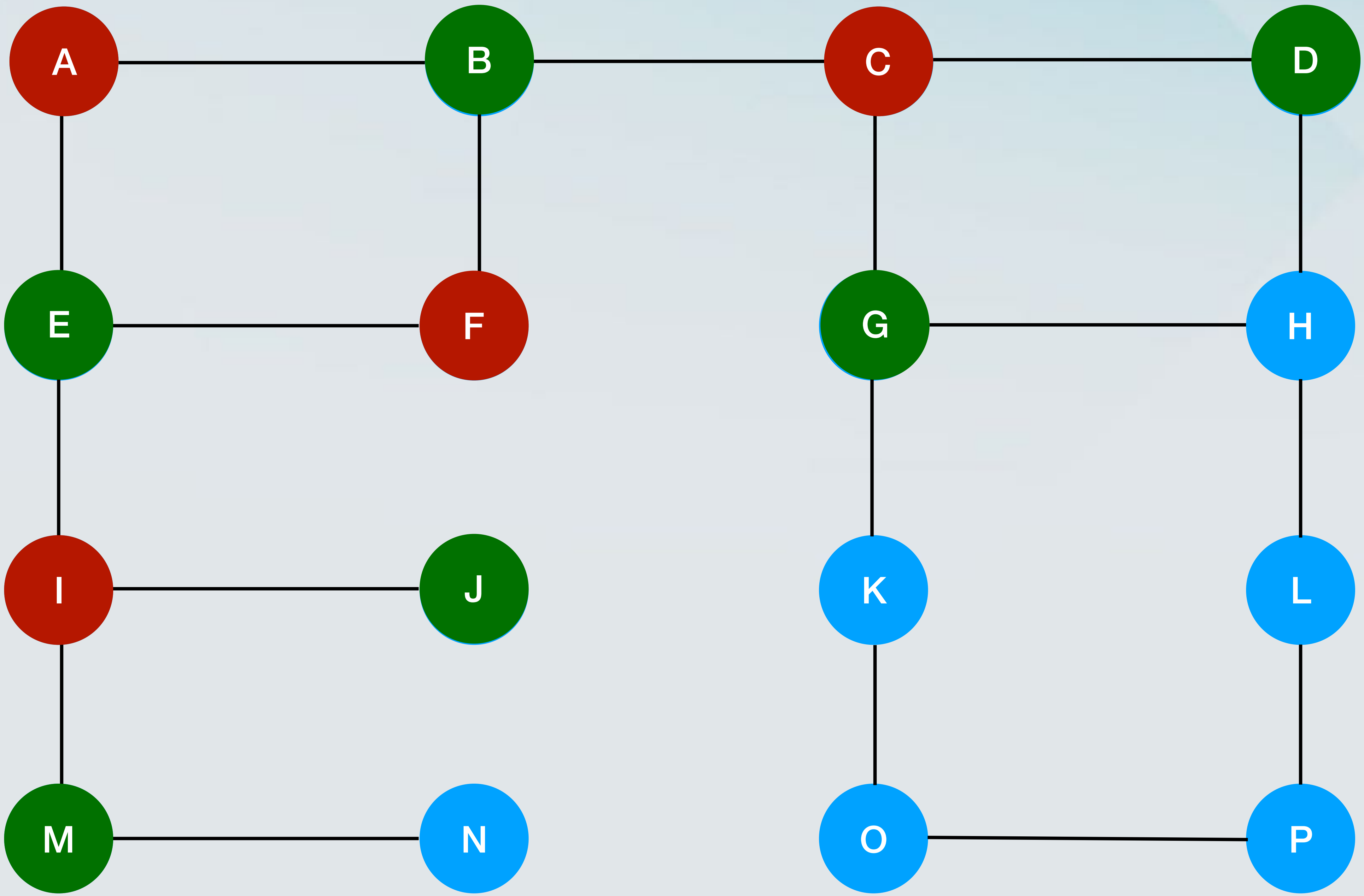

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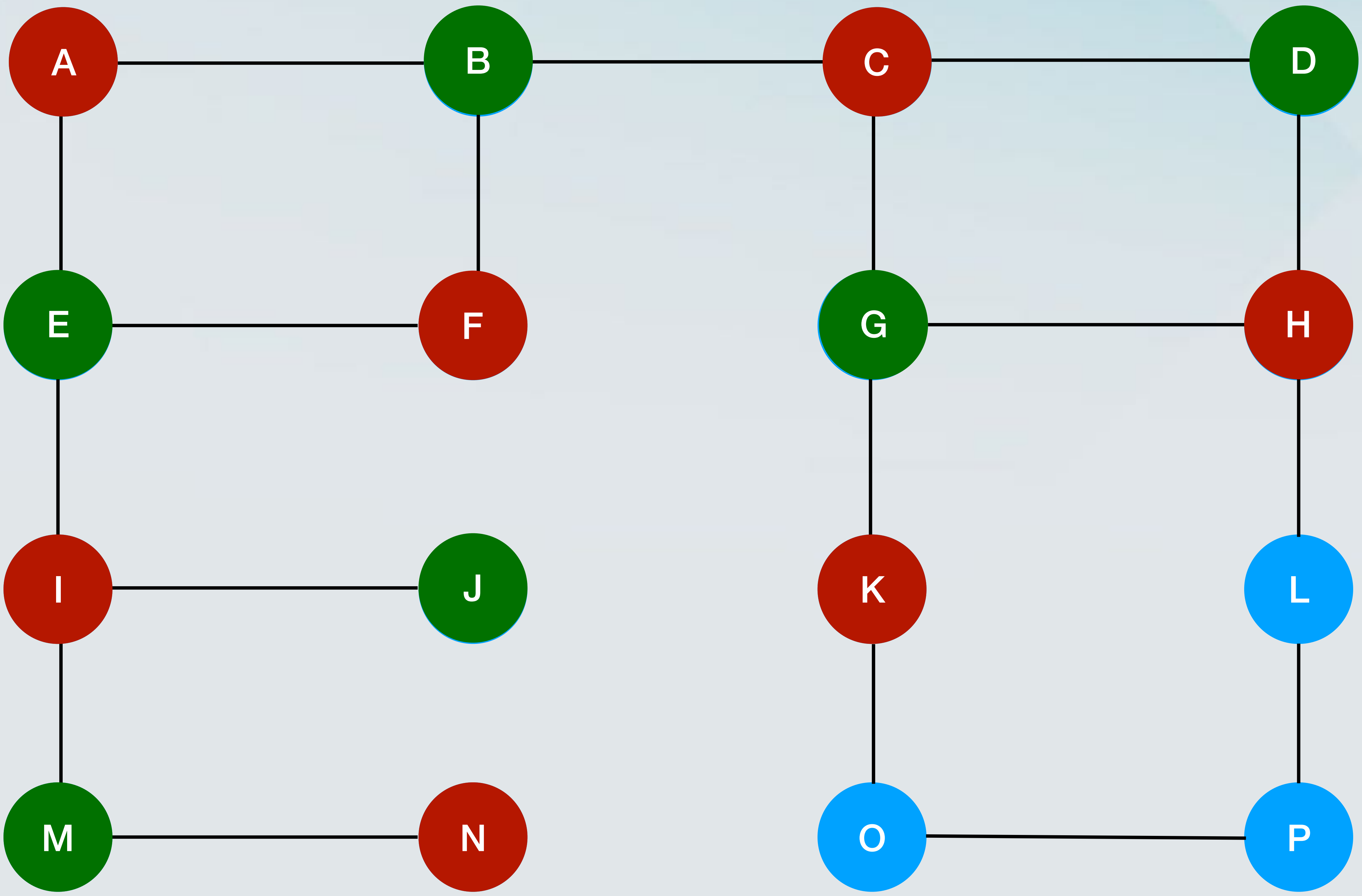

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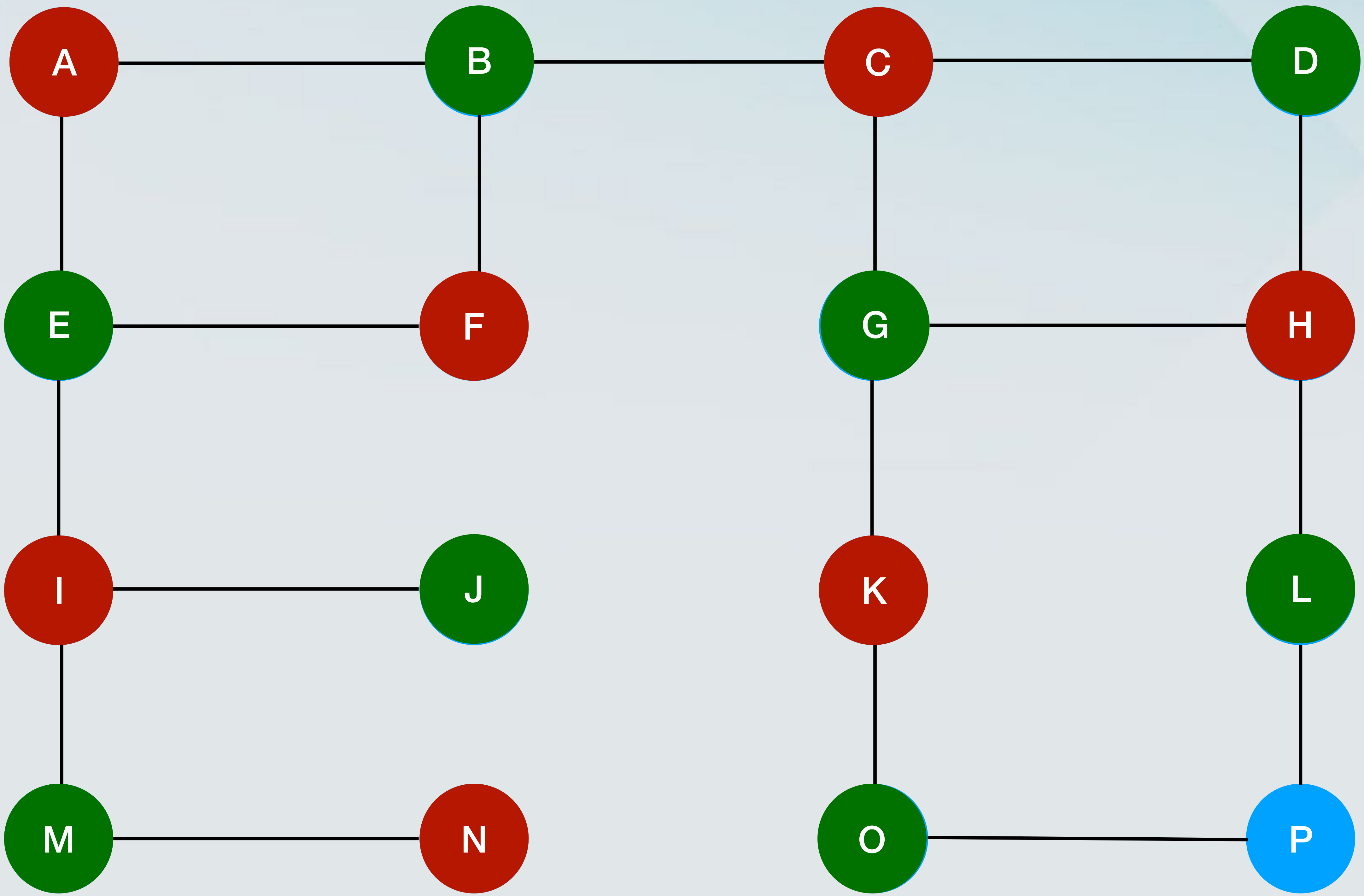

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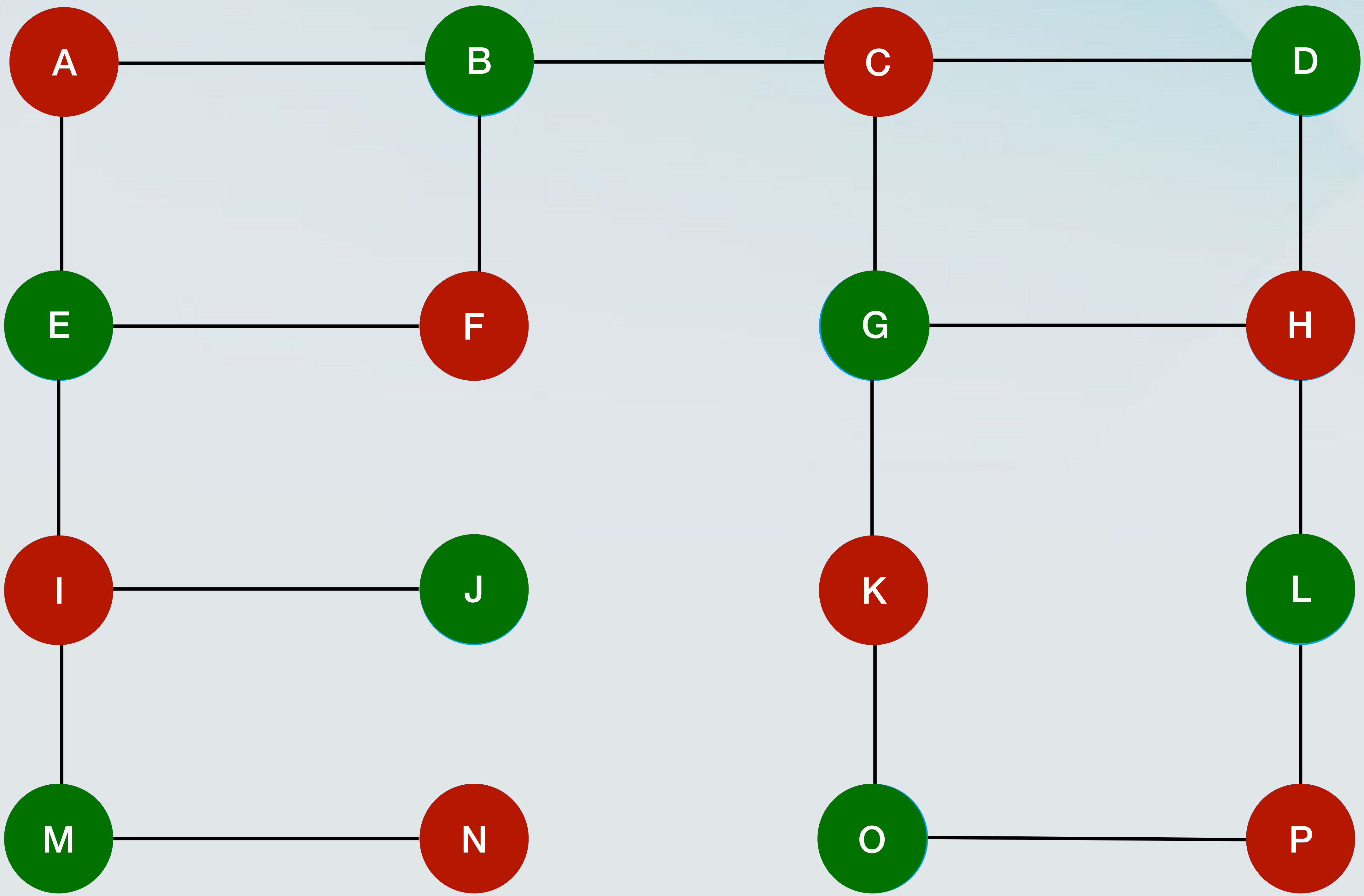

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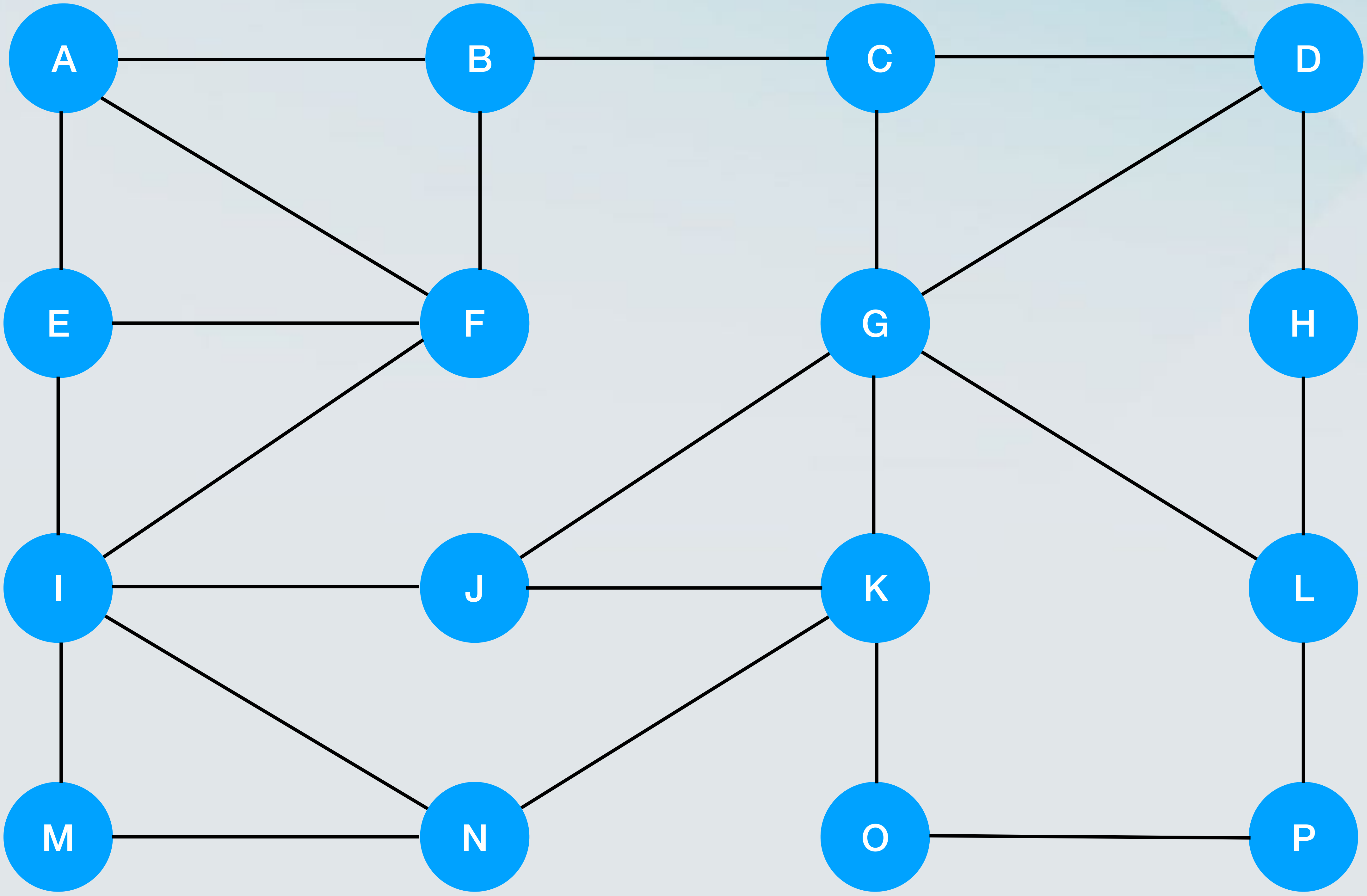

