Advanced Algorithmic Techniques (COMP523)

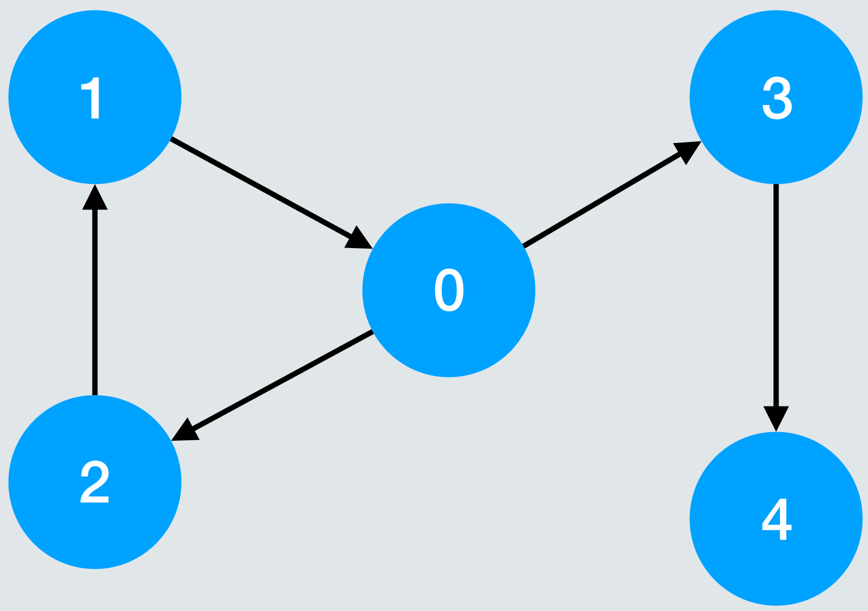
Graph Algorithms #3

Recap and plan

- **Last lecture:**
  - Testing bipartiteness
  - DFS and BFS on directed graphs
  - Testing connectivity
- **This lecture:**
  - Directed Acyclic Graphs (DAG)
  - Topological Ordering
  - Finding strongly connected components

Directed Acyclic Graphs

- A directed acyclic graph (DAG) G is a graph that does not have any cycles.



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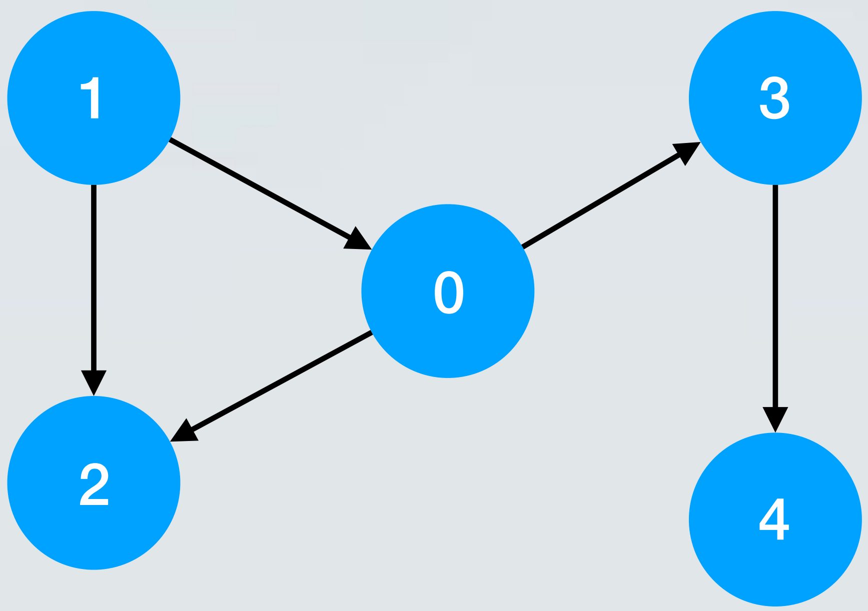

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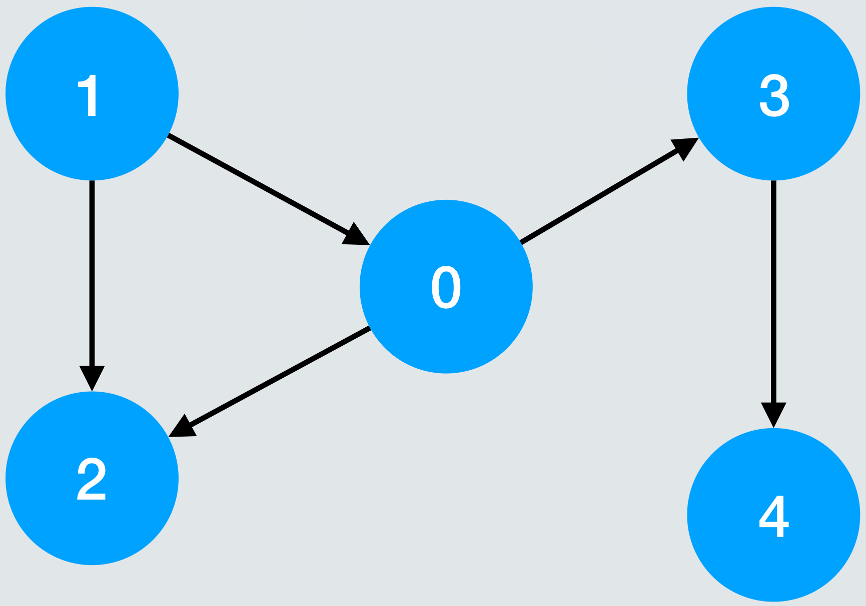
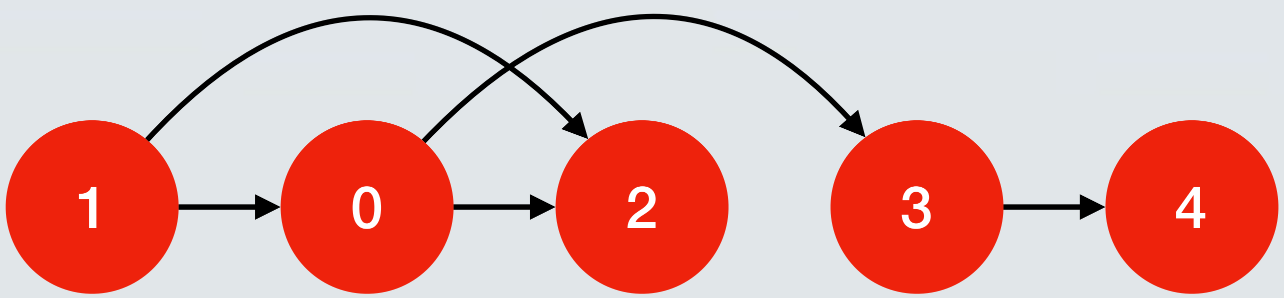
- Example - prerequisite modules: To take module A you need to have taken module B and module C.
- If the module prerequisite relation has a cycle, then it is impossible to get a degree!

Topological Ordering

- Given a directed graph G, a topological ordering of G is an ordering of the nodes $u_1, u_2, \ldots, u_n$, such that for every edge $e=(u_i, u_j)$, it holds that $i < j$.

- Intuitively, a topological ordering orders the nodes in a way such that all edges point "forward".





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- u_i must appear before u_i in the topological order, by the presence of this edge.
- This **contradicts** the fact that u_j was the smallest element of C according to the topological ordering.