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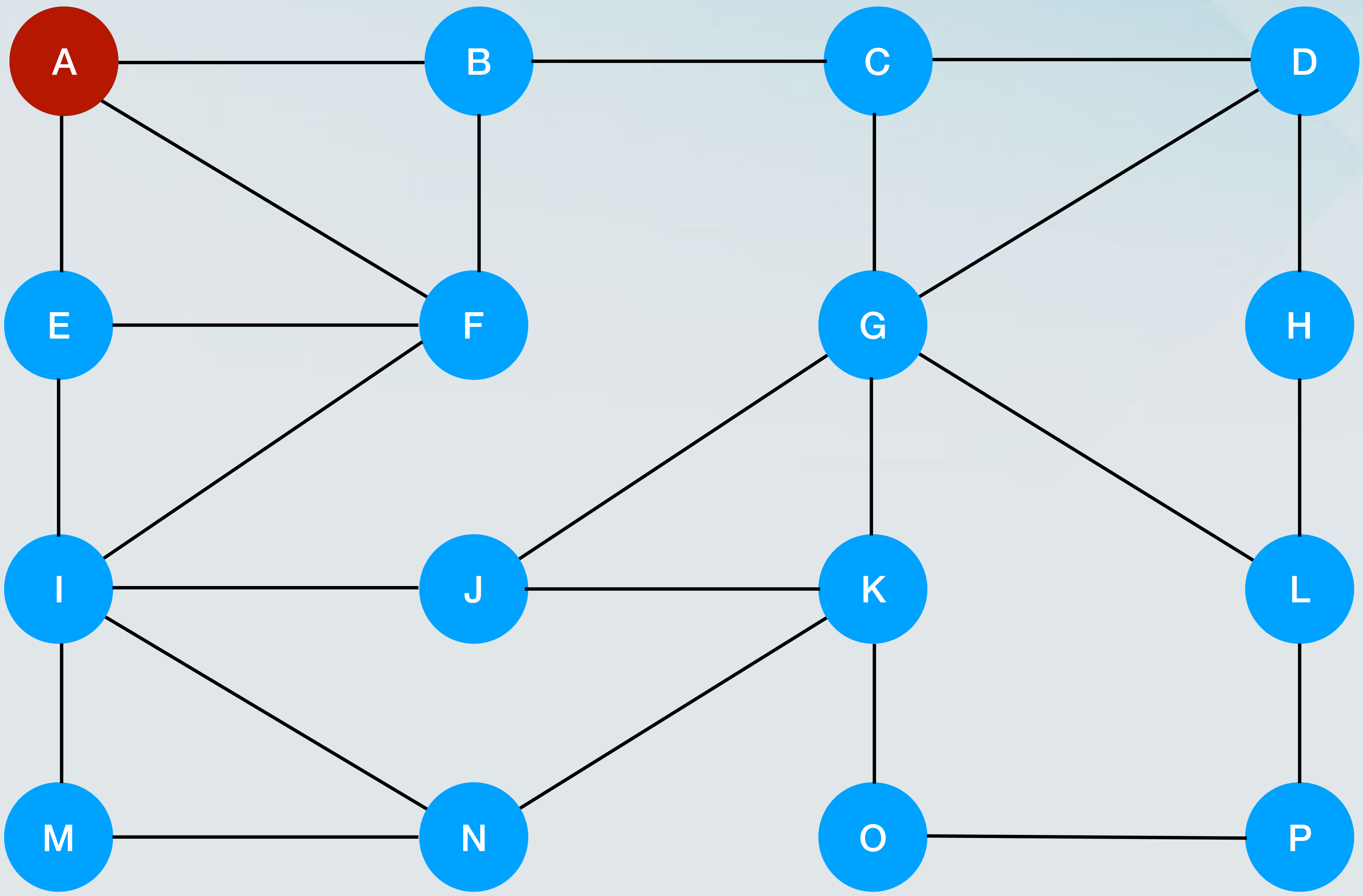

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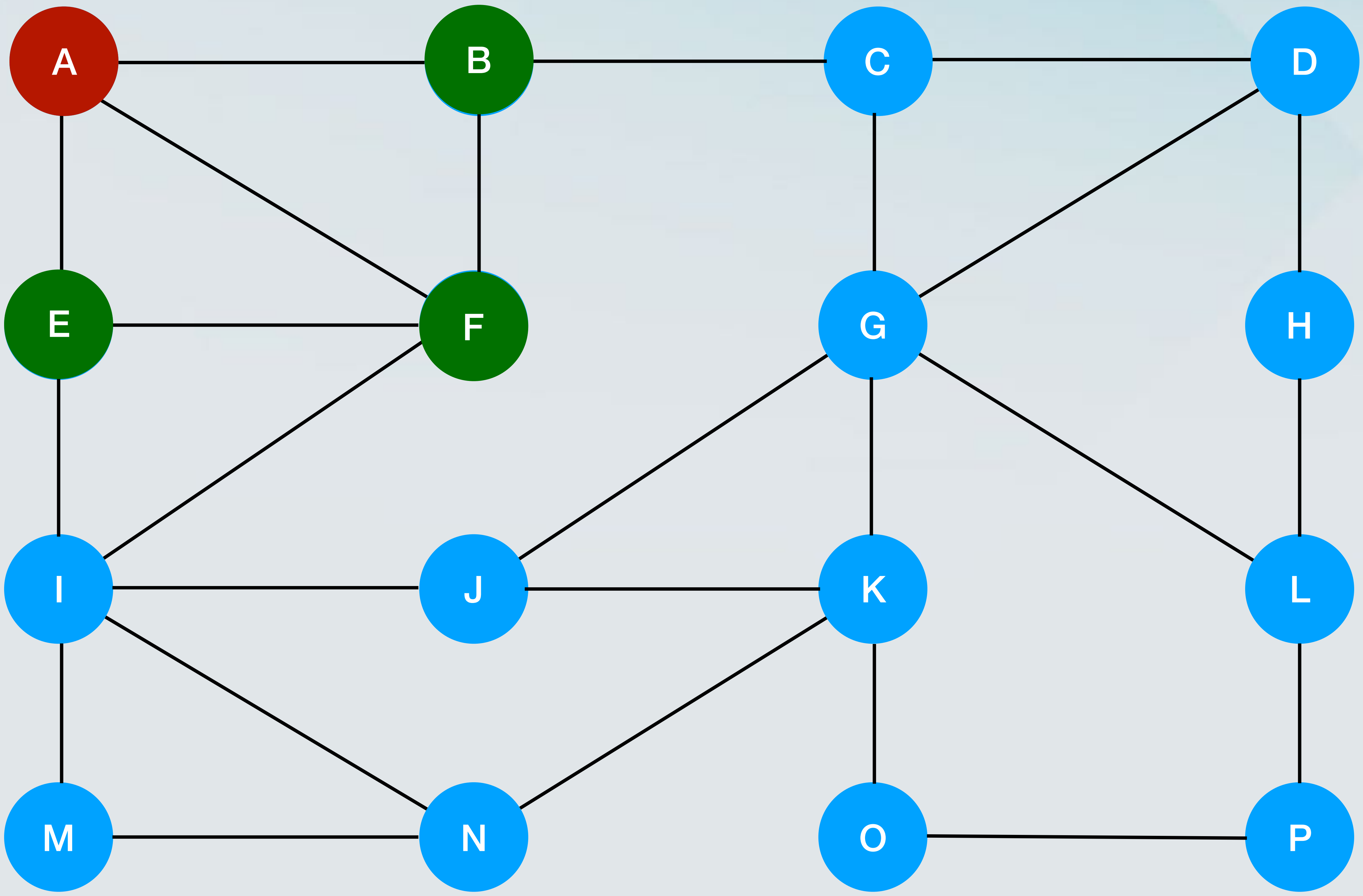

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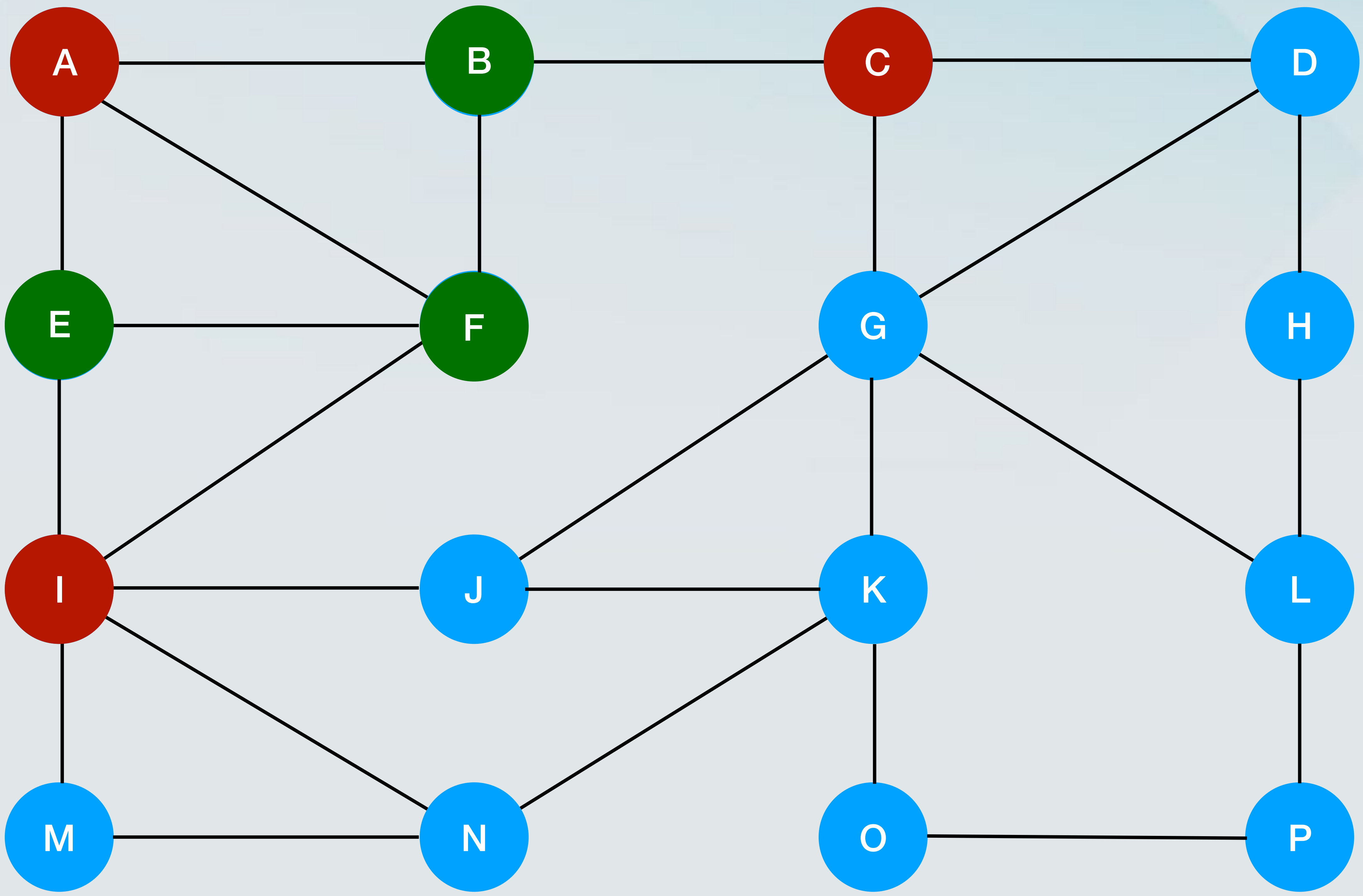

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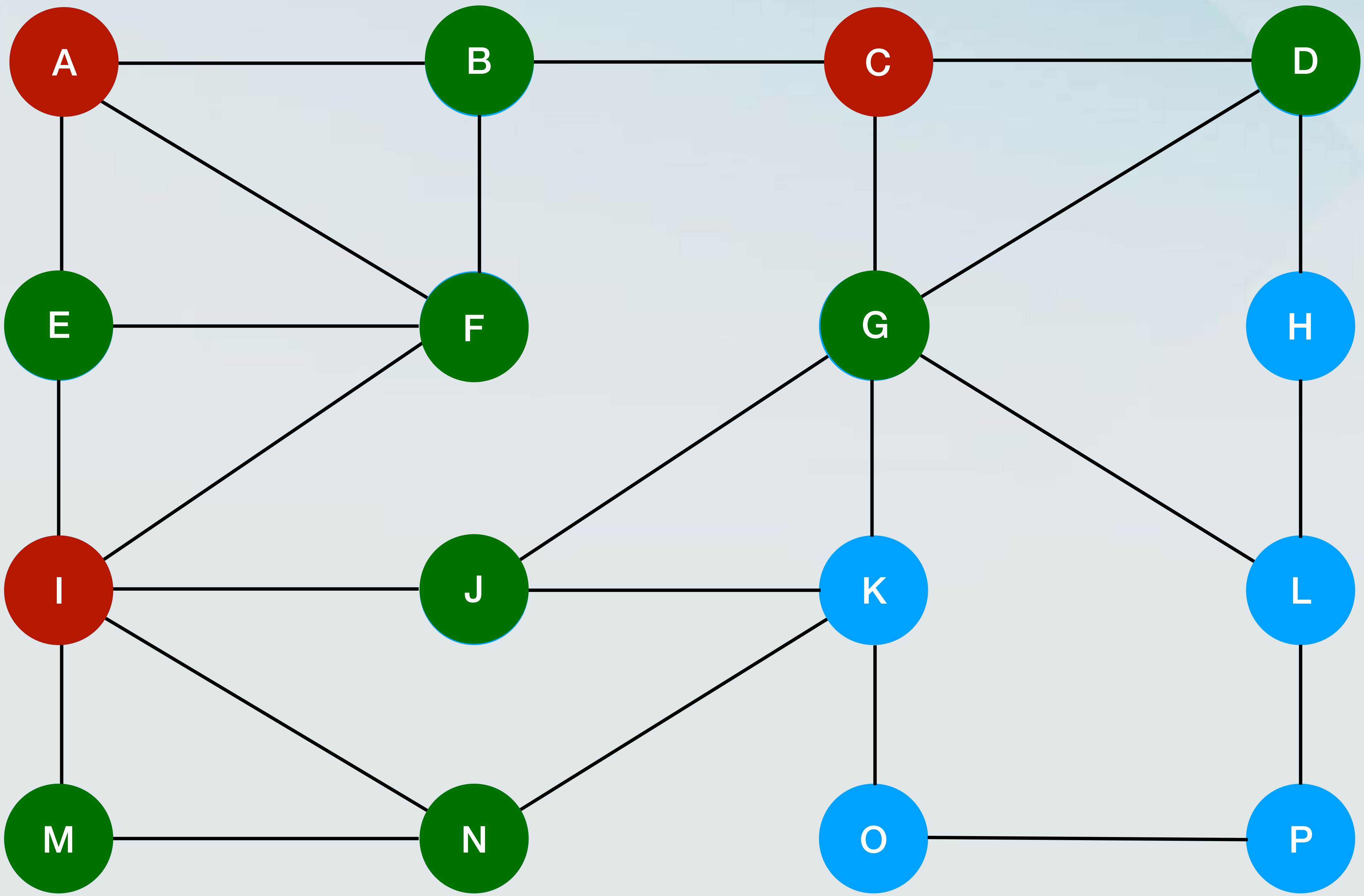

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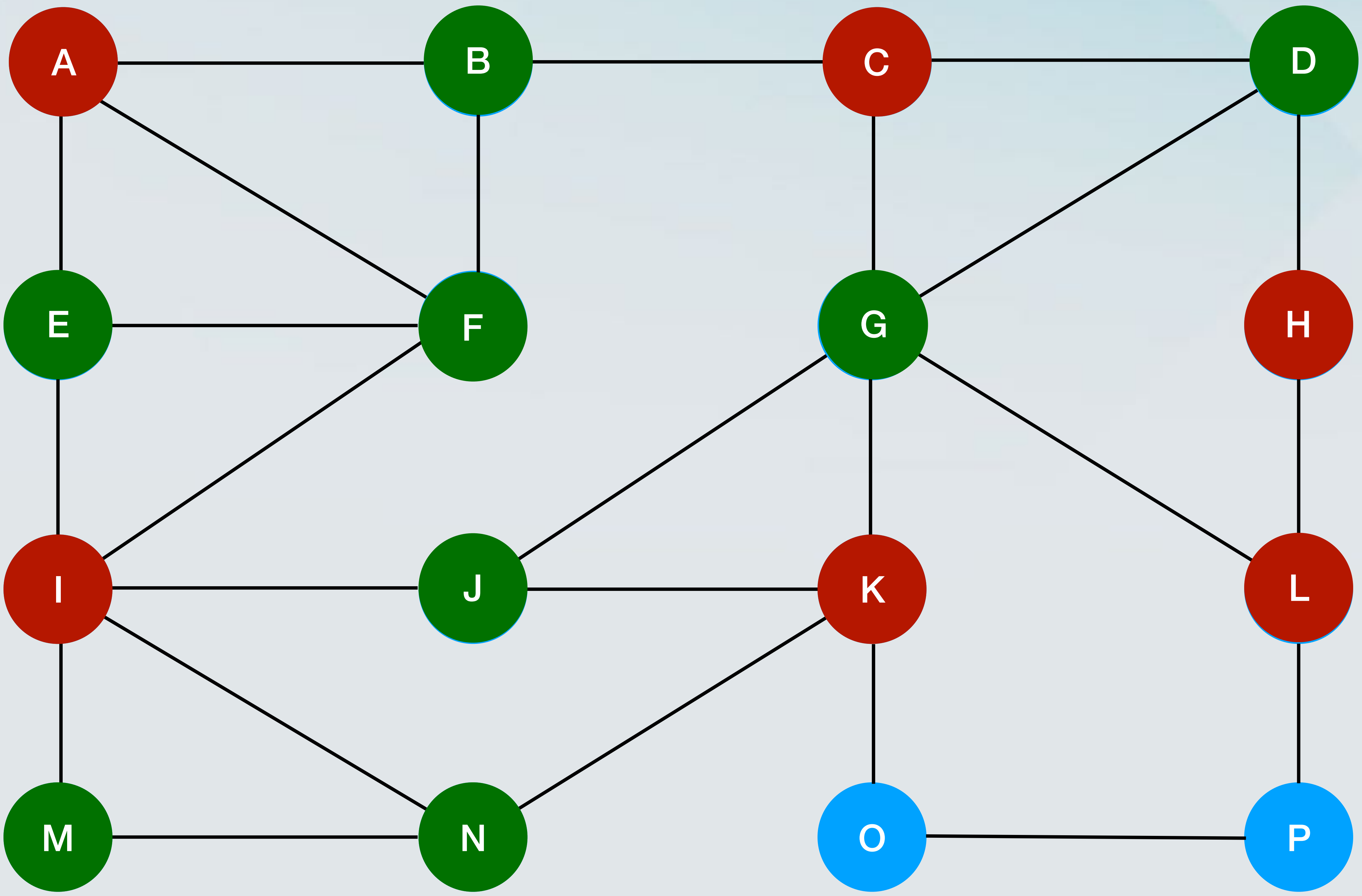

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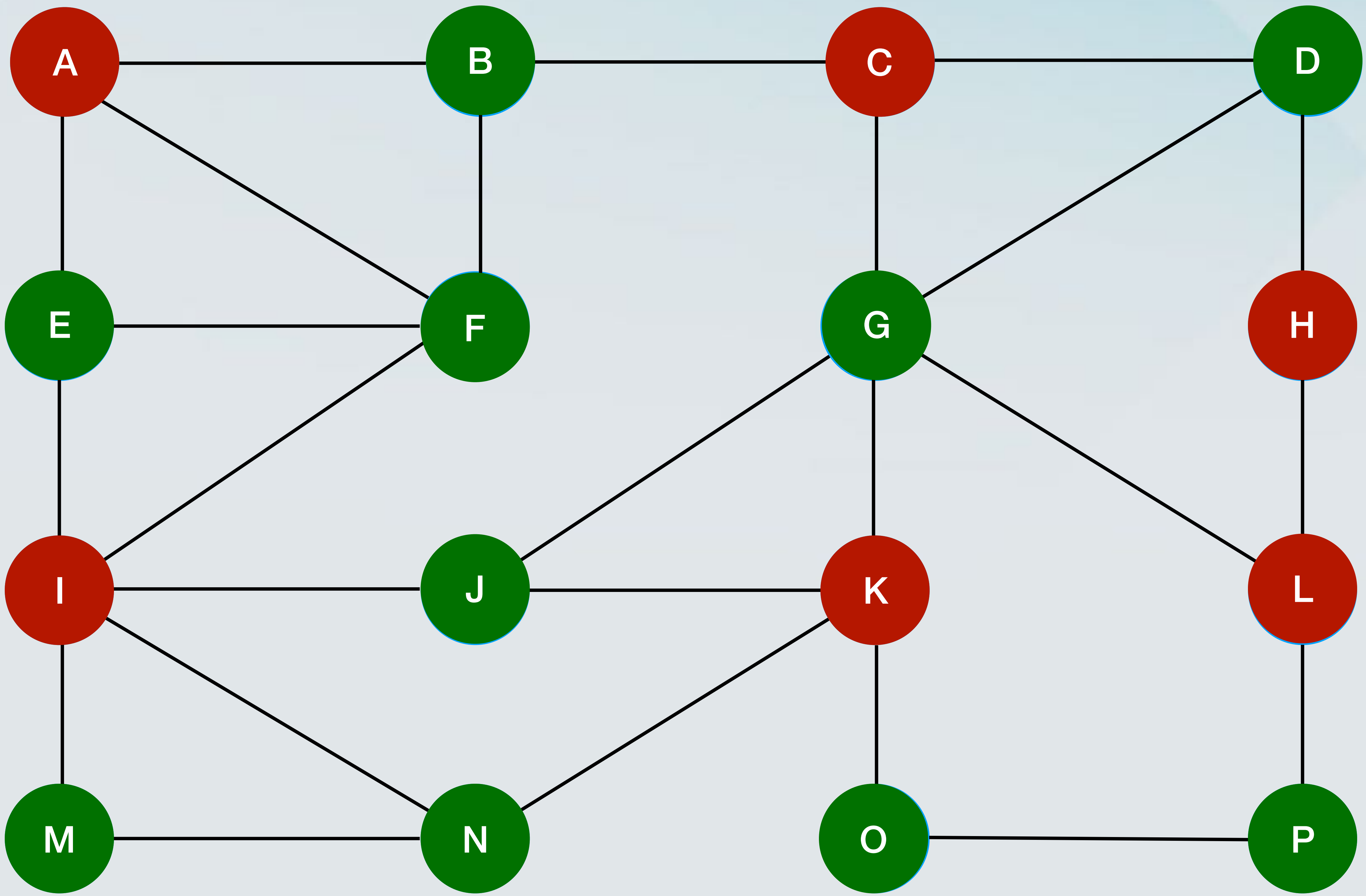