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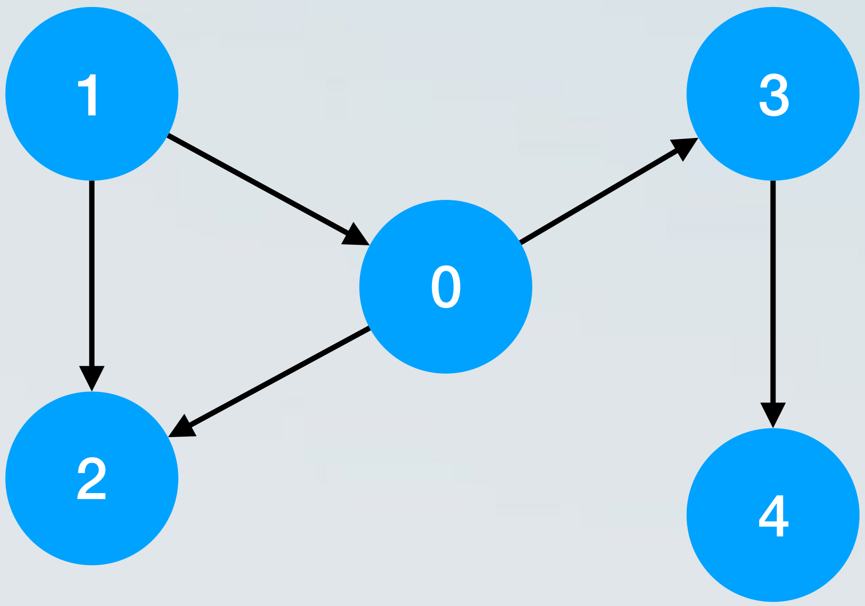
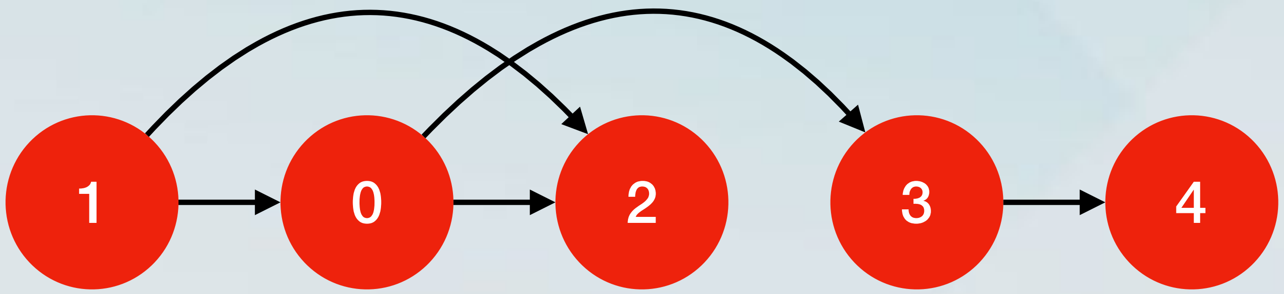
- DAG => TO will be proved via "proof-by-algorithm".
- We will design an *efficient* algorithm that, given a DAG G, finds a topological ordering of G.

How do we start?





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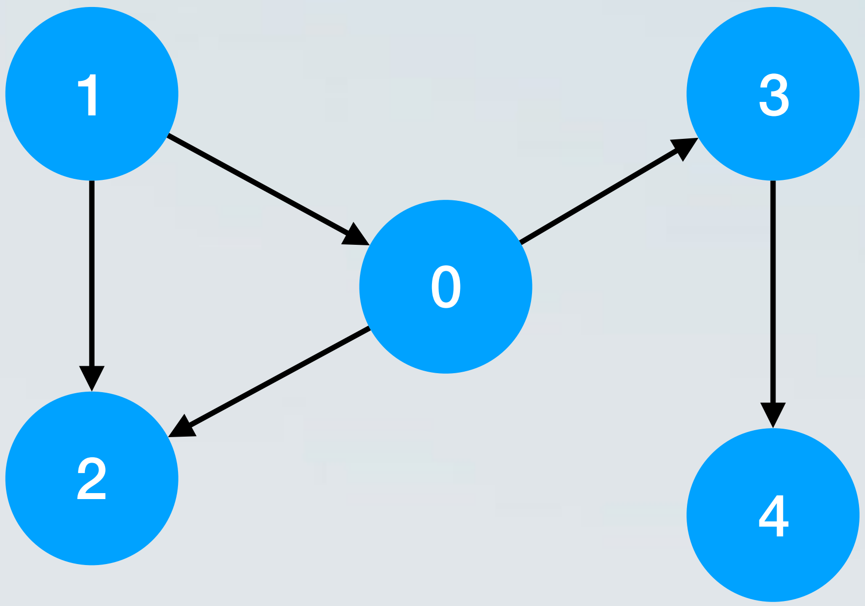

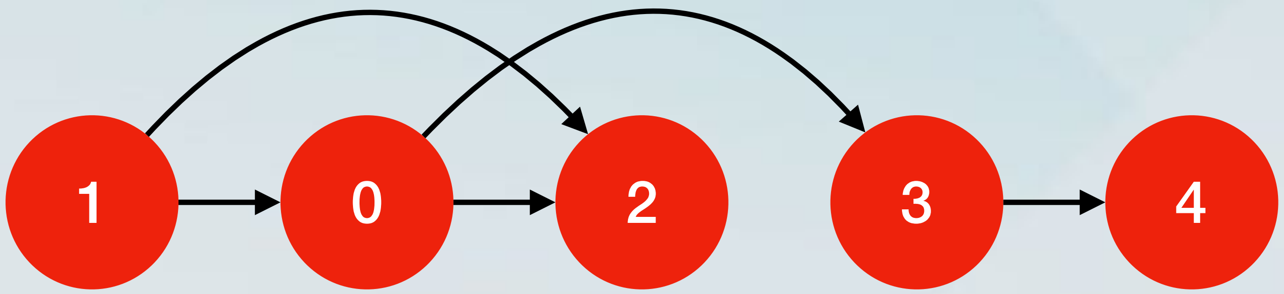

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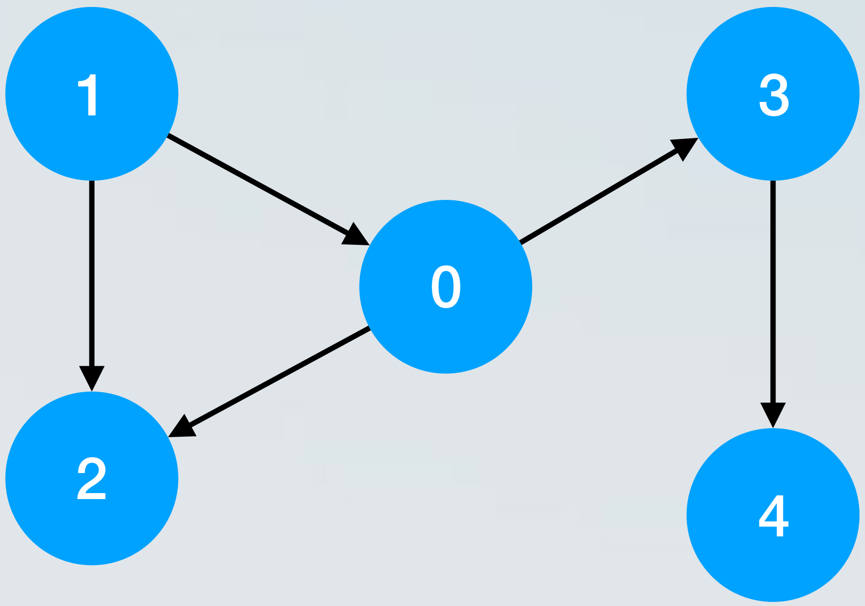

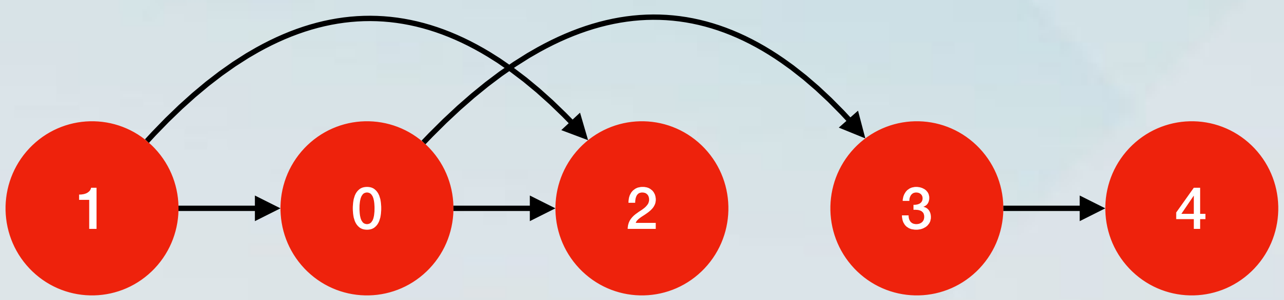

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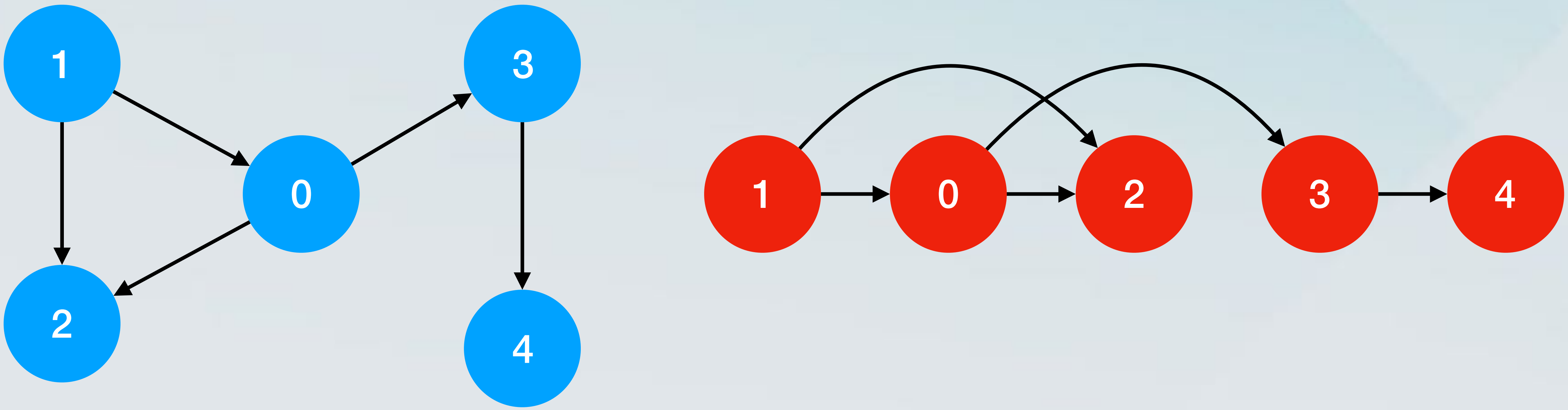

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- The starting node must have no incoming edges!
- Can we always find such a node?

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  - The graph has a cycle, therefore it can't be a DAG. **Contradiction!**

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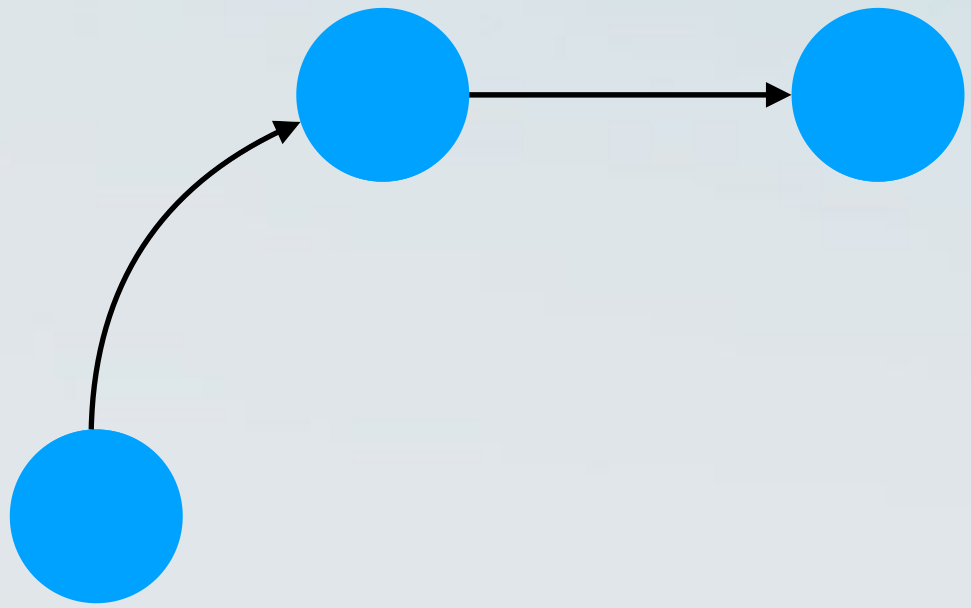

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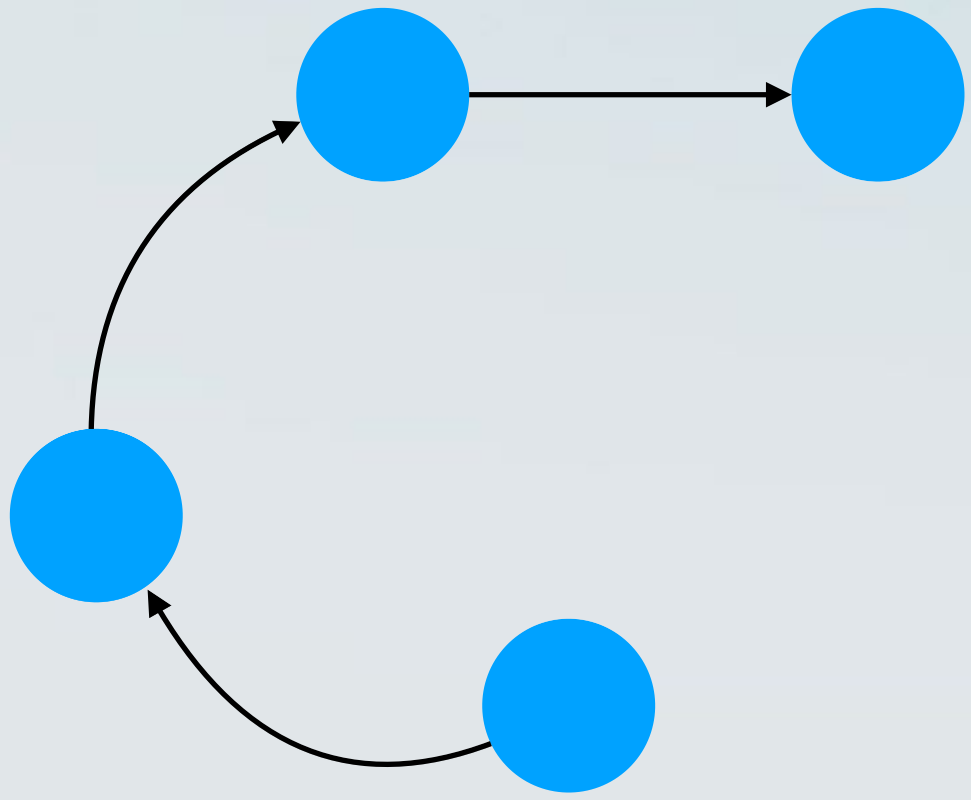