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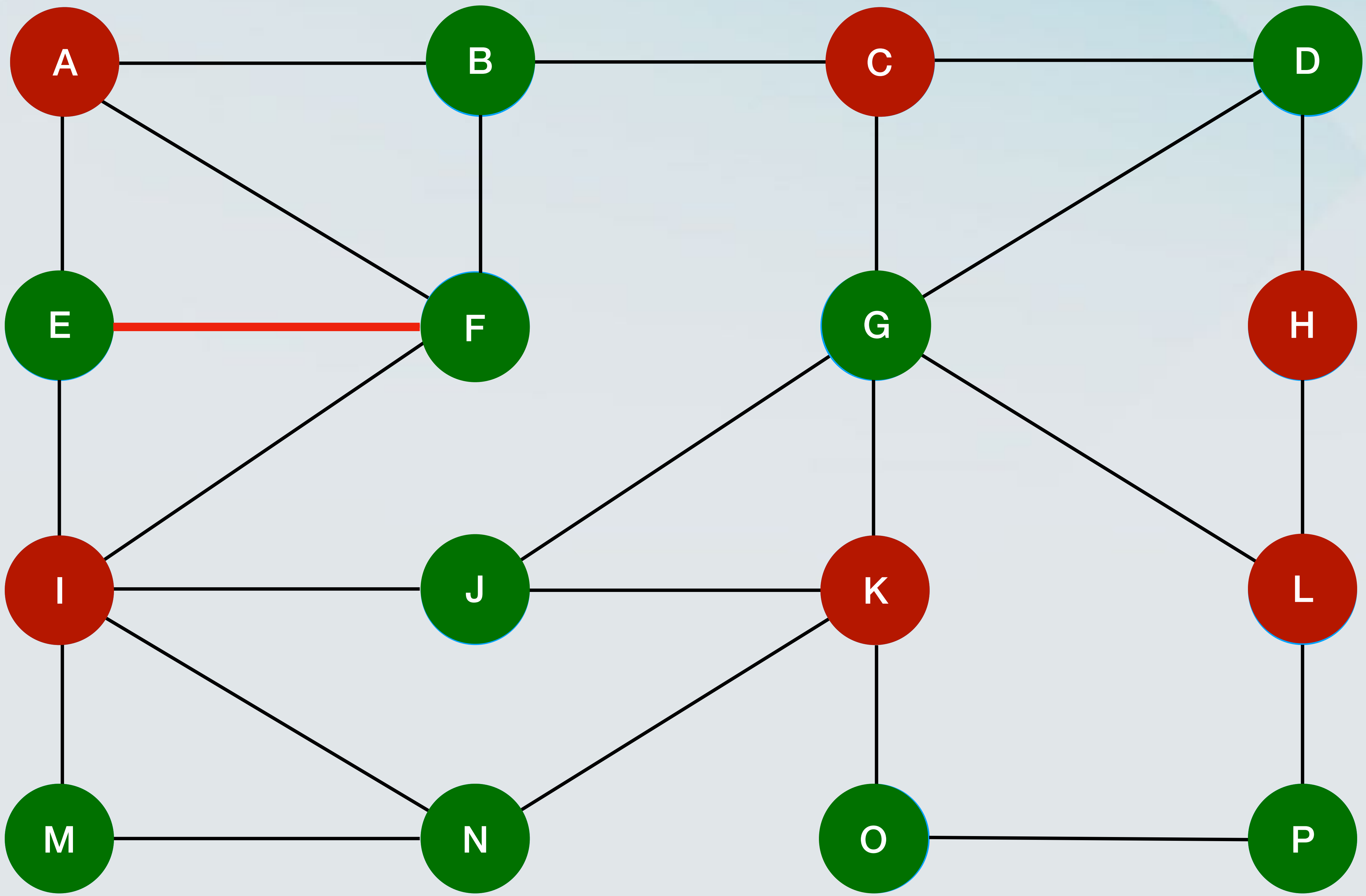

Colouring the nodes

- Does this remind you of something?
  - It is essentially BFS!
  - We label the nodes of *level 1* red, the nodes of *level 2* green, and so on.
- Implementation:
  - Add a check for odd/even and assign a colour accordingly.
  - In the end, check all edges to see if they have endpoints of the same colour.

Breadth-First Search Pseudocode

```
Algorithm BFS(G,s)

Initialise empty list L0
Initialise colour list C
Insert s into L0
Set C[s] = red

Set i=0
While Li is not empty
      Initialise empty list Li+1
      for each node v in Li
         for all edges e incident to v
           if edge e is unexplored
             let w be the other endpoint of e
             if node w is unexplored
                label e as discovery edge
                insert w into Li+1
                If i+1 is odd, set C[w] = red, else set C[w] = green
             else
                label e as cross edge
i = i+1

For all edges e=(u,v) in G
     if C[u] = C[v] return "not bipartite"
Return "bipartite"
```

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  - Nothing!
- Running time **O(m+n)**.

Correctness

- We started at an arbitrary node s.
- Maybe we were lucky / unlucky?

Properties of BFS

- For simplicity, assume that the graph is **connected.**
- The traversal visits all vertices of the graph.
- The *discovery edges* form a spanning tree.
- The path of the spanning tree from **s** to a node **v** at level *i* has *i* edges, and this is the shortest path.
- If **e**=(**u**,**v**) is a *cross edge*, then the **u** and **v** differ by at most one level.

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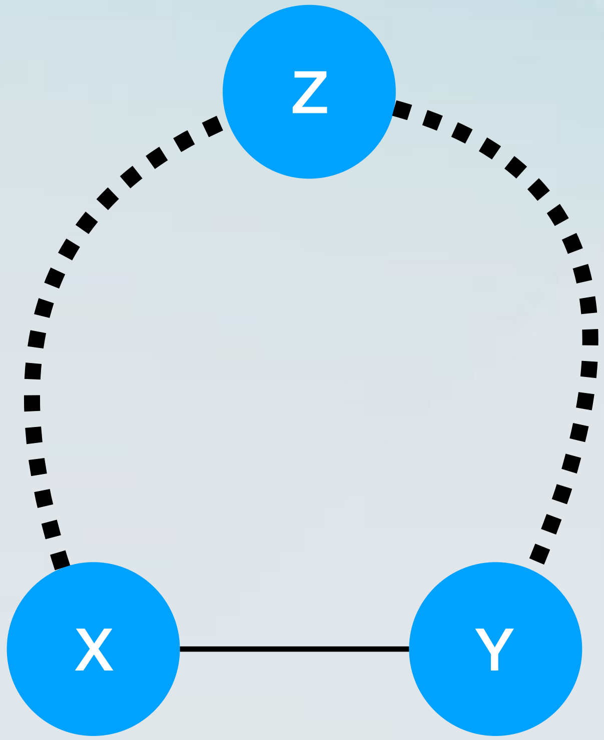
  - This means that it has found an edge e=(x,y) with endpoints of the same colour.
  - Since the endpoints of any edge can not differ by more than one layer and layers have alternating colours, x and y must be in the same layer.

Correctness

- Consider the lowest common ancestor z of x and y in the BFS tree.



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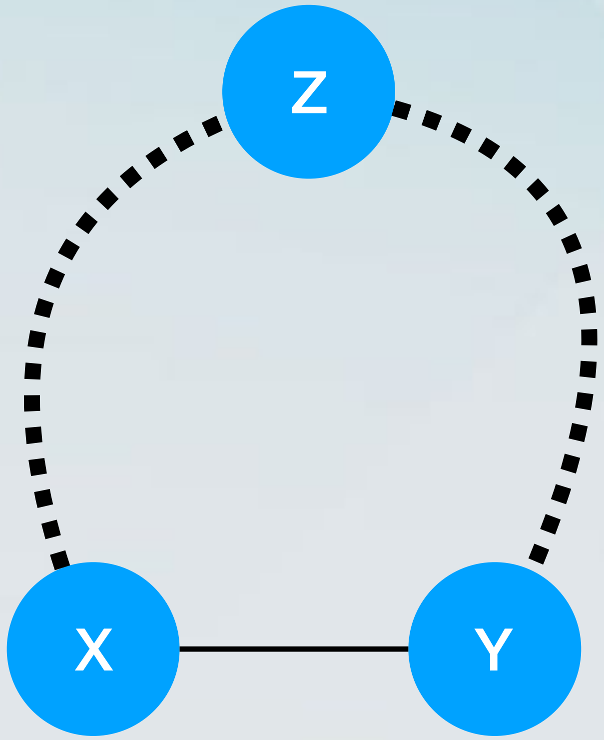

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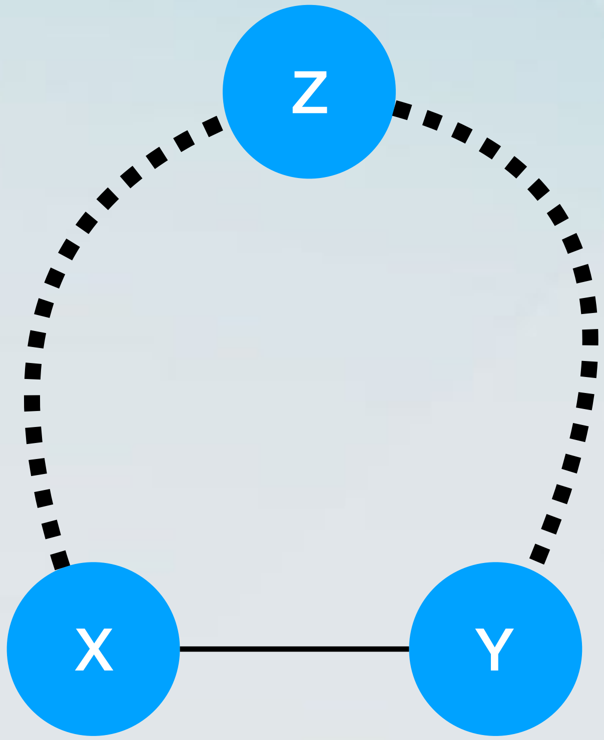

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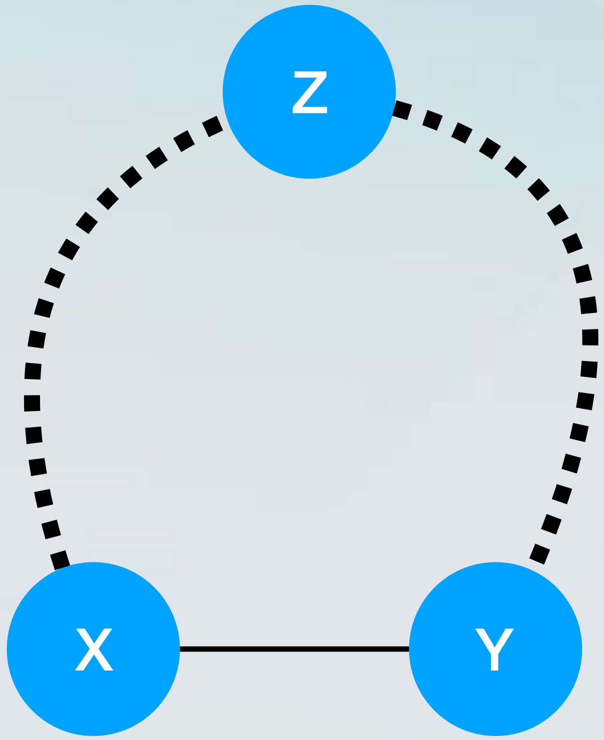

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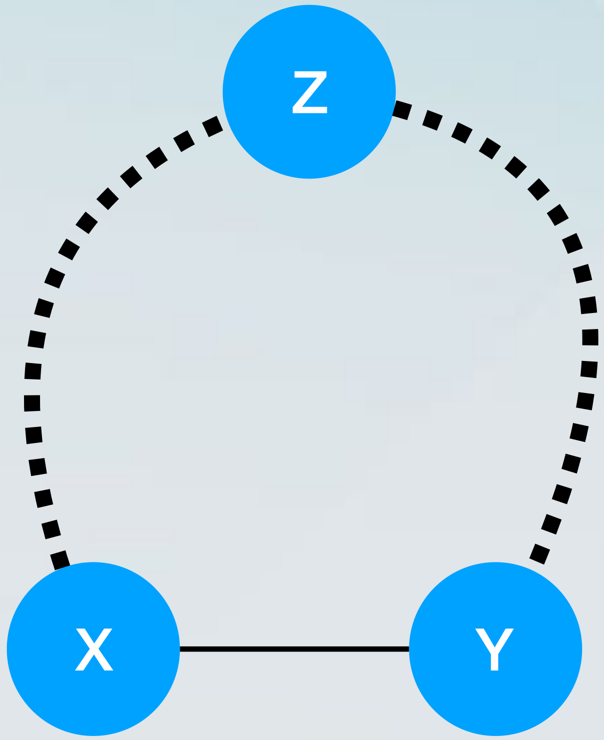

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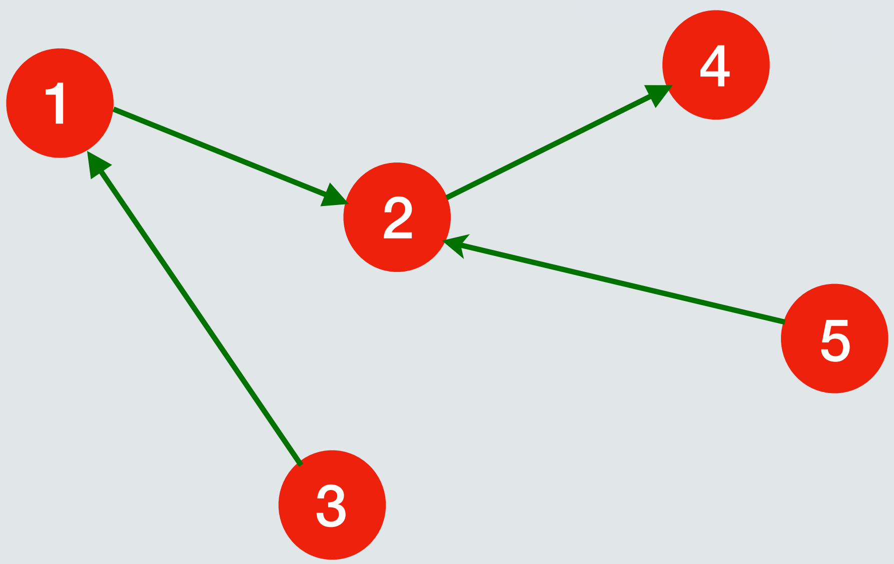
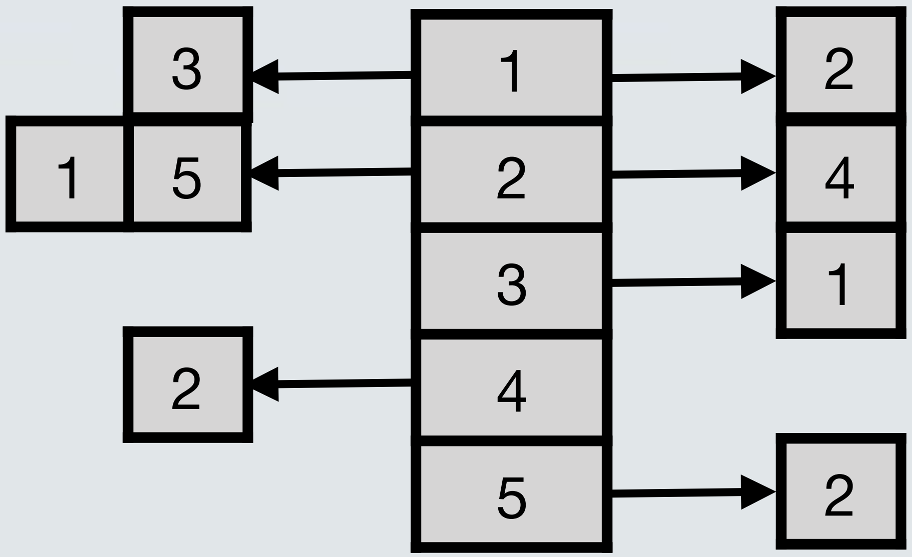
Directed graphs

- Nodes are arranged as a list, each node points to the neighbours.
- For directed graphs, the node points in two directions, for in-degree and for out-degree.

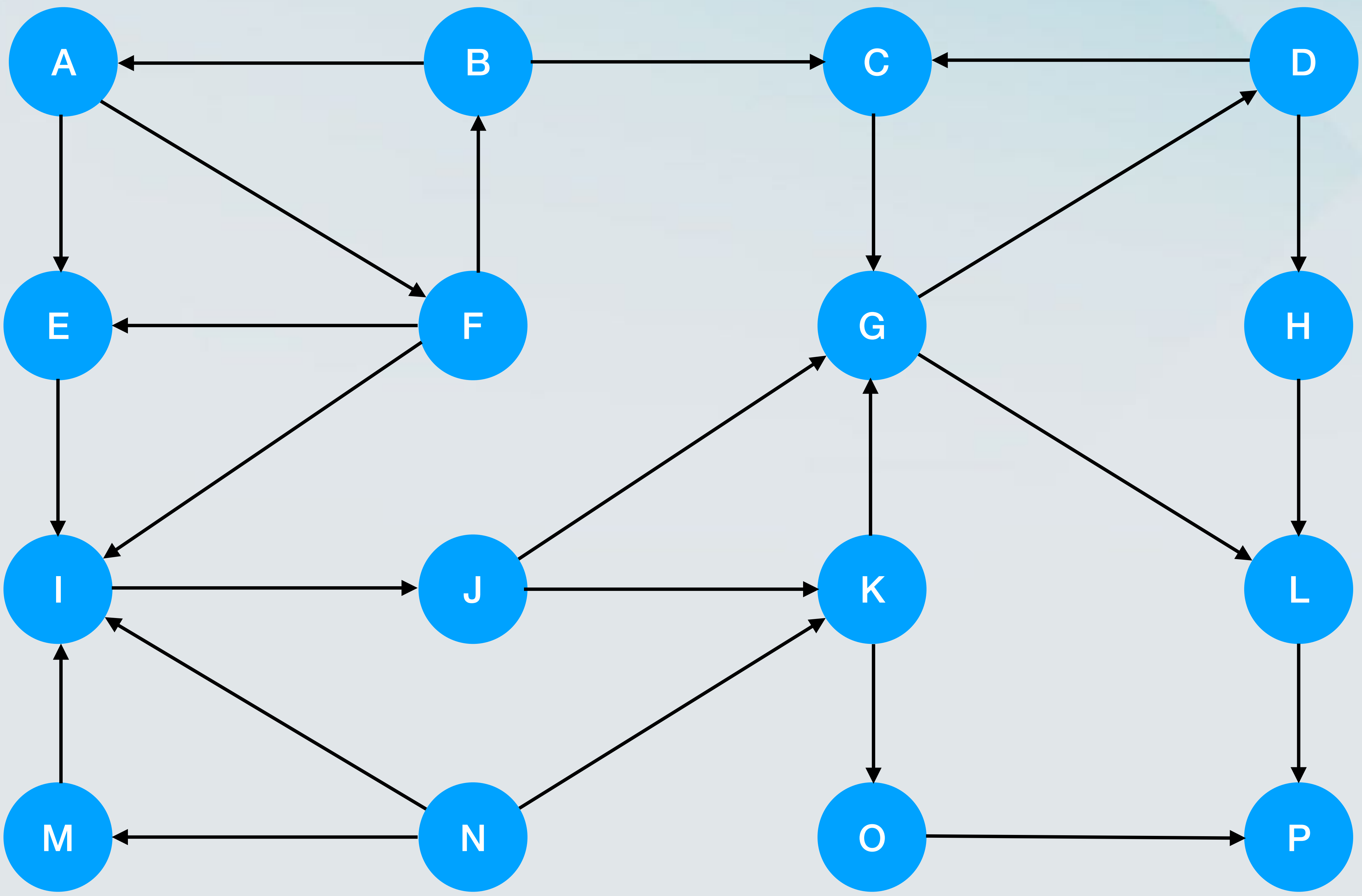




DFS and BFS on directed graphs

- Very similar to their version on undirected graphs.
- When we are at a node and we examine its neighbours, a neighbour is now only a node that we can reach with a directed edge.
- The running time is still **$O(n+m)$**.