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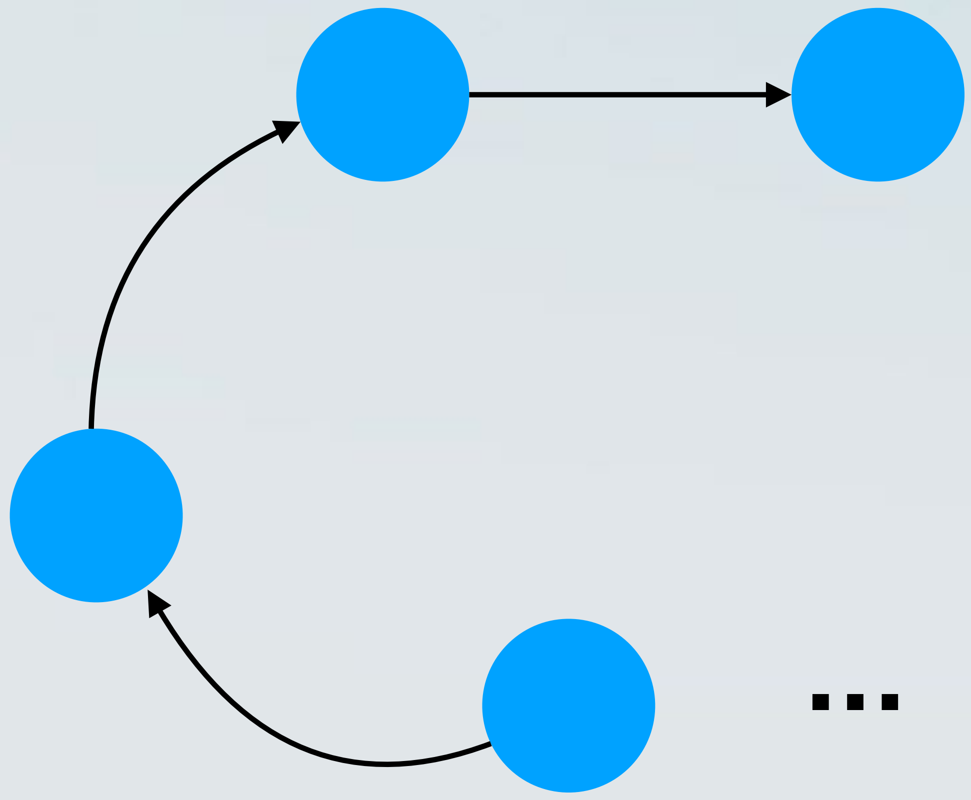

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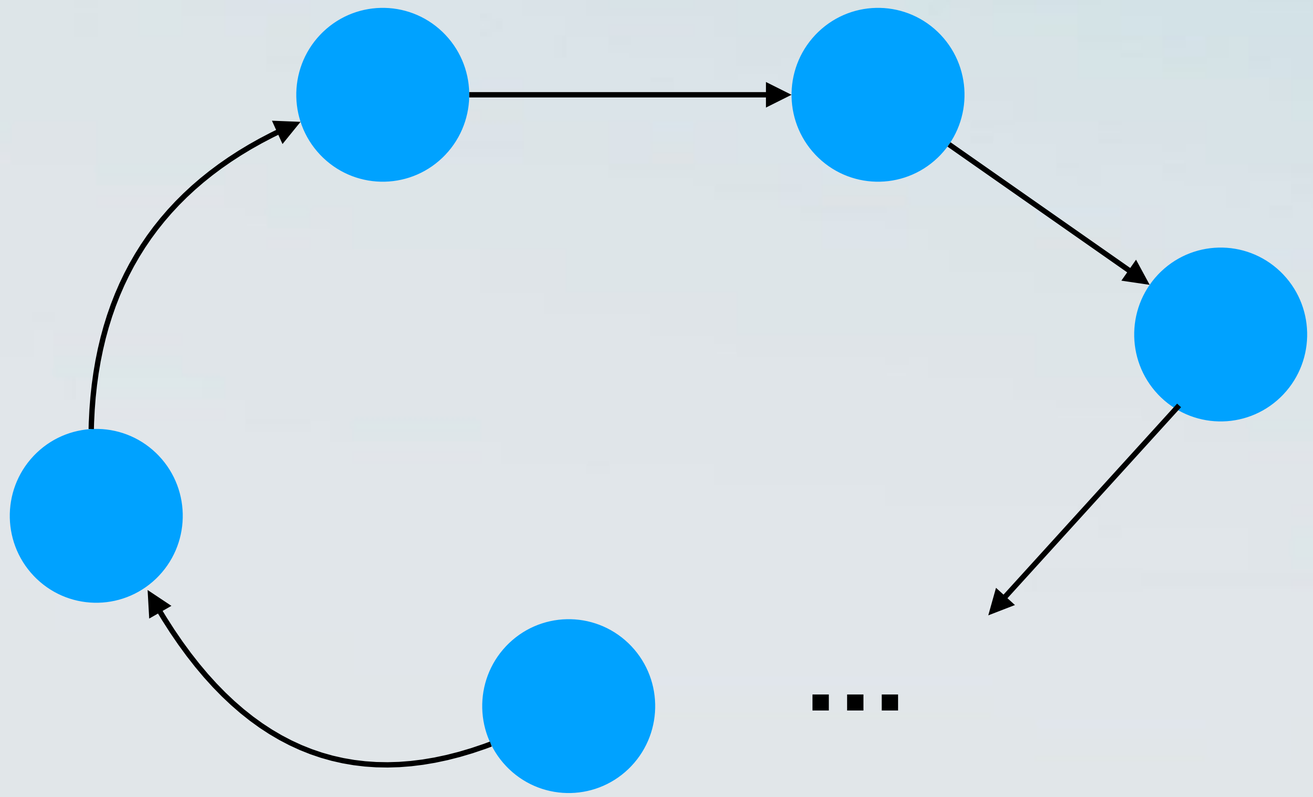

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  - If G’ had a cycle, the same cycle would be present in G.

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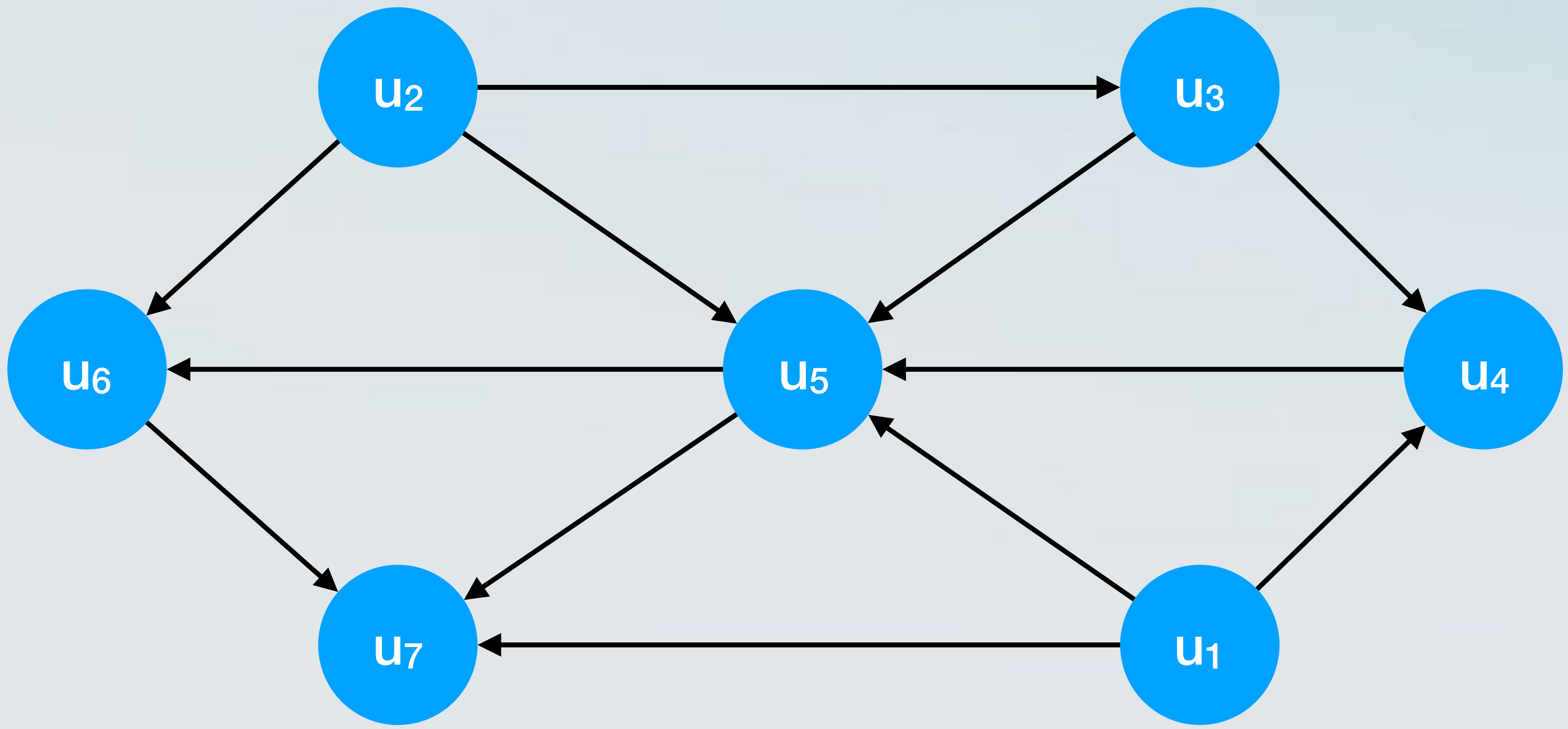
Algorithm TopologicalSort(G)

    Find a source vertex u and put it first in the order.
    Let G'=G-{u}
    TopologicalSort(G')
    Append this order after u

Example



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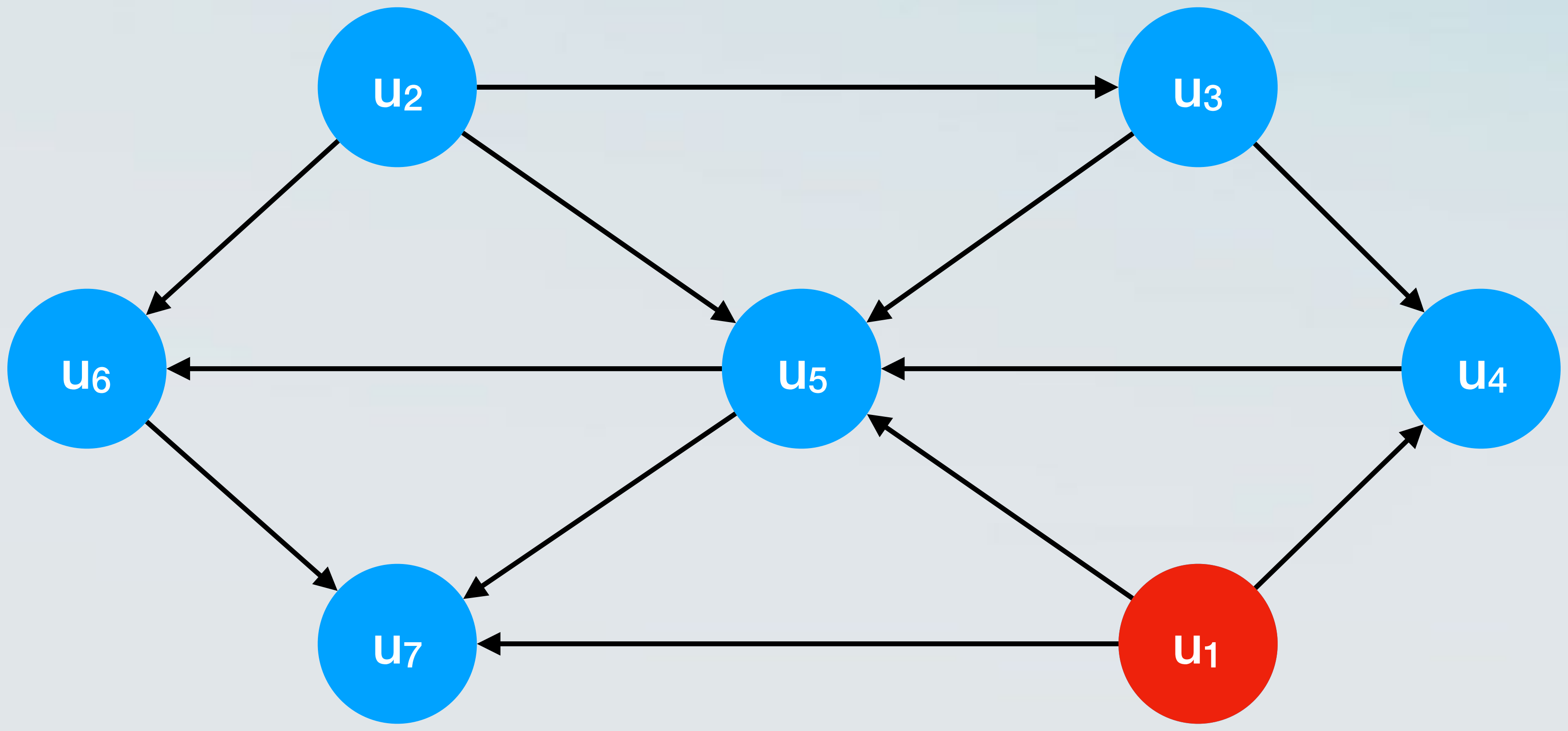