Breadth-First Search



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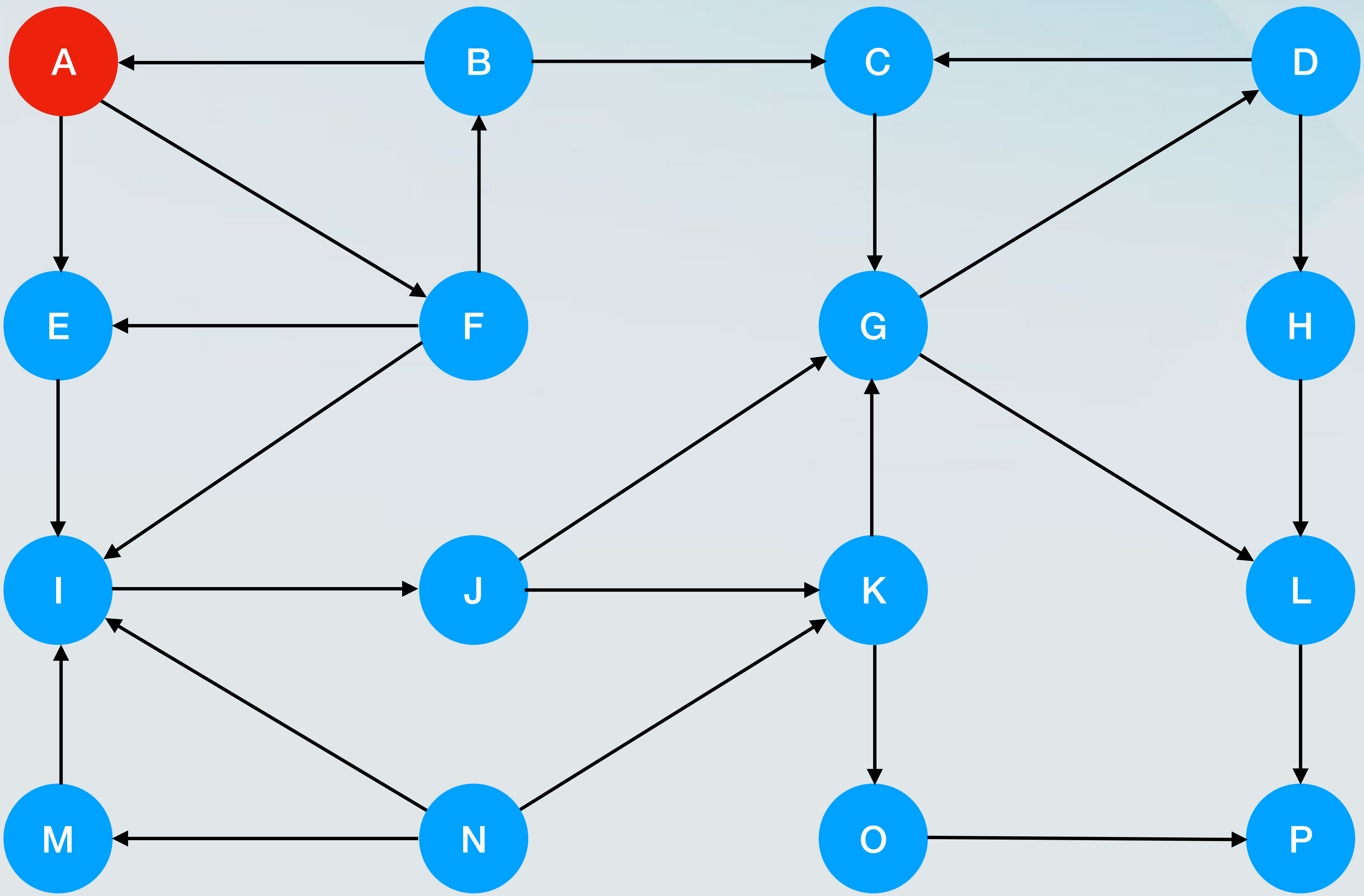

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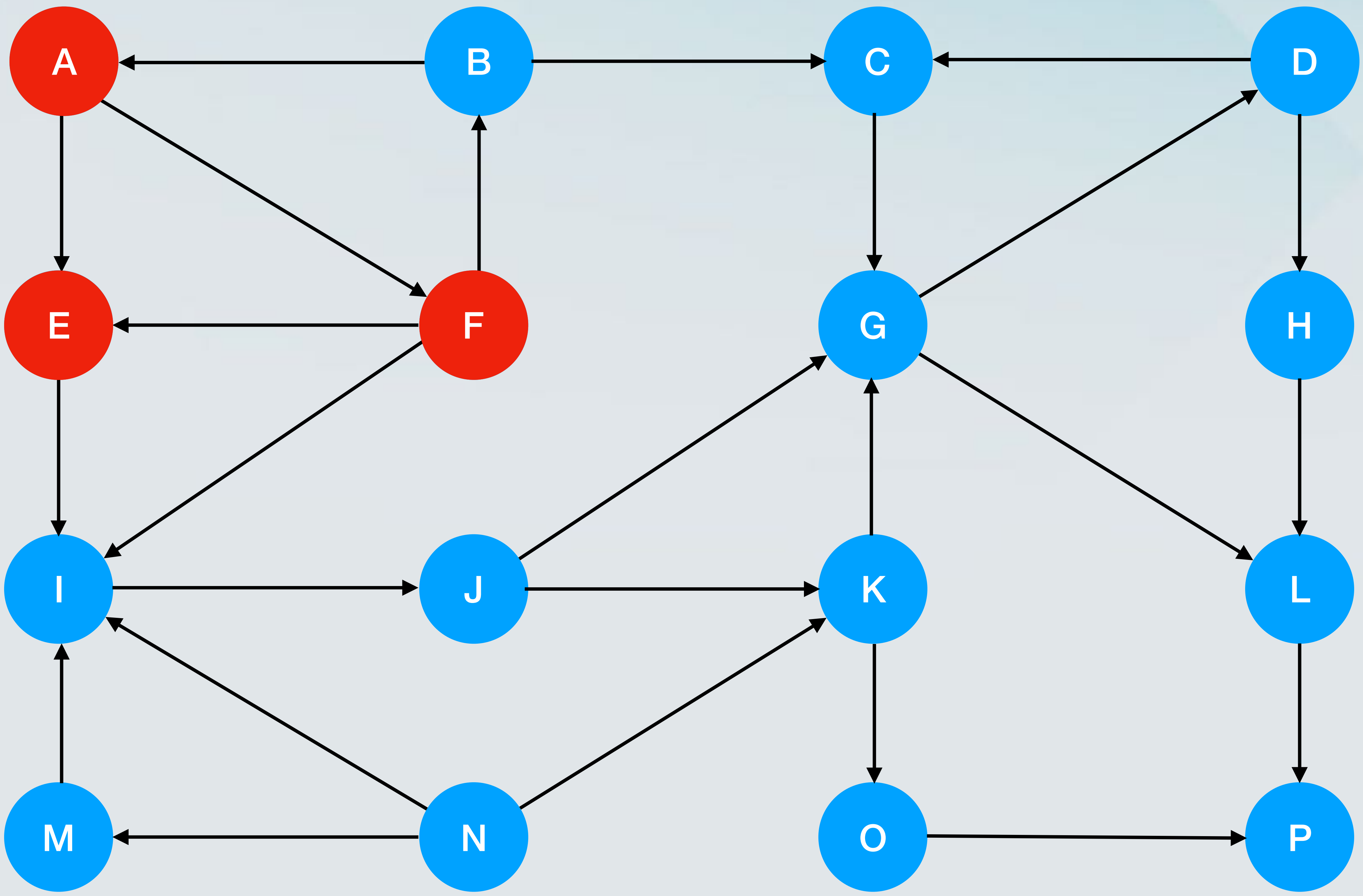

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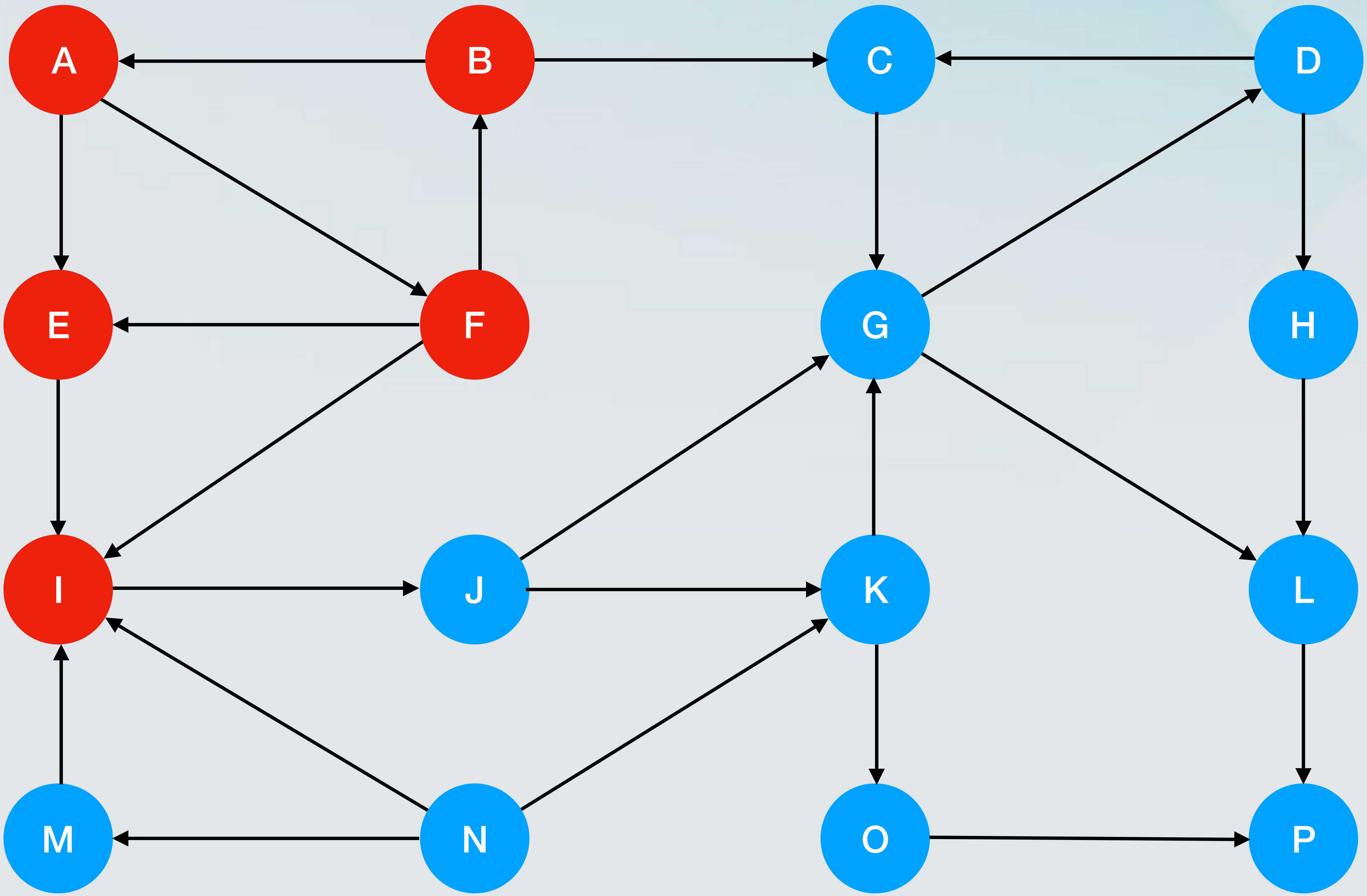

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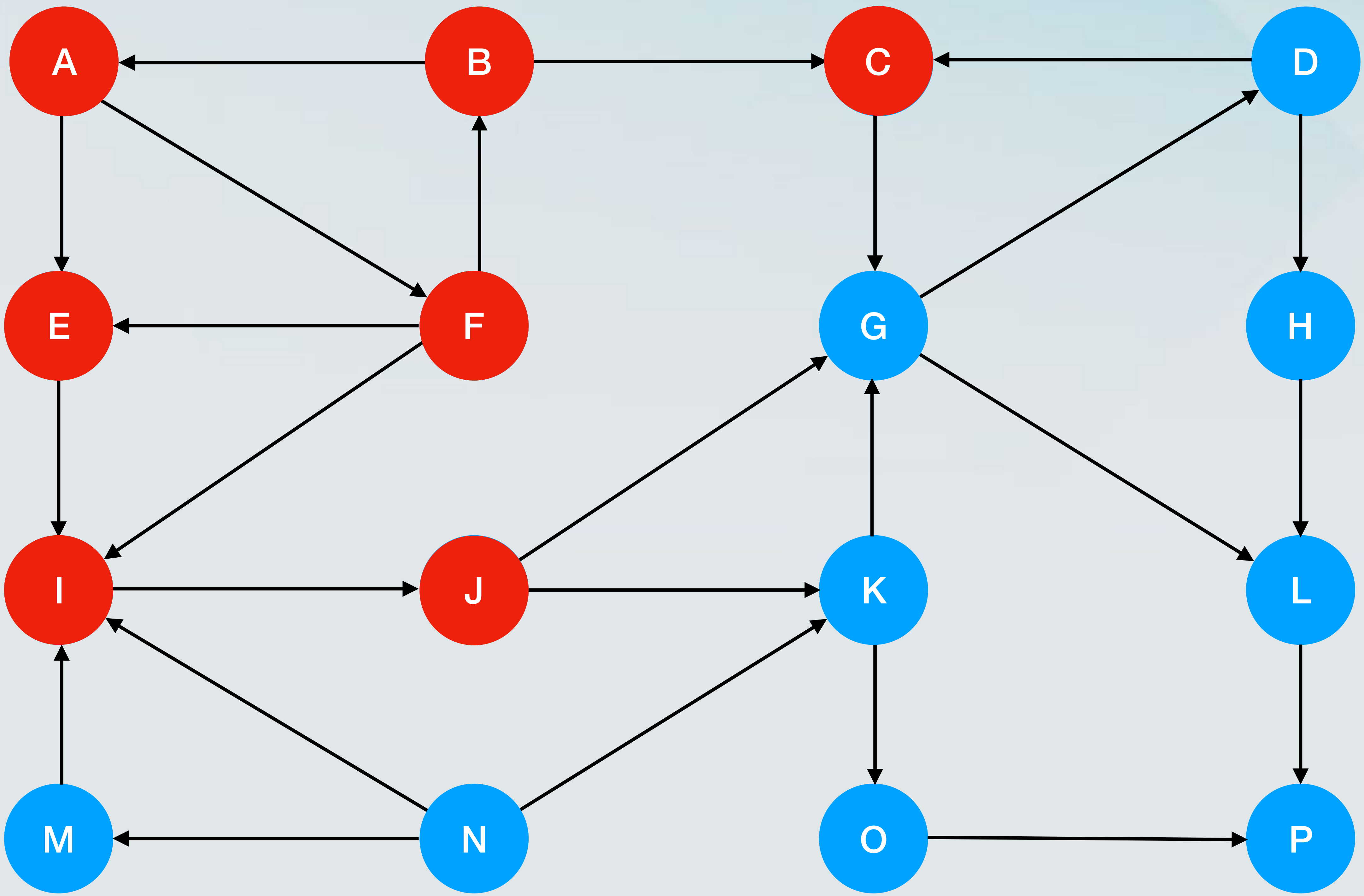

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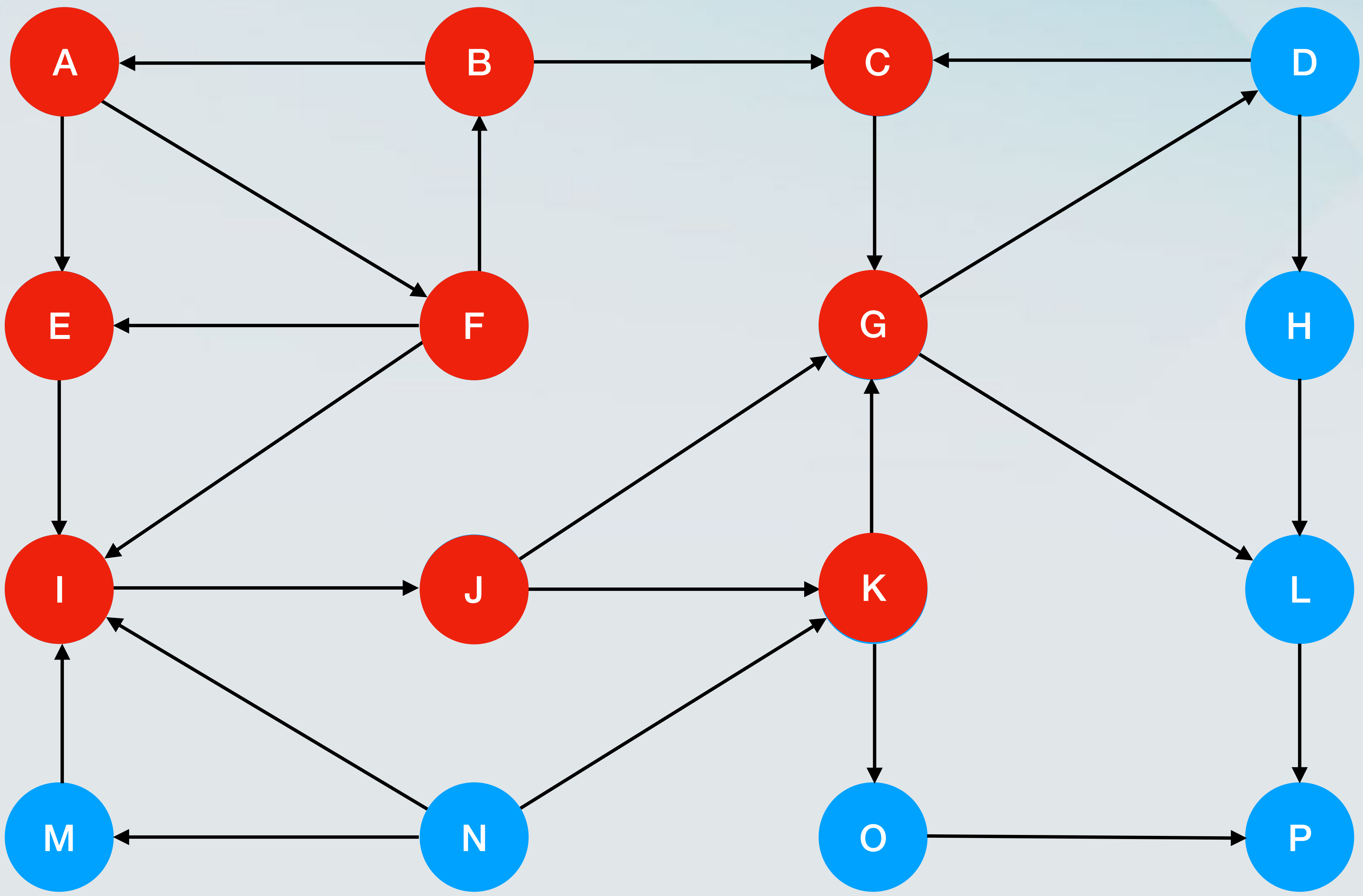

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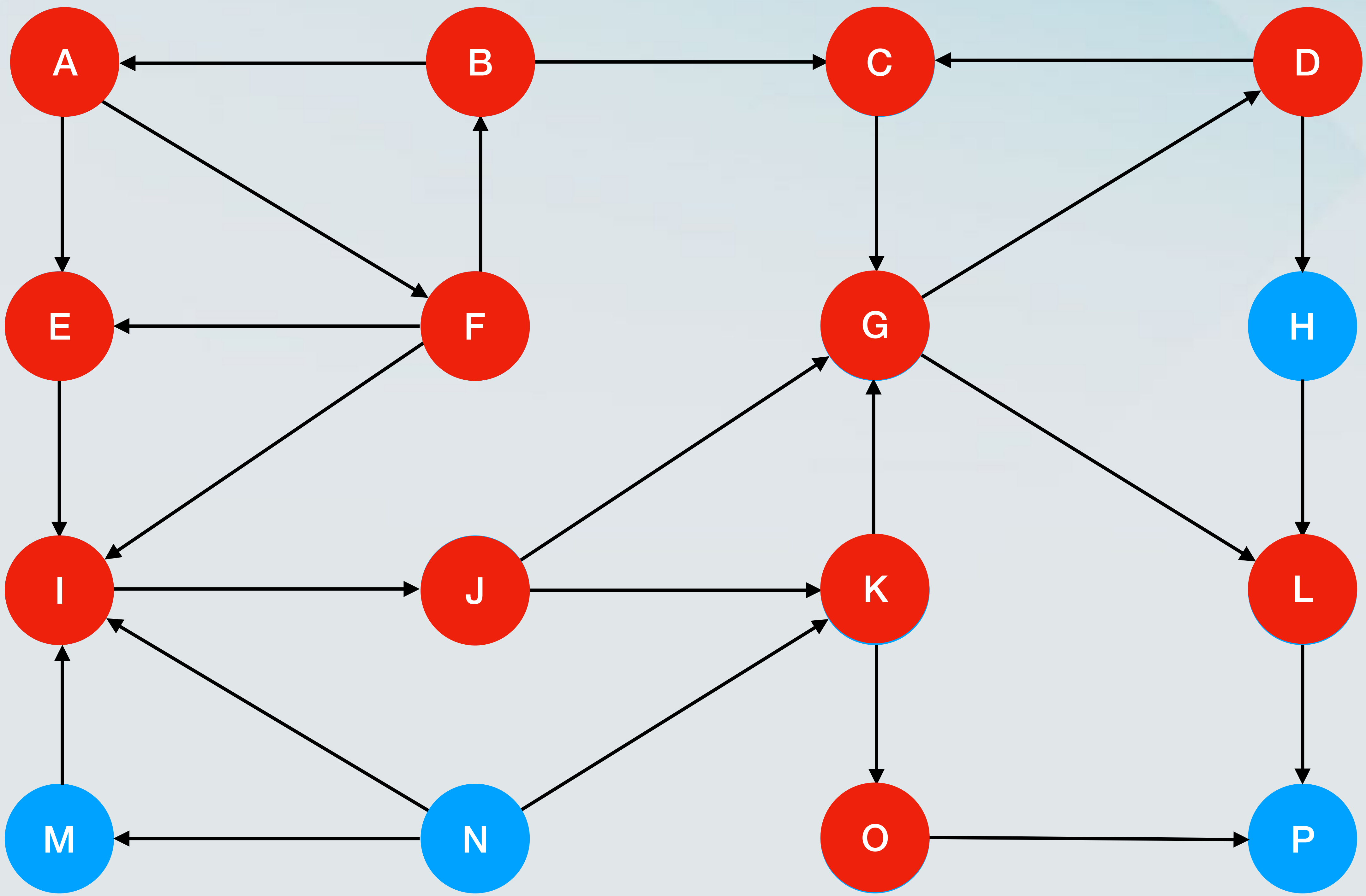