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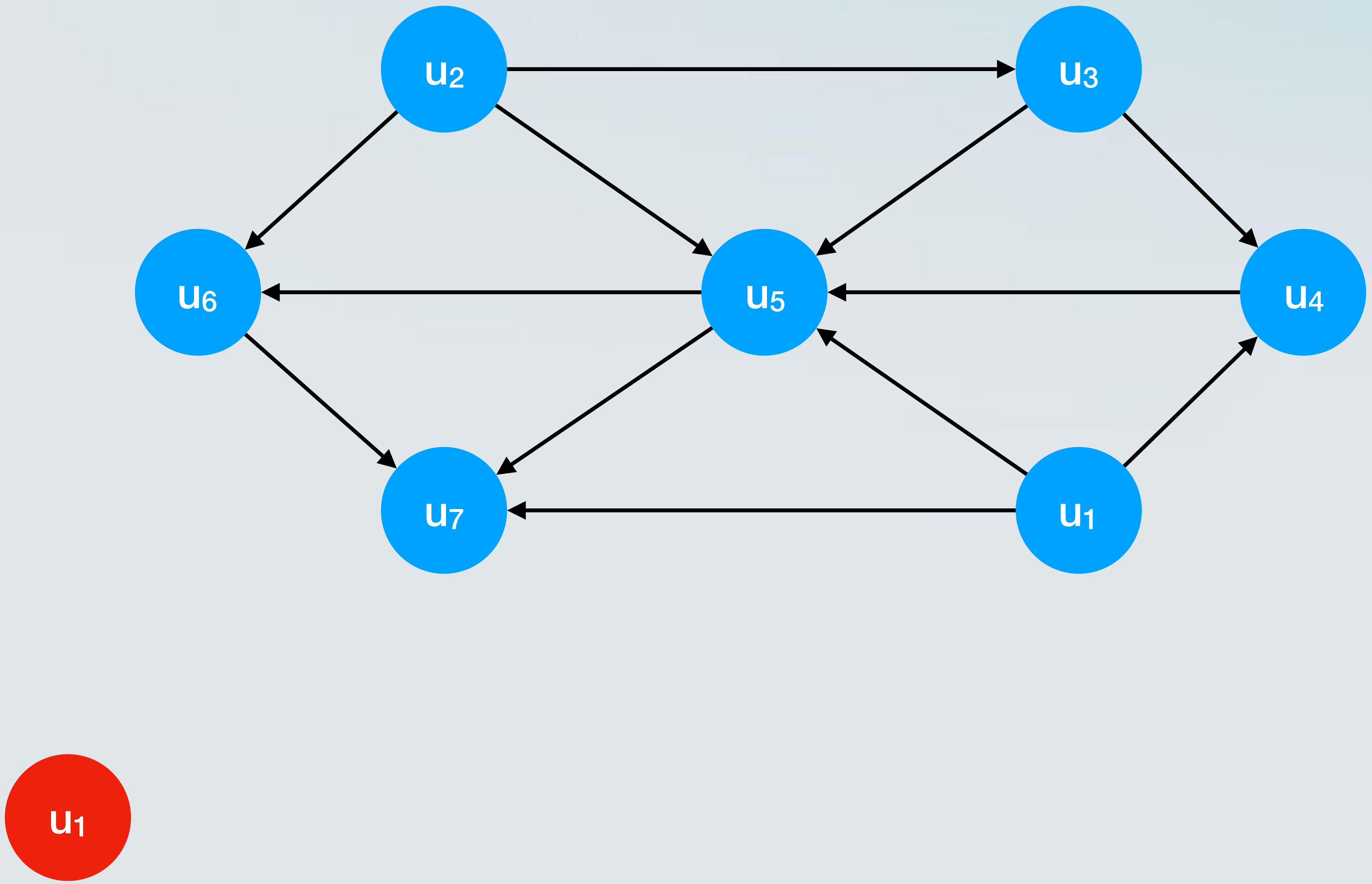

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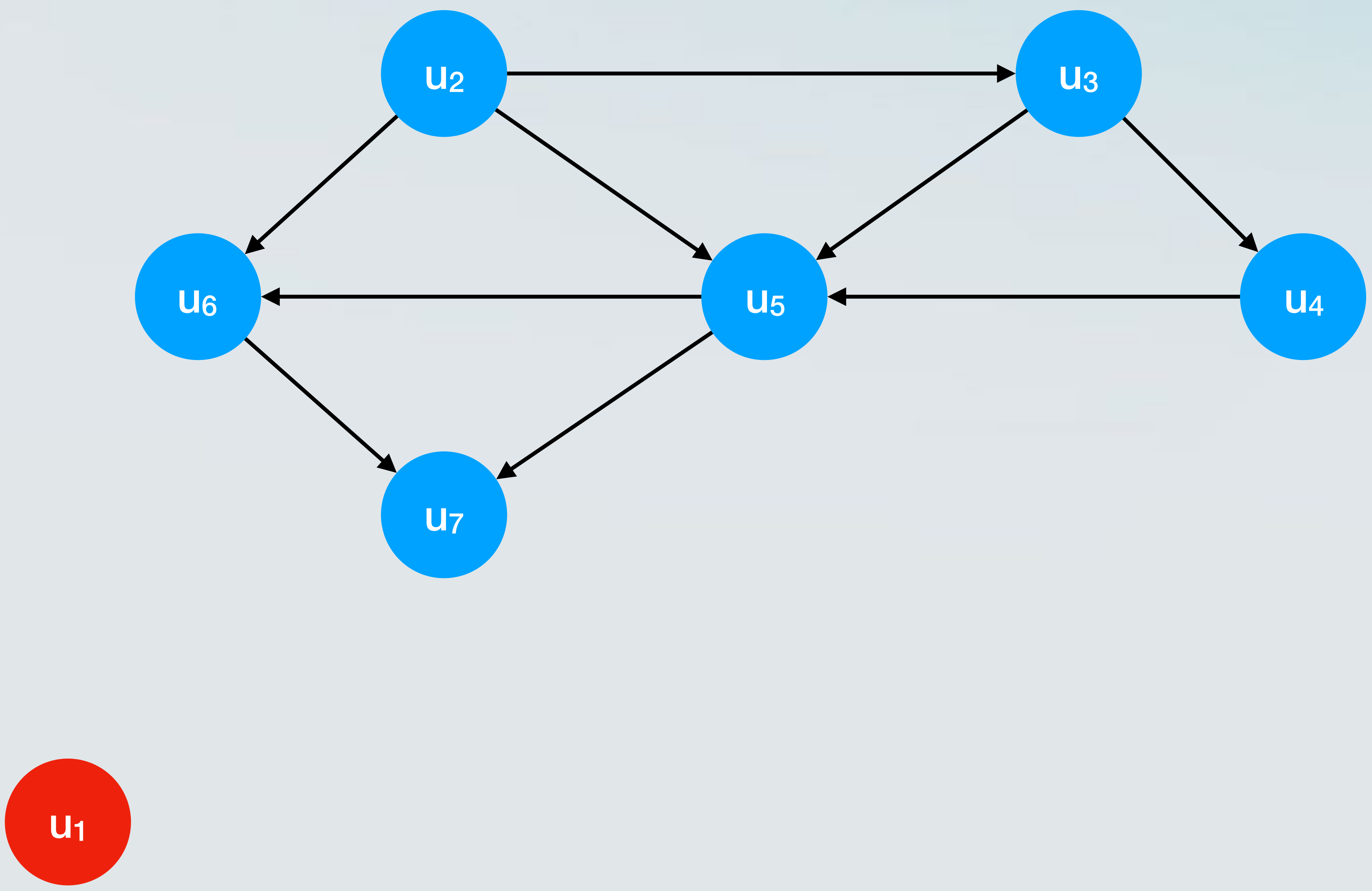

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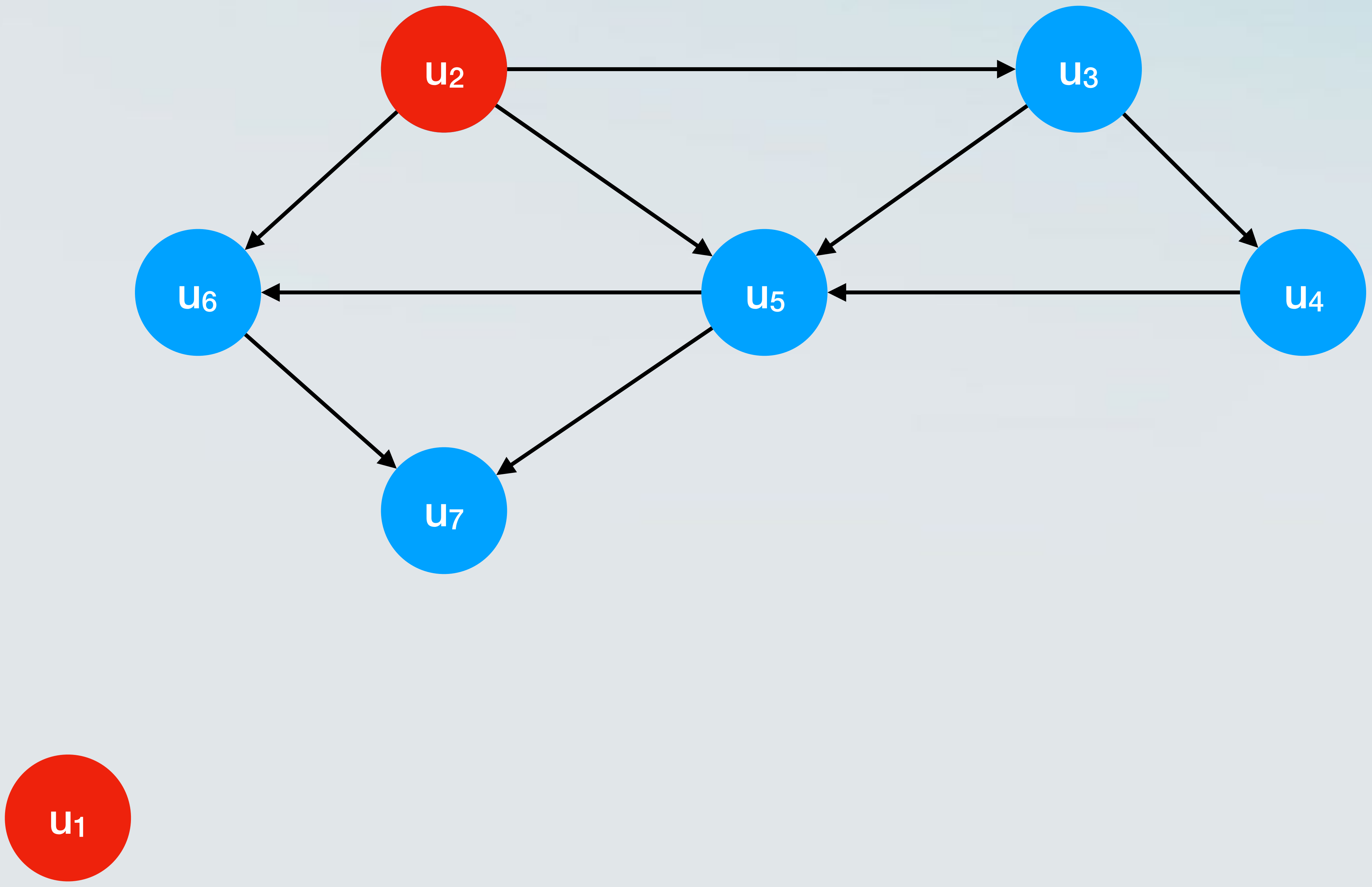

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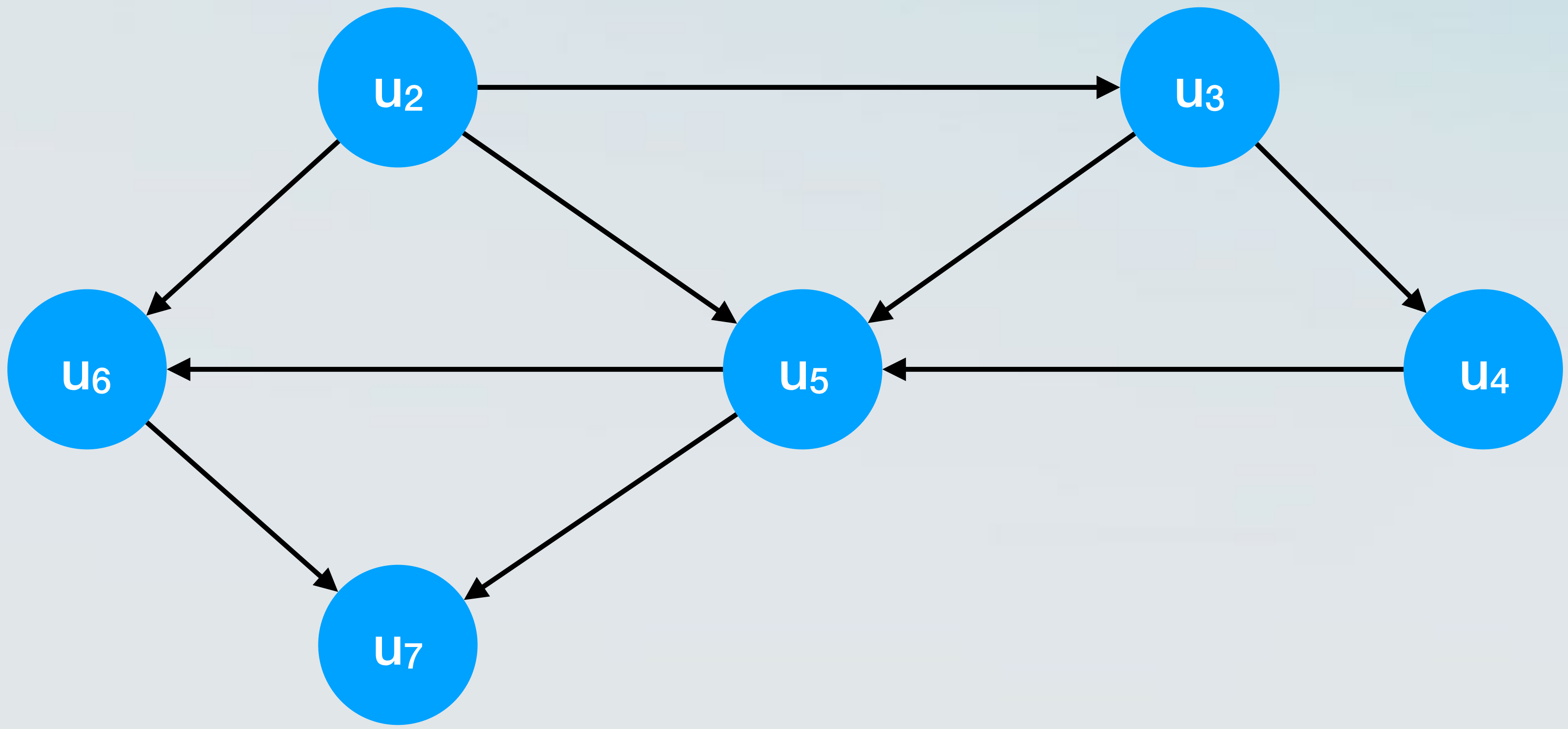