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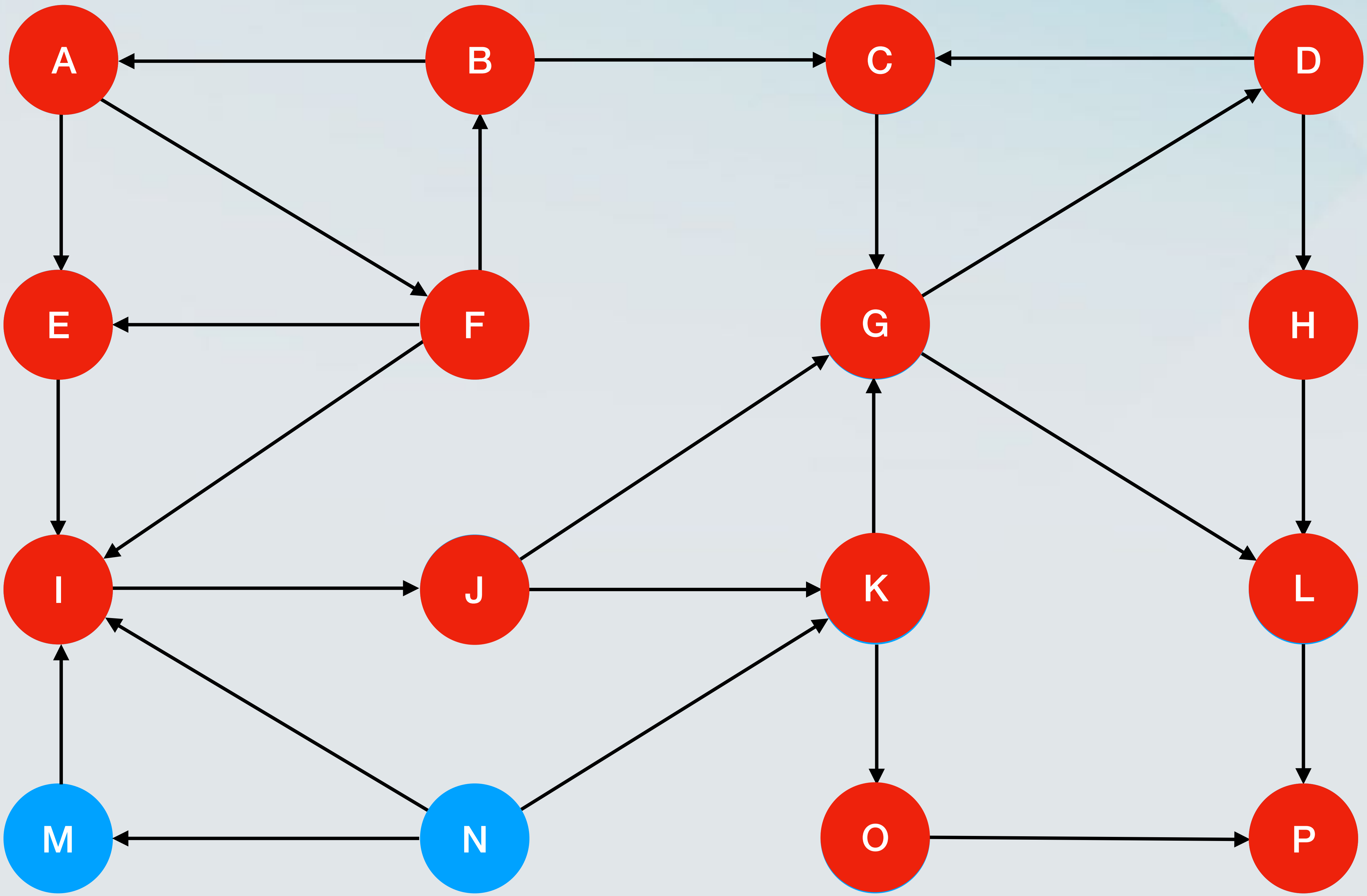

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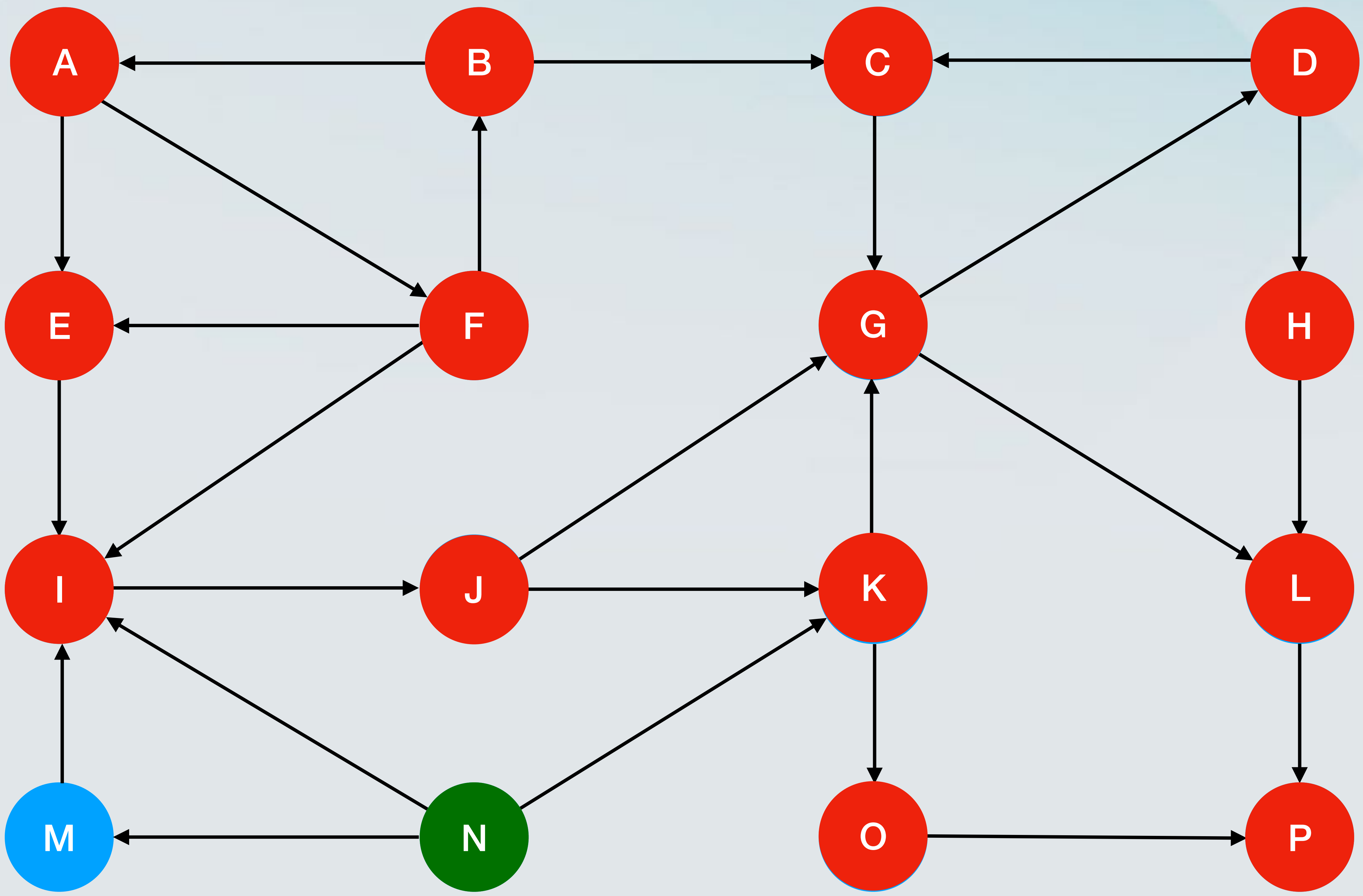

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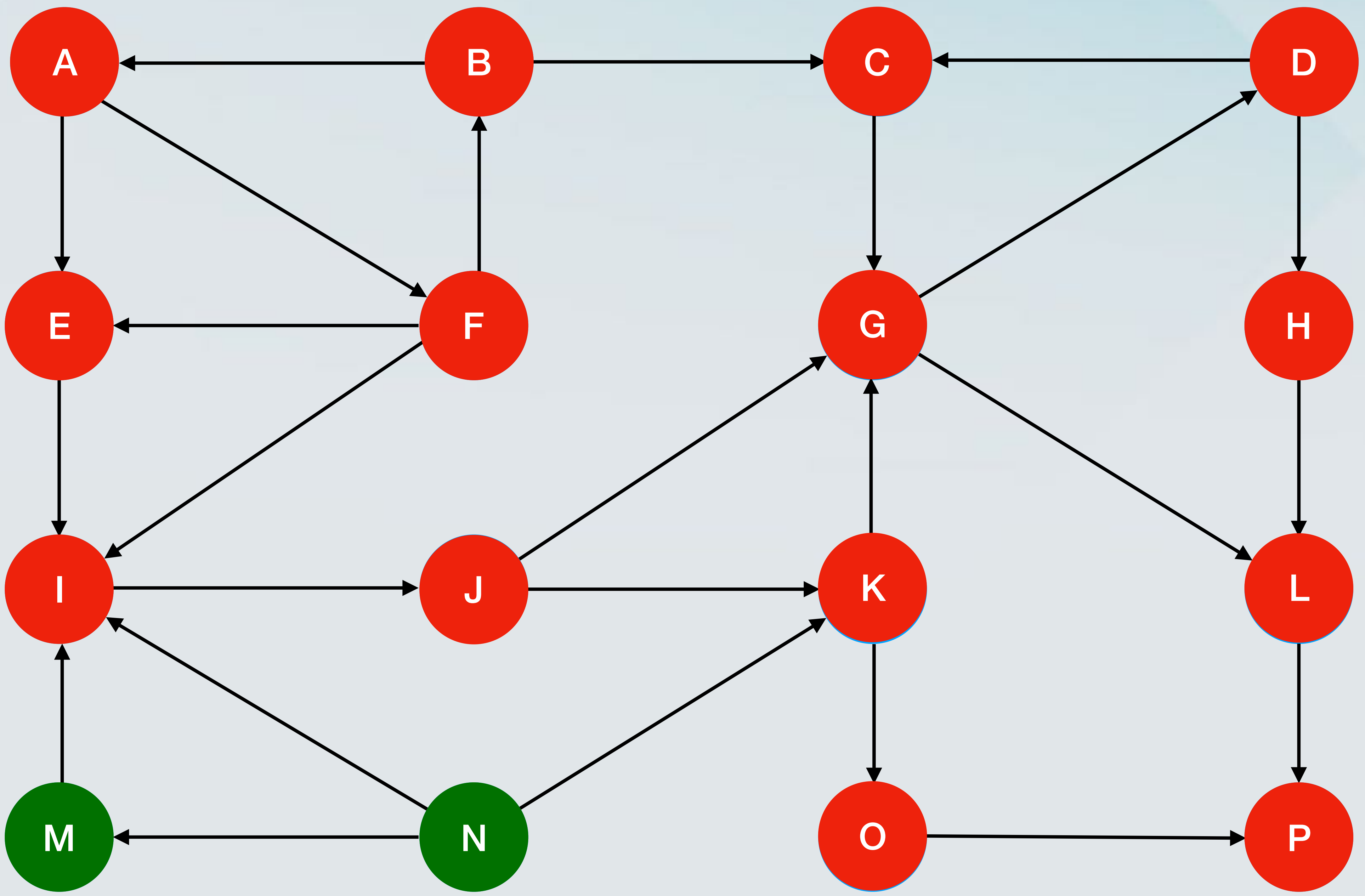

Connectivity

- What BFS is computing is the set of nodes t such that there is a path from s to t.
- A path from s to t does not mean that there is path from t to s.
- (Weak) connectivity: If we ignored the directions for all edges, there would a path from any node to any node.
- Strong connectivity: For every two nodes u and v, there is a path from u to v and a path from v to u.
- **Question: Given a graph G=(V,E), is it strongly connected?**

Mutual reachability

- Two nodes u and v are mutually reachable, if there is path from u to v and a path from v to u in G.
- Strong connectivity: For every pair of nodes u and v, these nodes are mutually reachable.
- Transitivity: If u and v are mutually reachable and v and w are mutually reachable, then u and w are mutually reachable.

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  - For any node u, s and u are mutually reachable.

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- Pick any node s in V and run BFS(G,s) and BFS(G^{rev},s).

- If one of the two searches does not reach every node, then the graph is definitely not strongly connected.

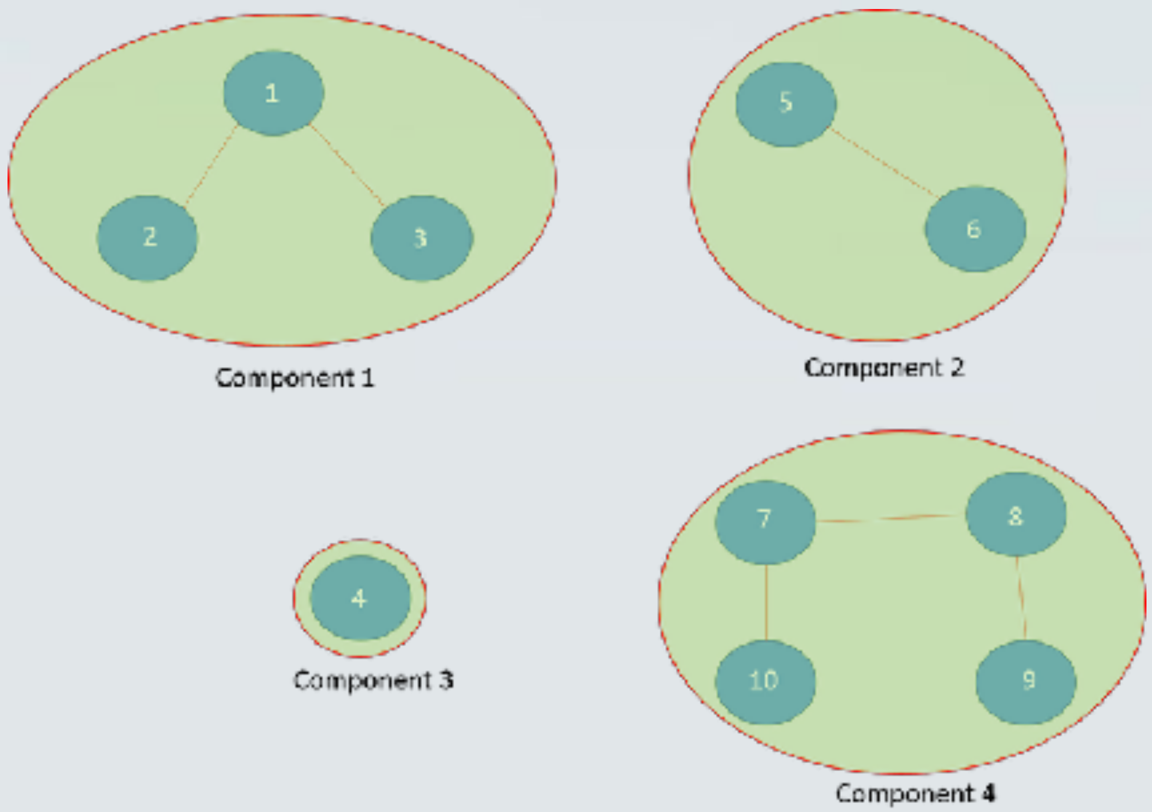
- Assume that both searches reach every node. This means that there is a path from s to any node u and a path from any node u to s.

  - For any node u, s and u are mutually reachable.

- Pick any other node v. Since s and v are also mutually reachable, by transitivity, v and u are mutually reachable and the graph is strongly connected.

Connected component

- A **connected component** of an *undirected* graph **G** is subgraph such that any two nodes are connected via some path.



Connected component

- A **connected component** of an *undirected* graph **G** is subgraph such that any two nodes are connected via some path.
- A strongly **connected component** of a *directed* graph **G** is subgraph such that any two nodes are mutually reachable.

Strongly connected components

- How do we find all strongly connected components of a graph G?
- We can run the "forward" and "backward" BFS for a node s and find the set of nodes that are mutually reachable from s.
  - This is the strongly connected component of s.
  - But BFS might produce different connected components, depending on how we visit the nodes.
  - We need a consistent way of visiting them in the "forward" and in the "backward" pass.

Kosajaru’s algorithm

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- Perform a DFS on G, starting from an arbitrary nodes s.
- Add the nodes that the DFS tree reaches to a stack.
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- Output the DFS trees of the second DFS as the strongly connected components.

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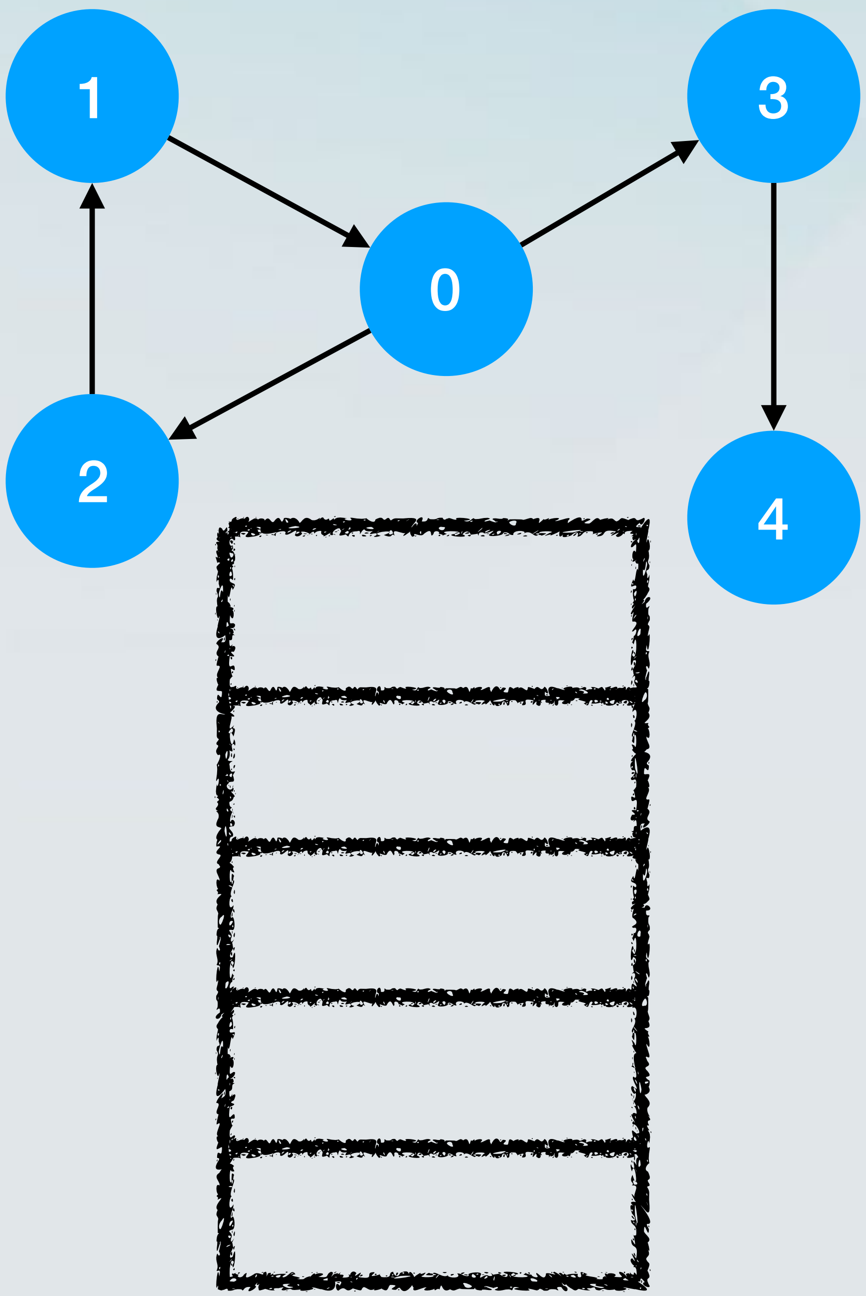

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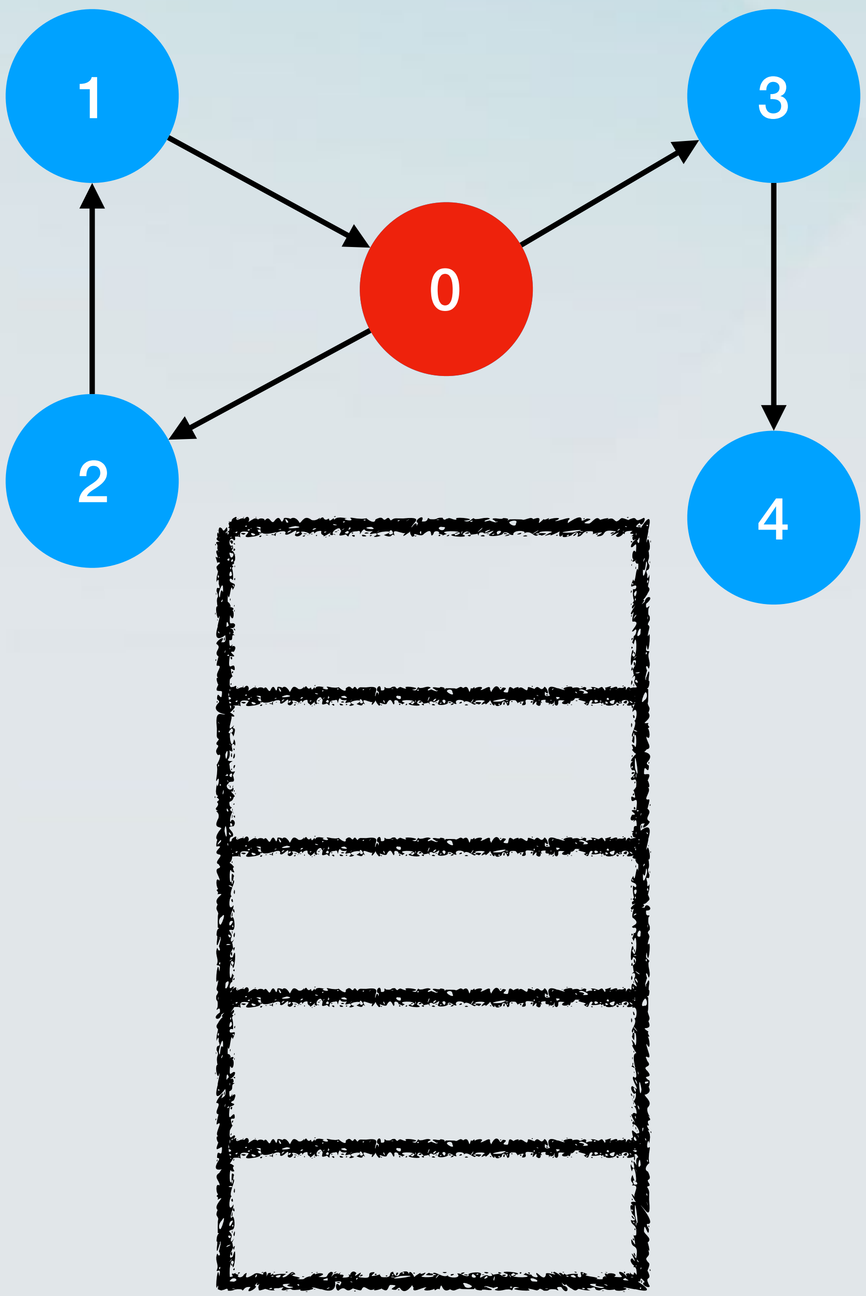

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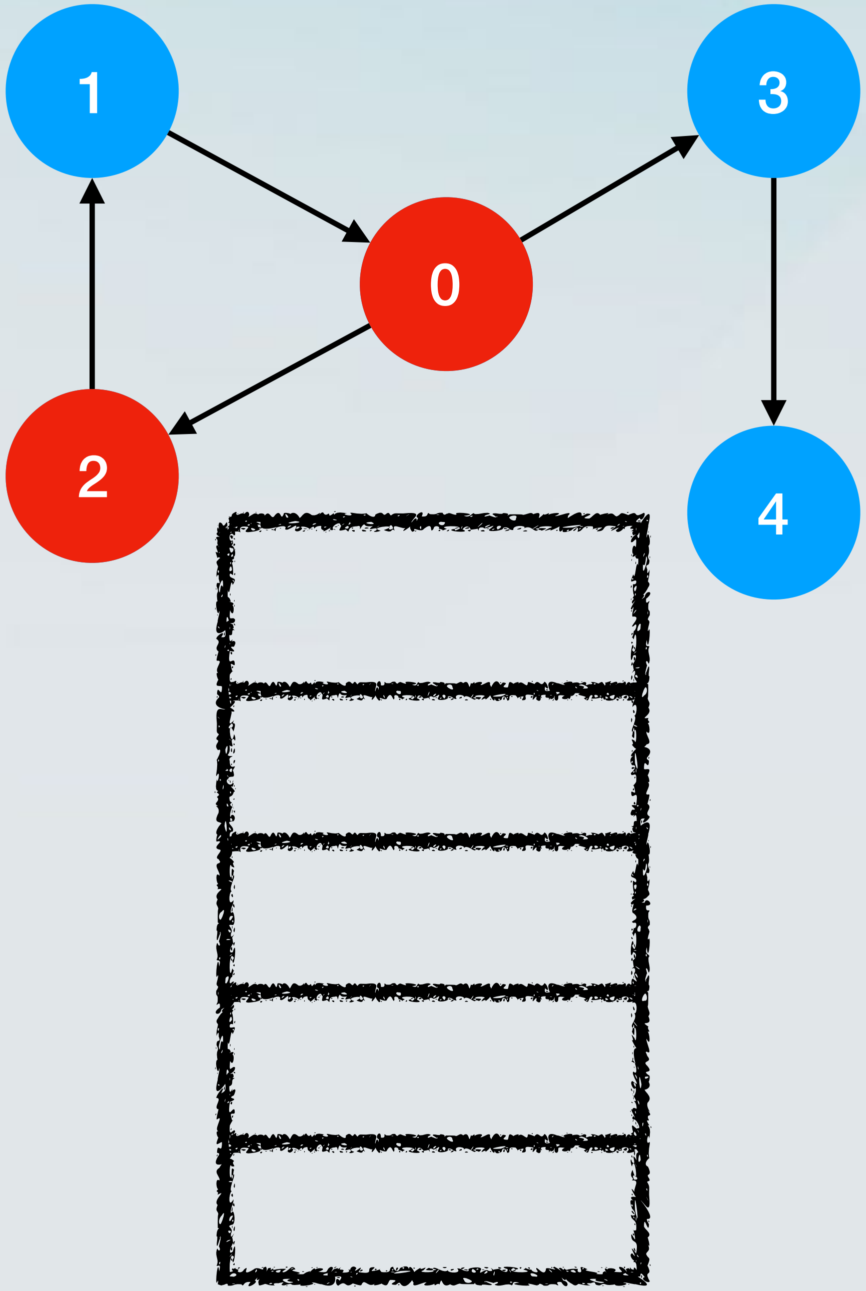

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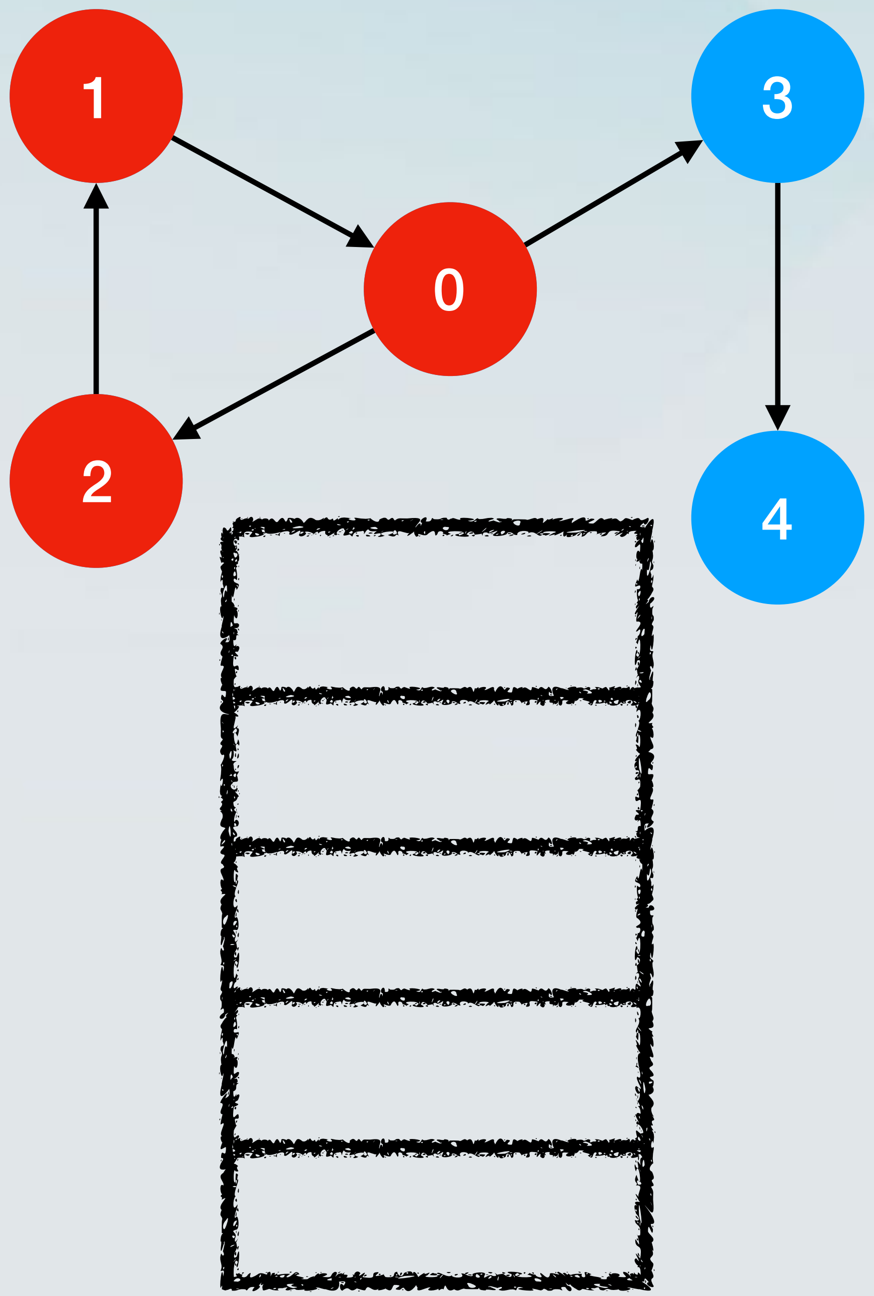

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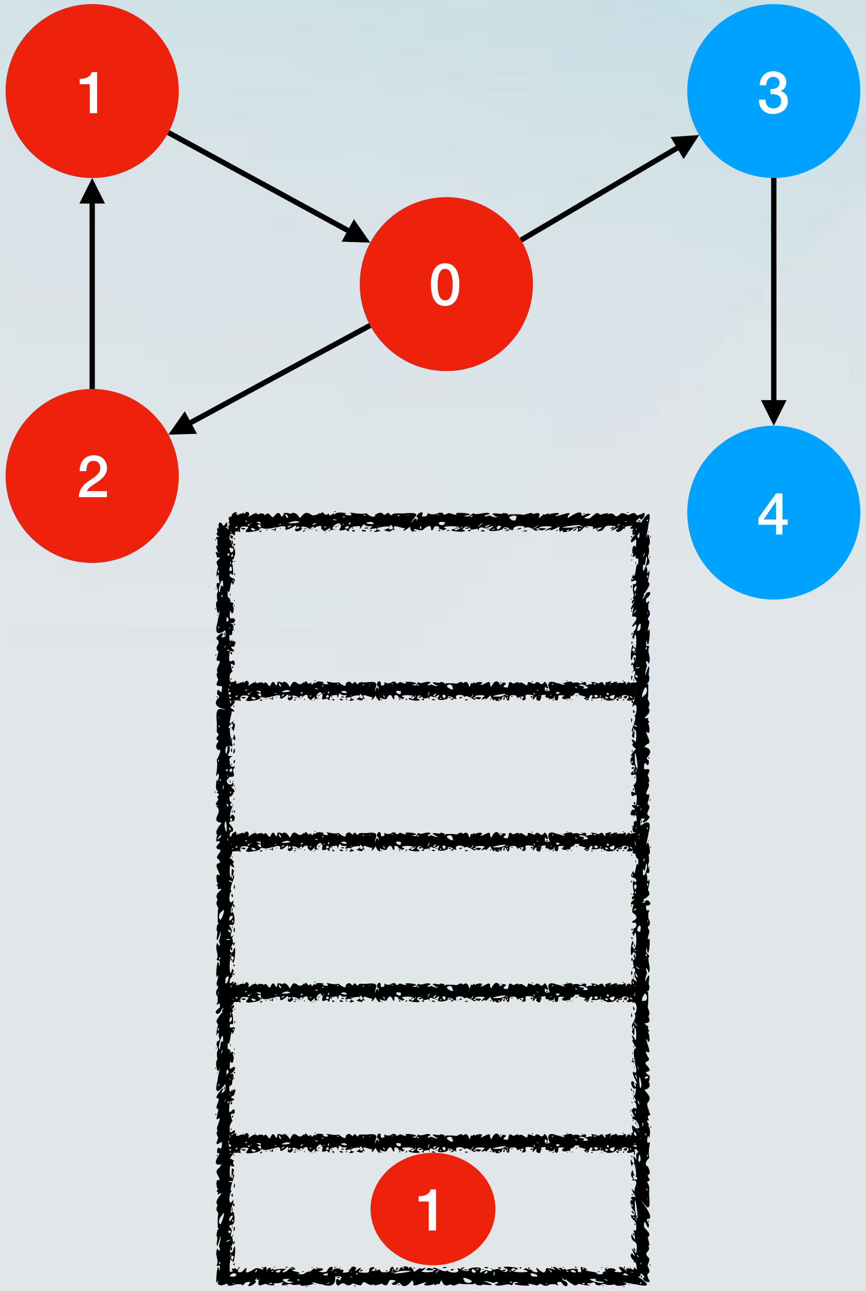

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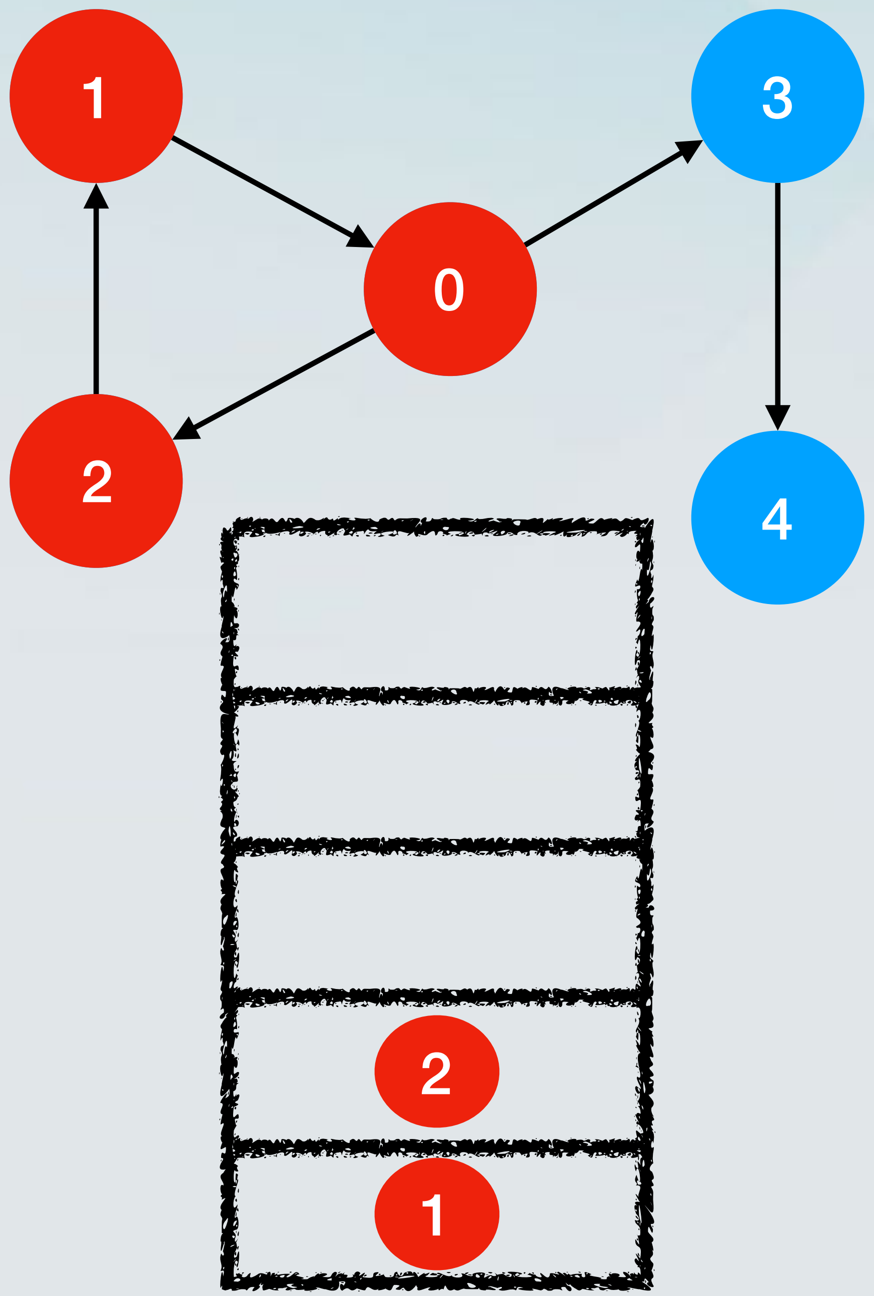

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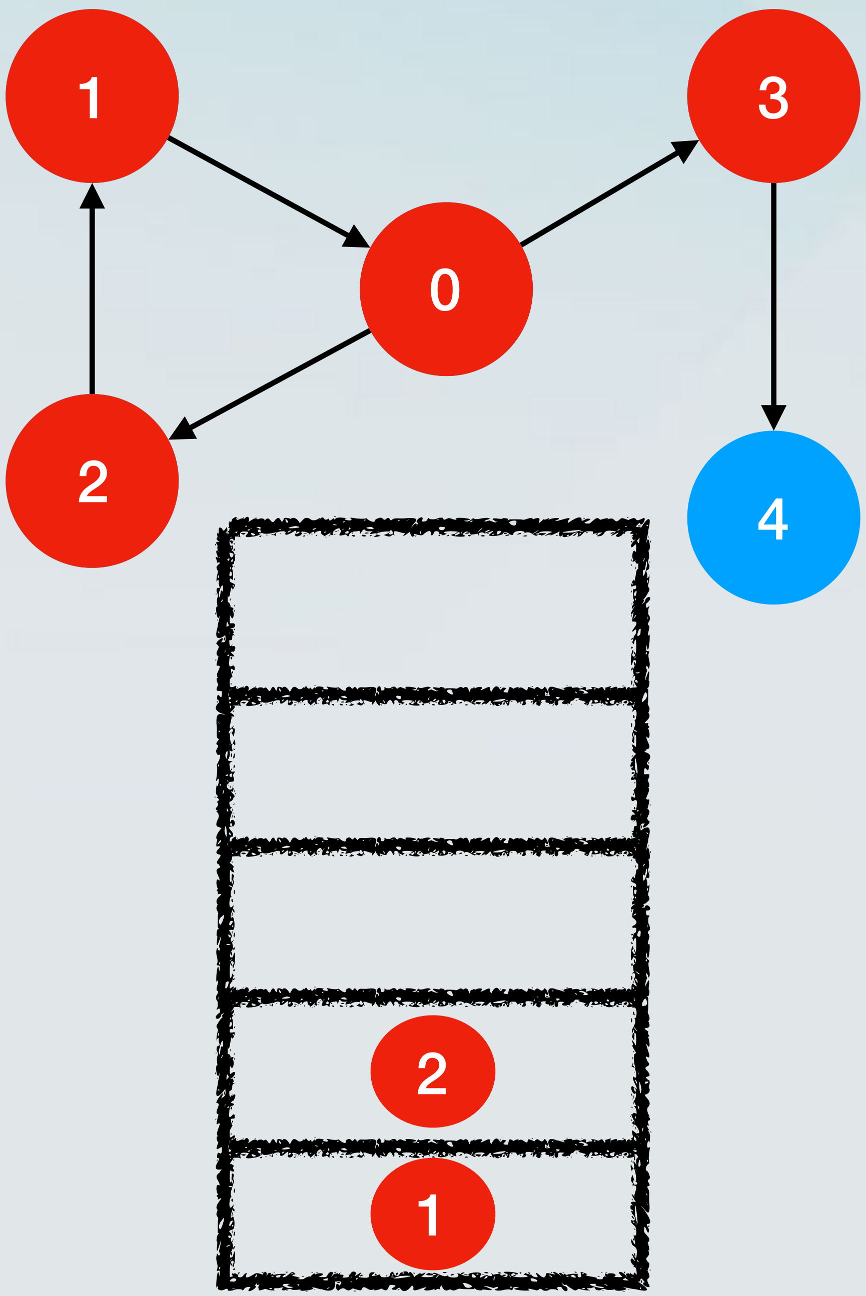

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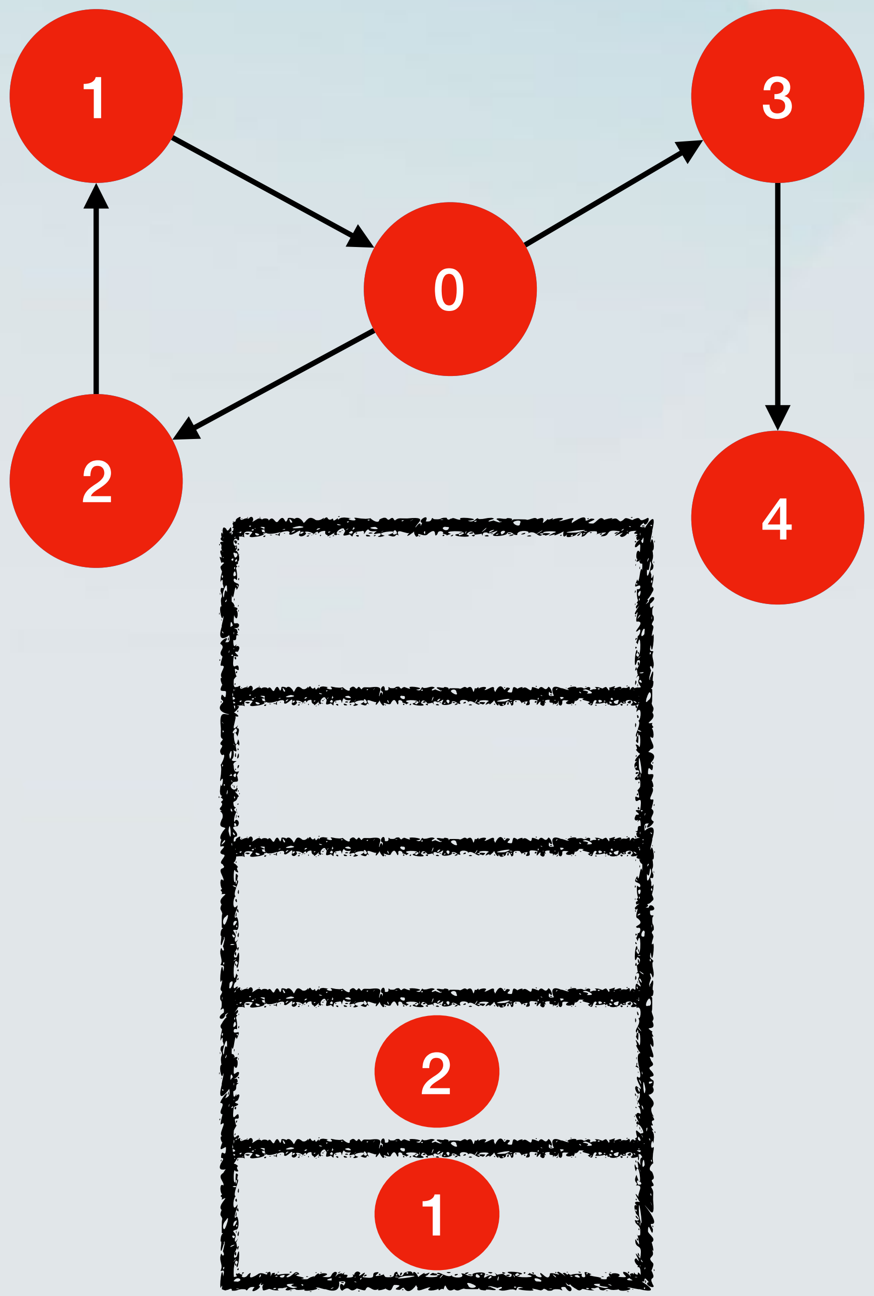

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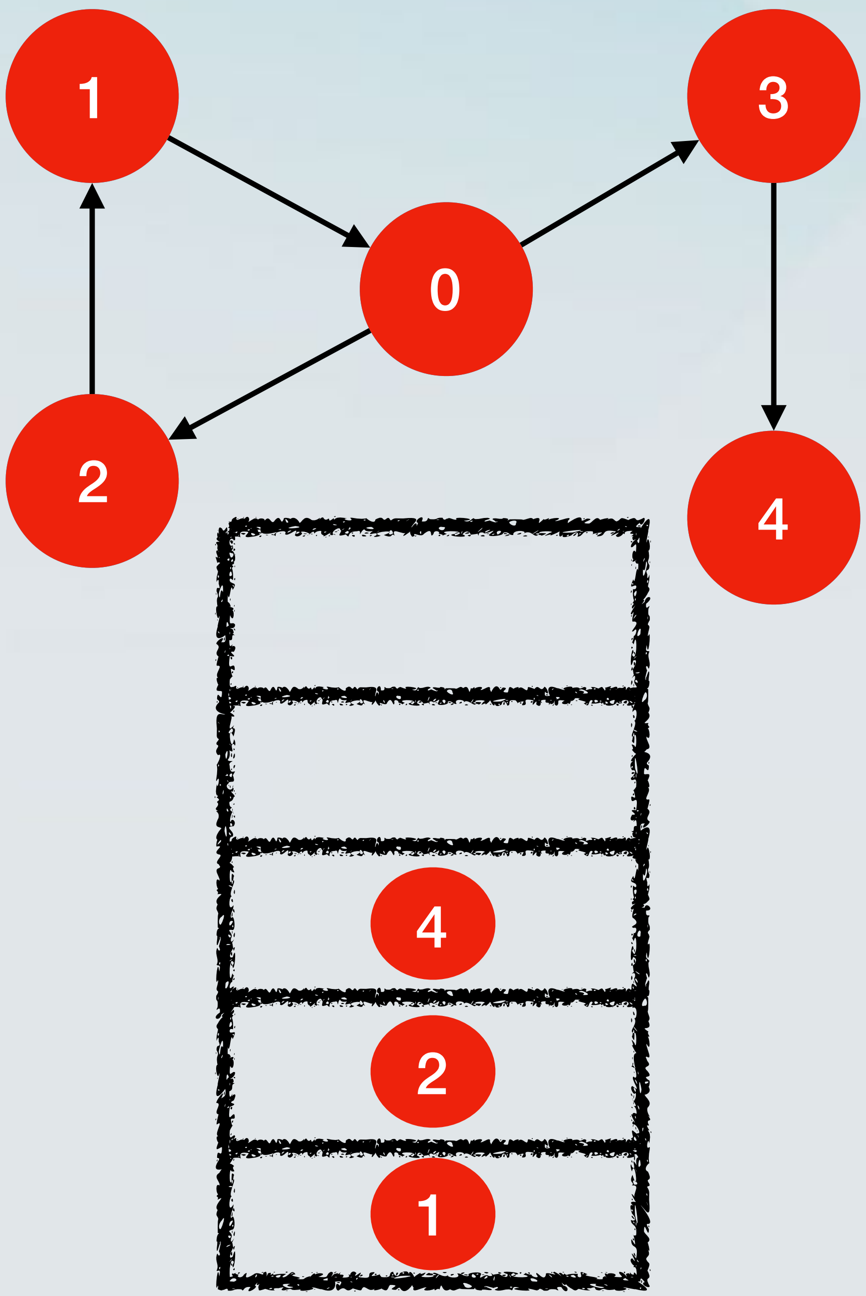

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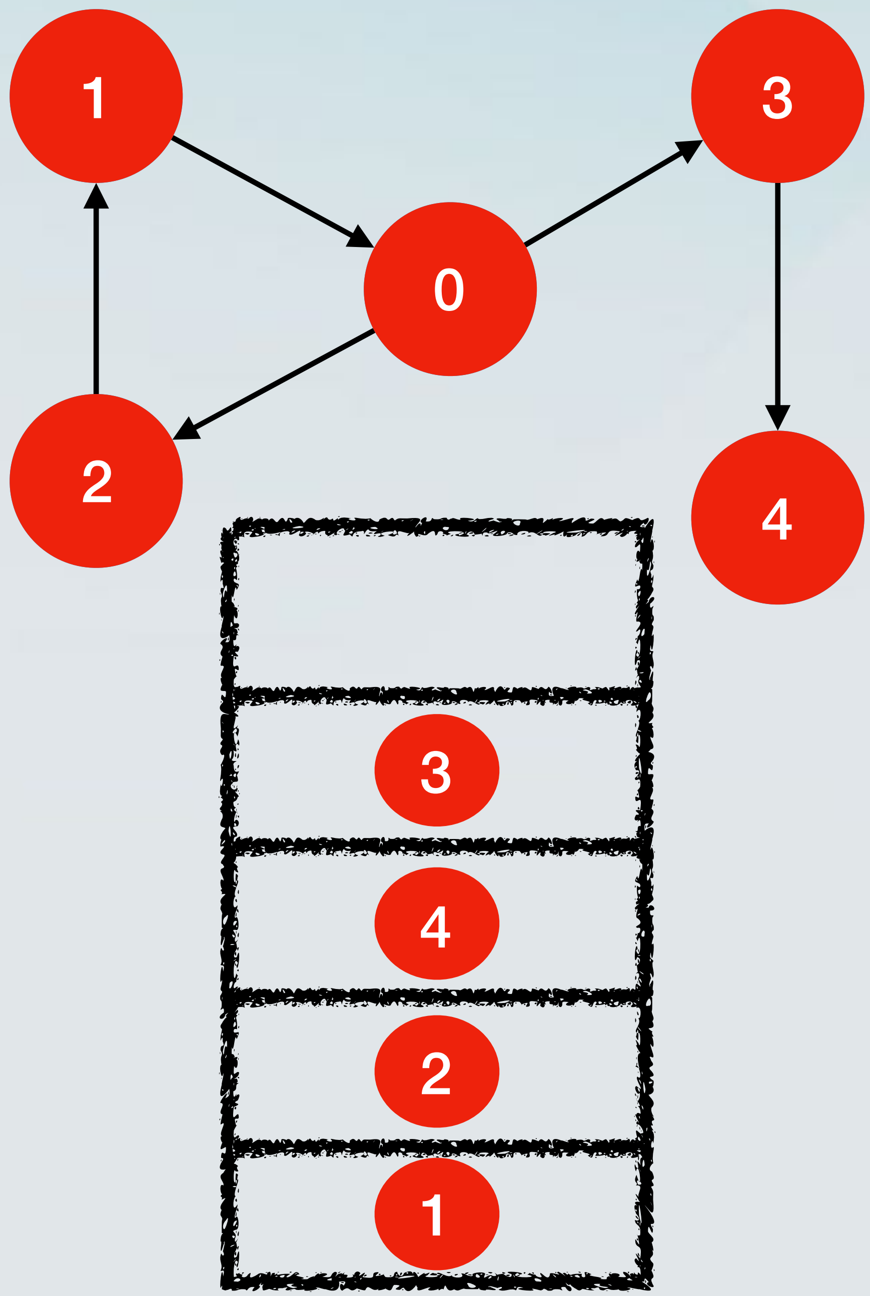

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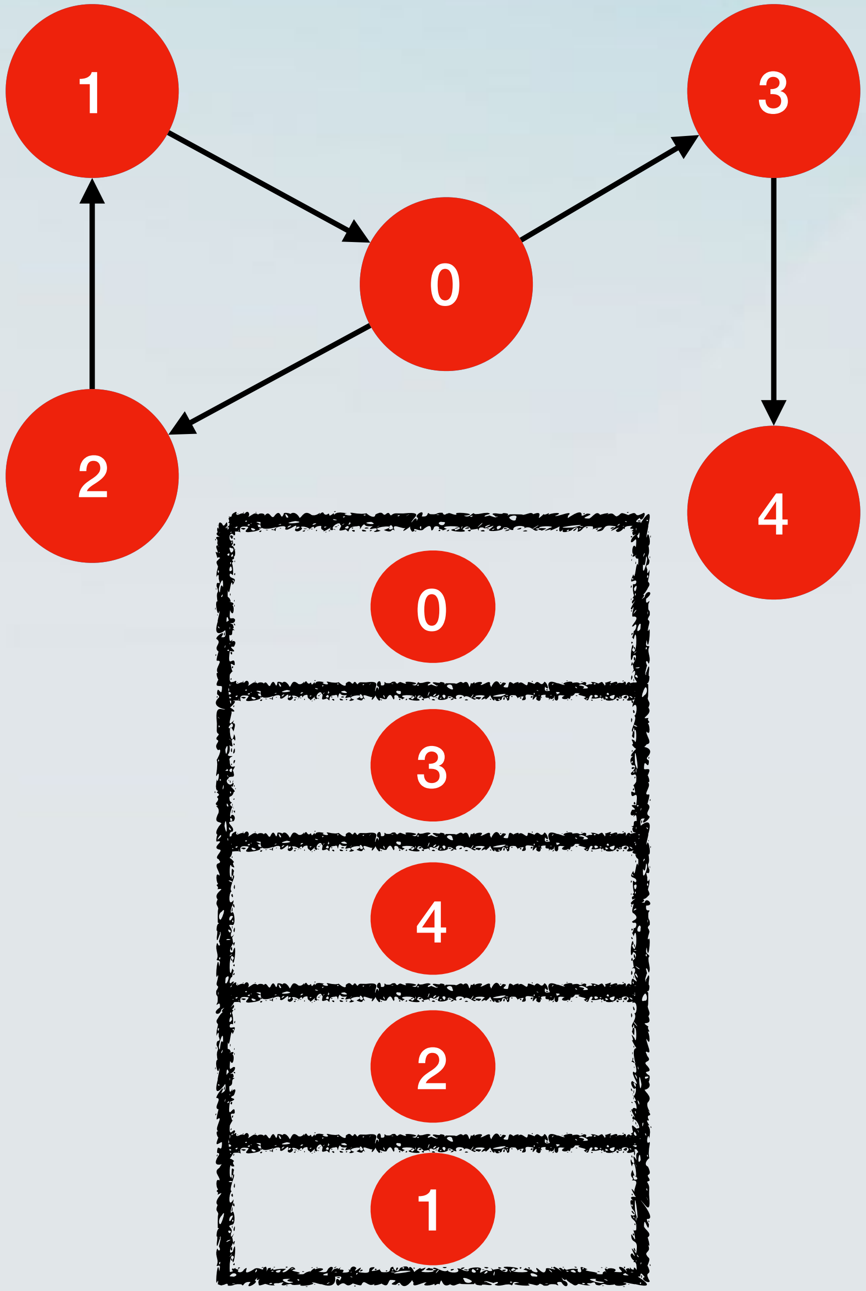

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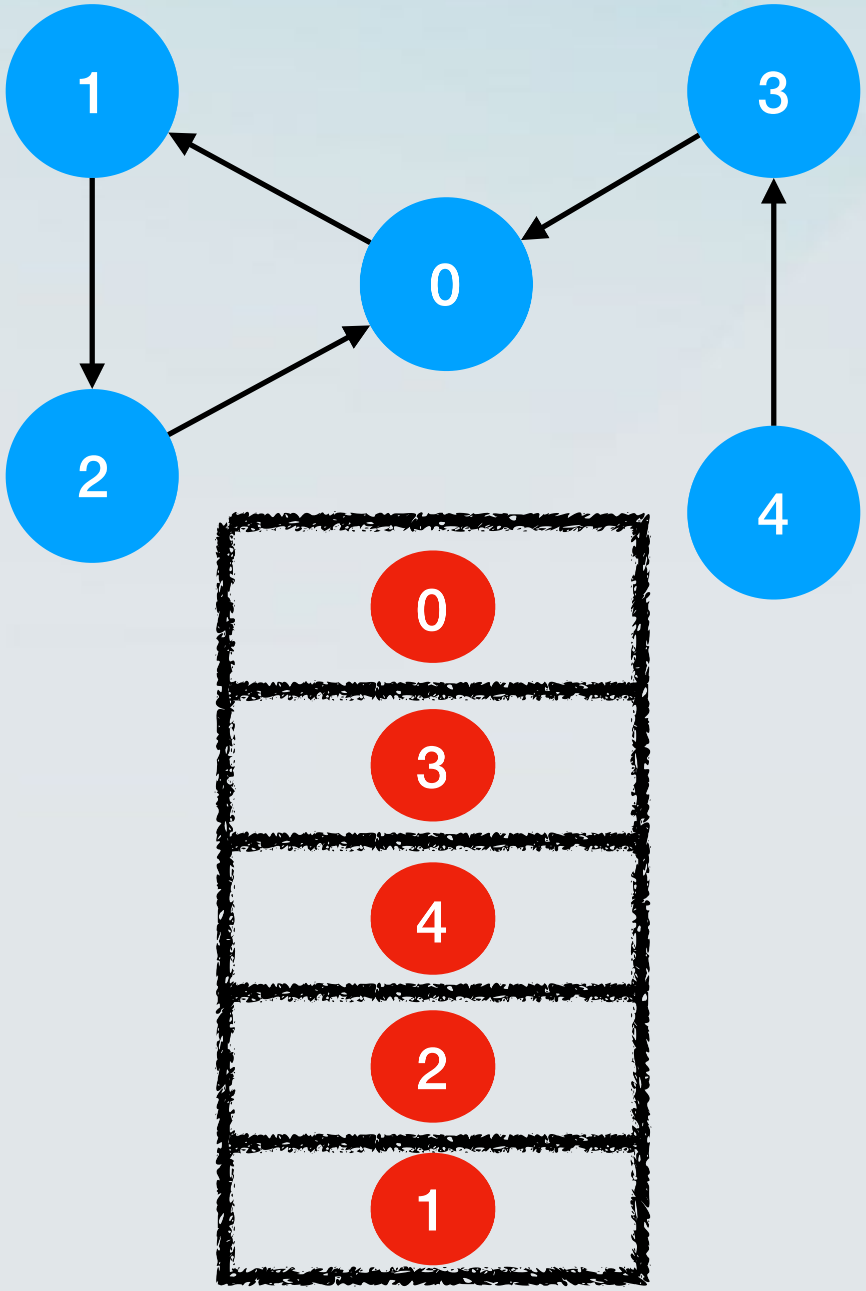

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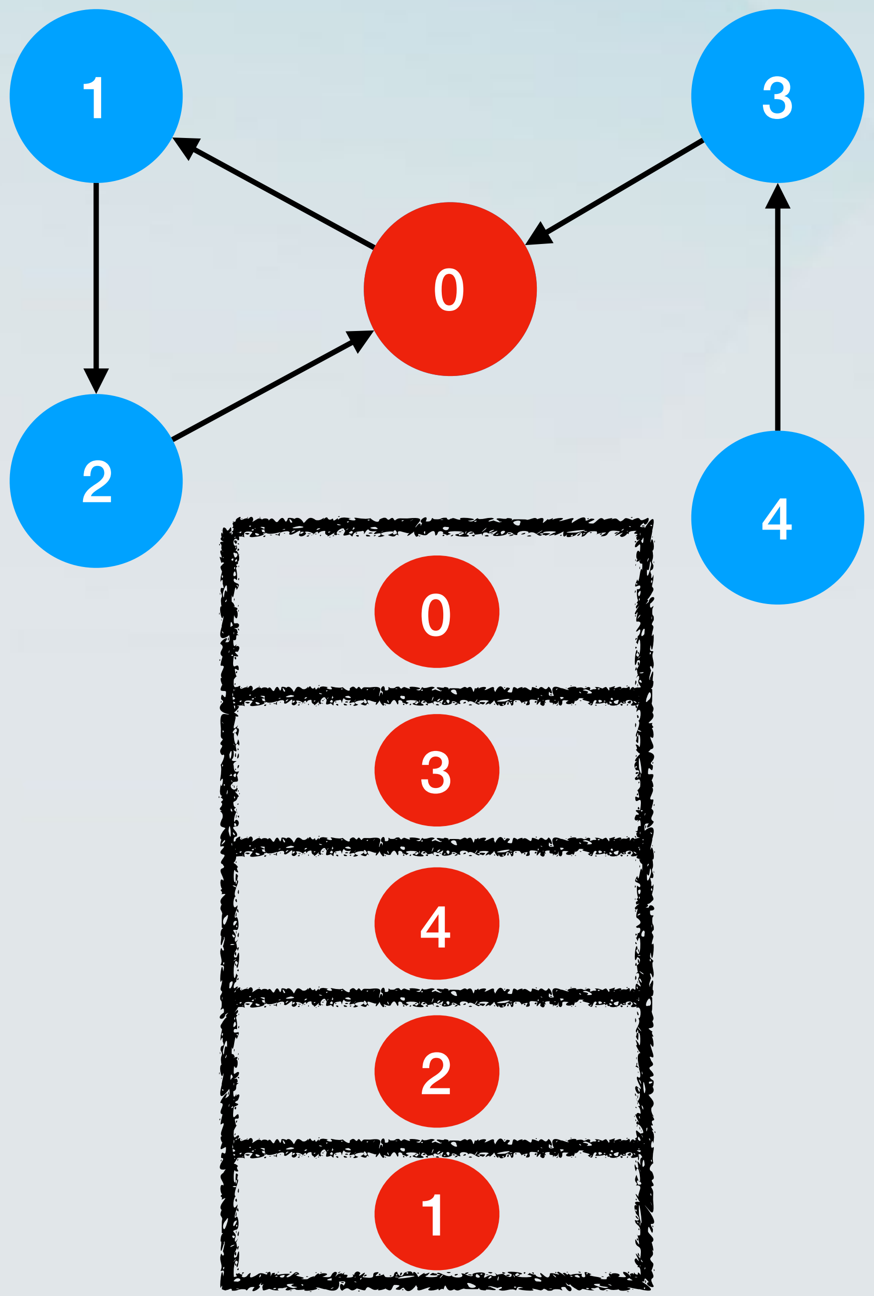

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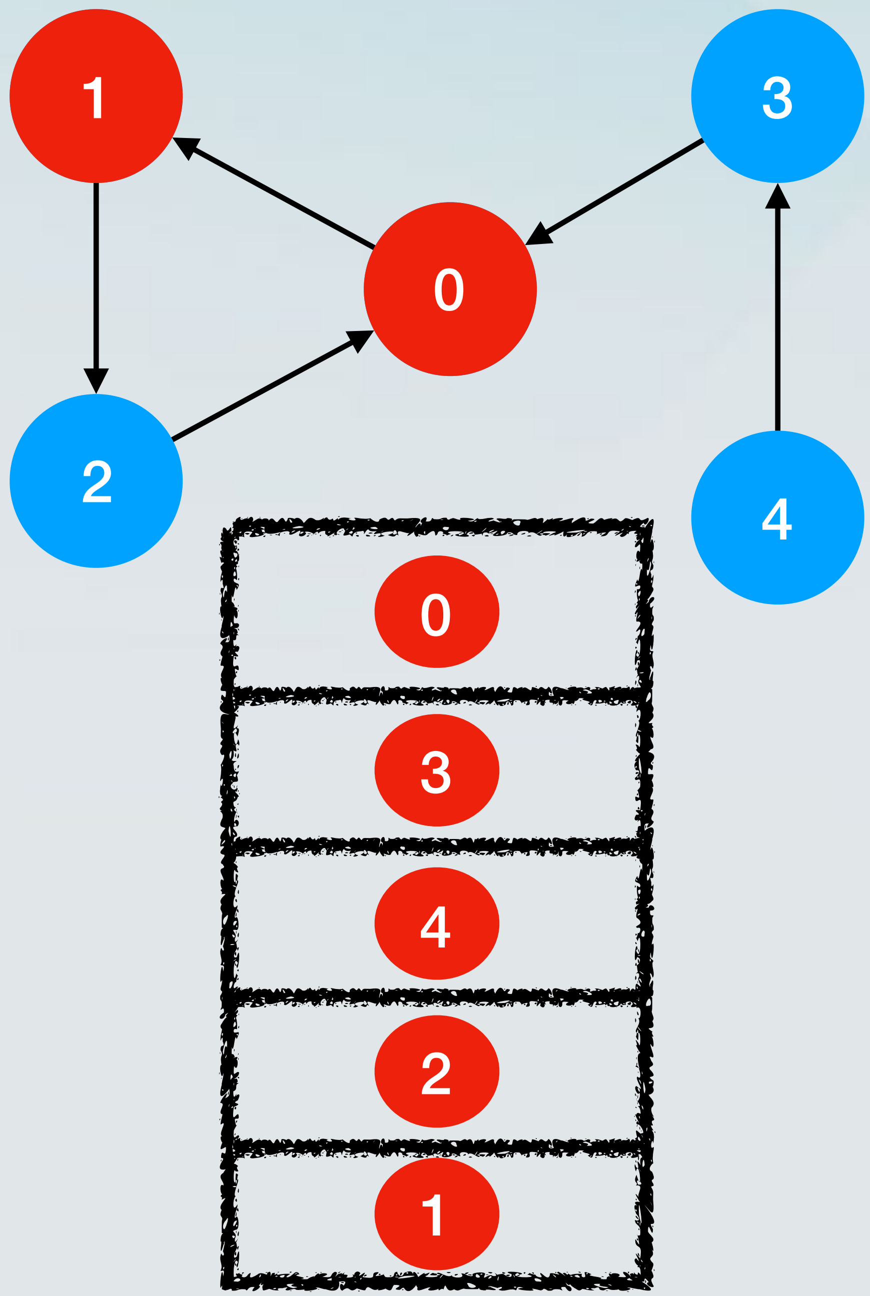

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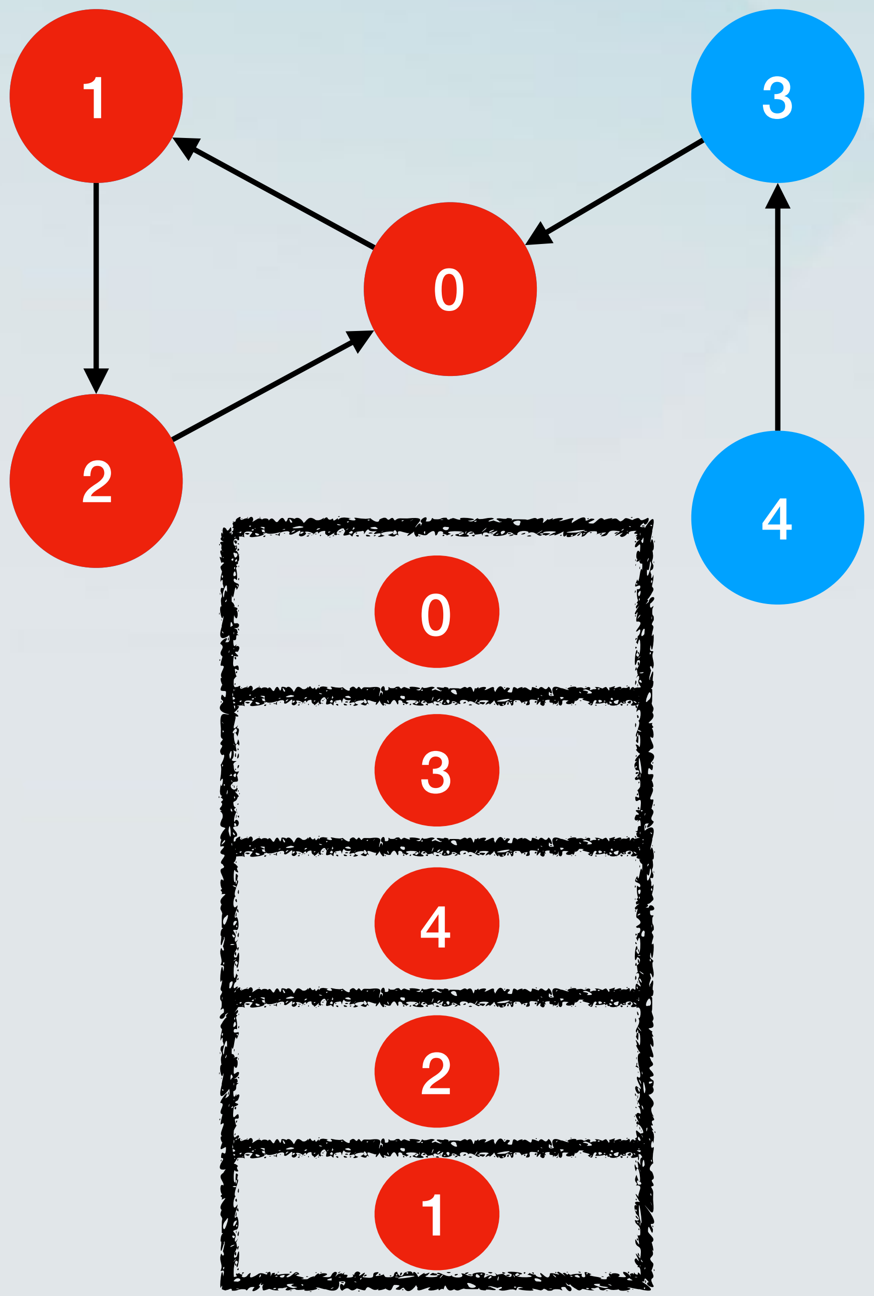

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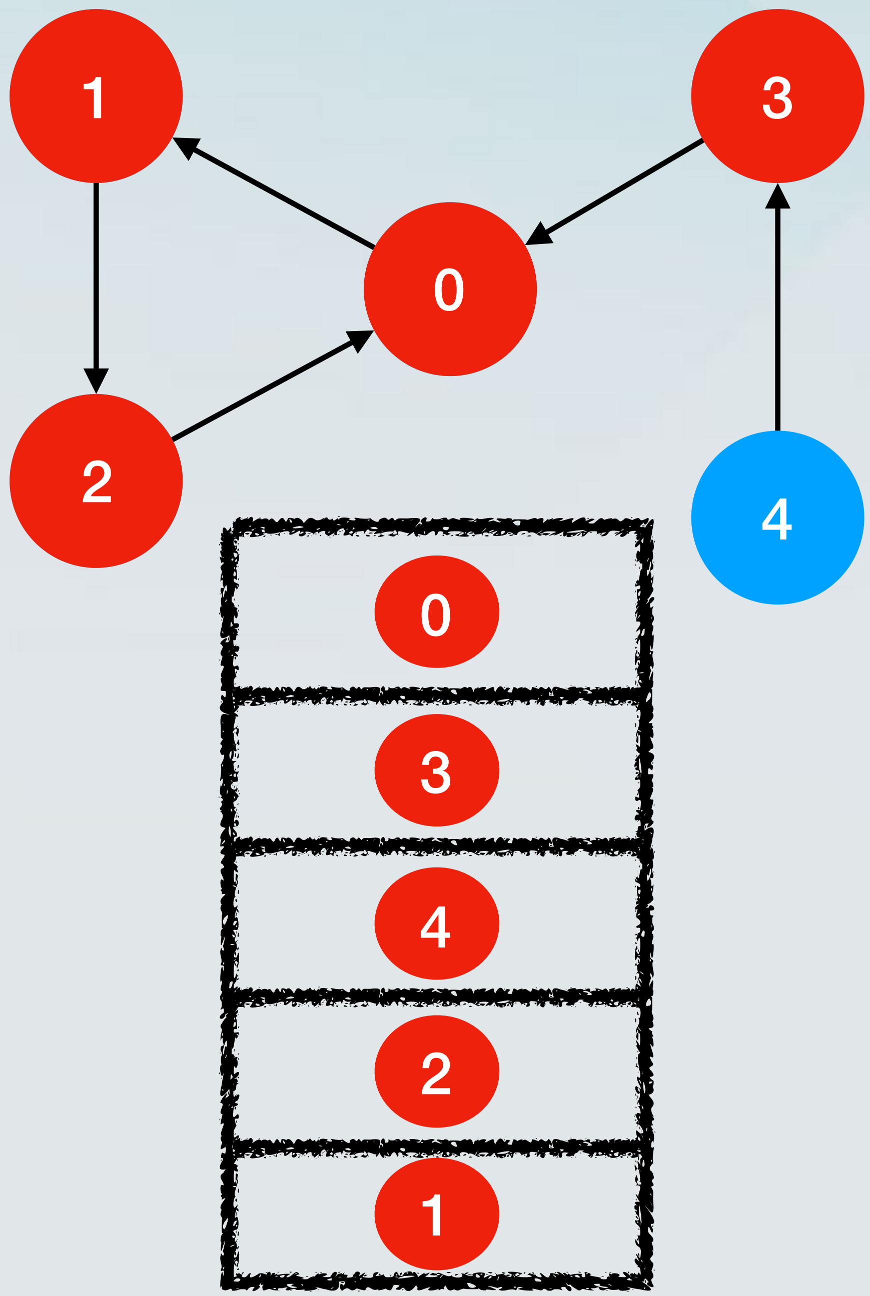

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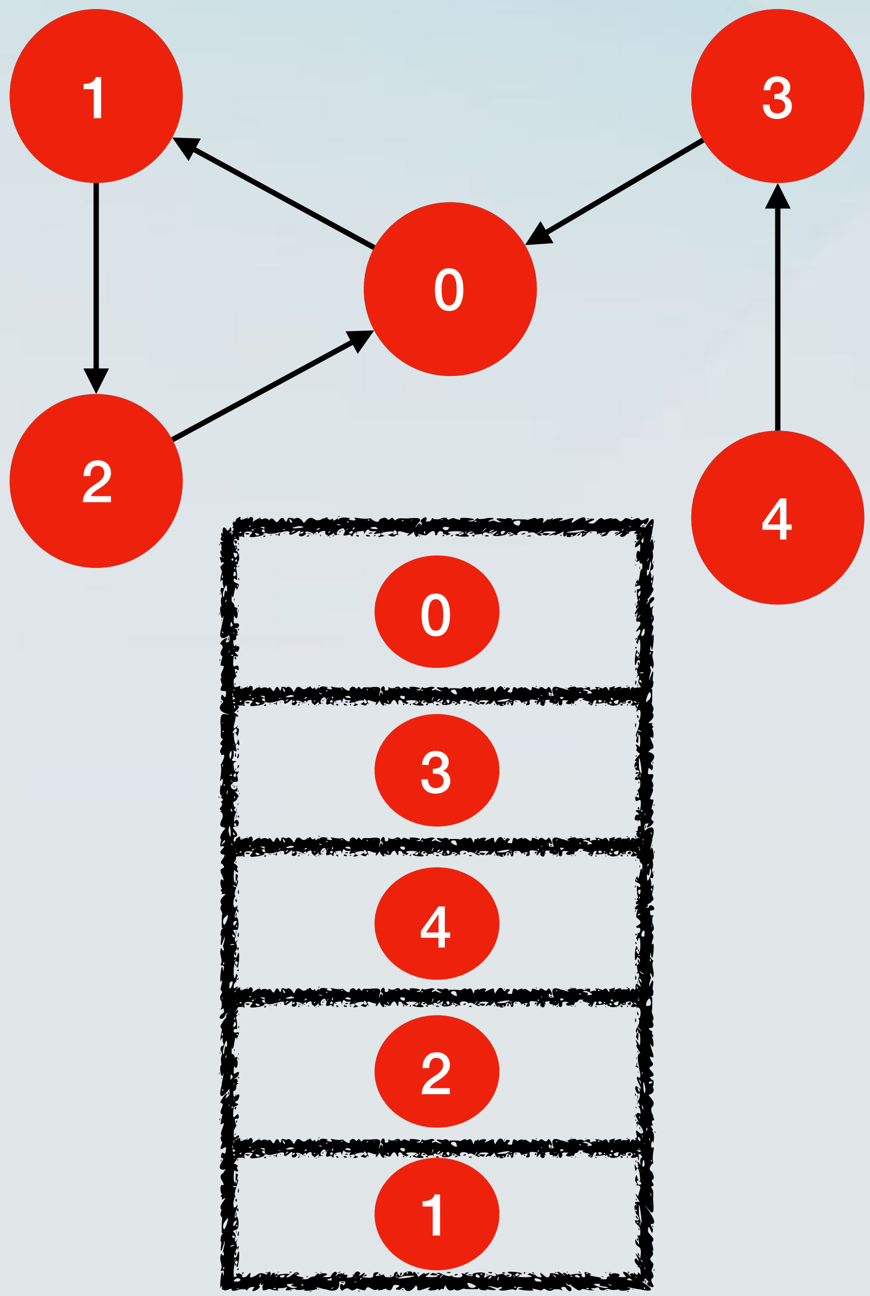

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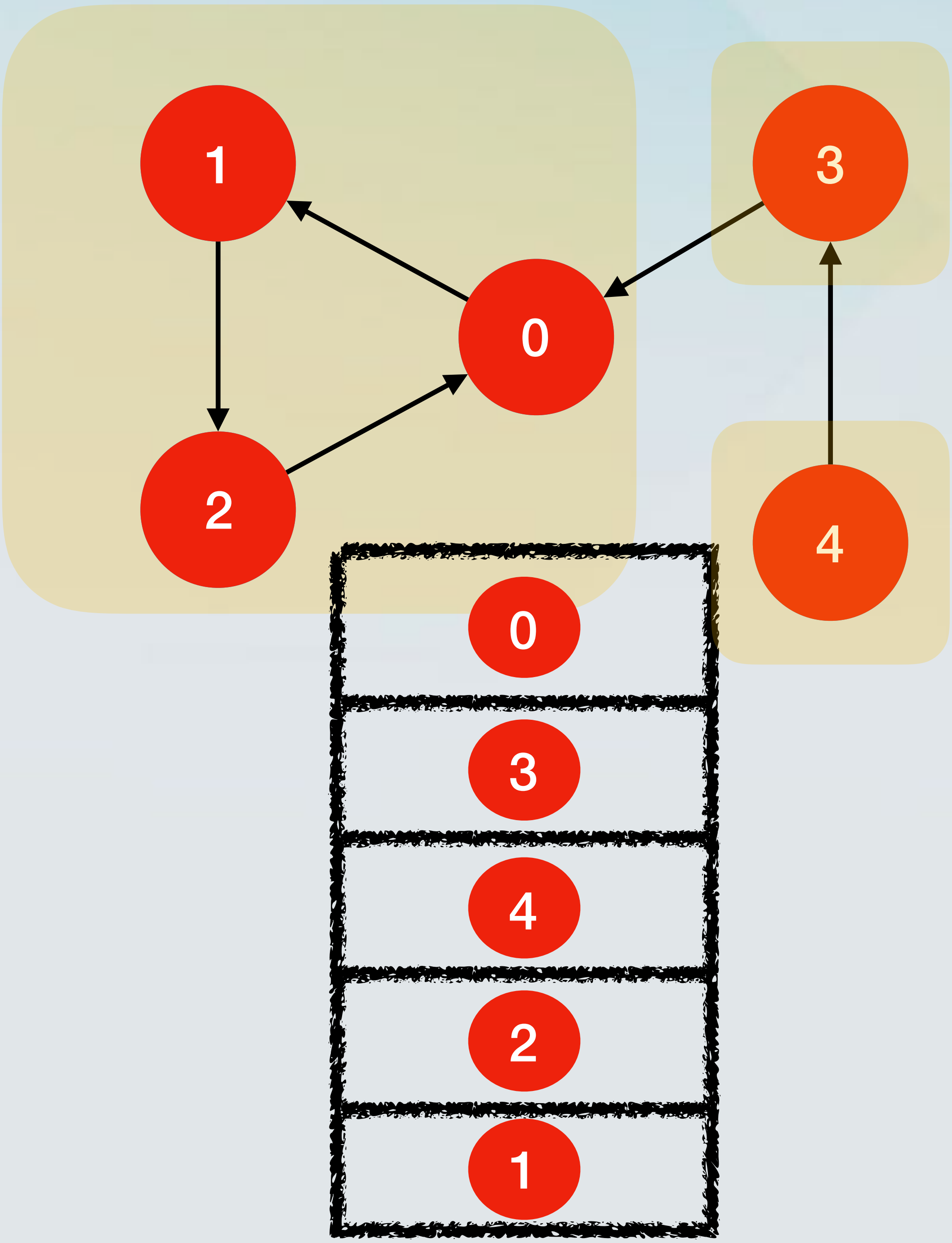

Running time

Running time

- We perform DFS twice.
- The running time is **O(m+n)**.

Correctness

- Next lecture.