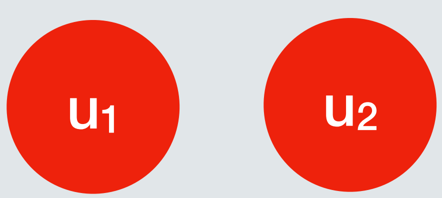

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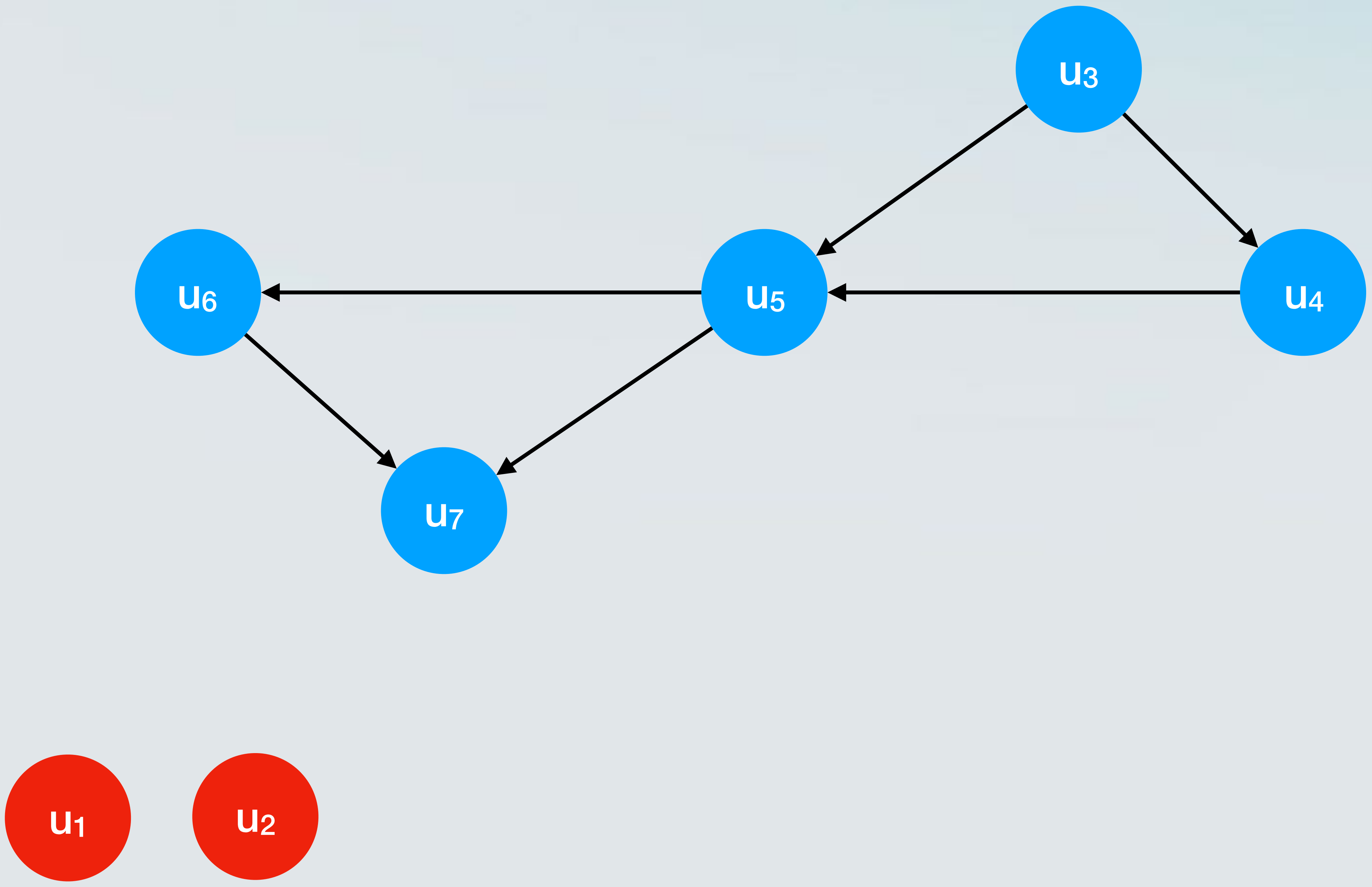

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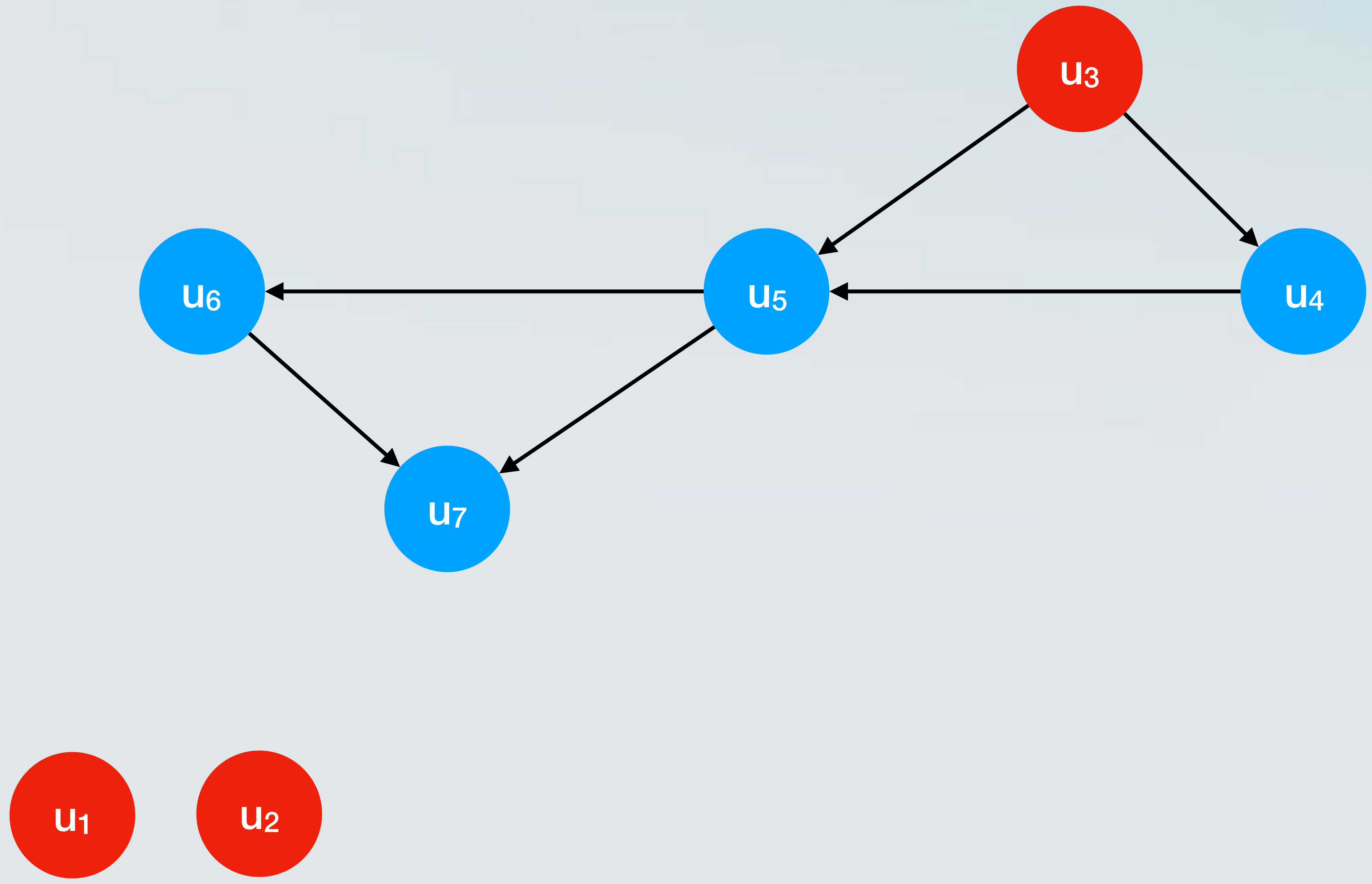

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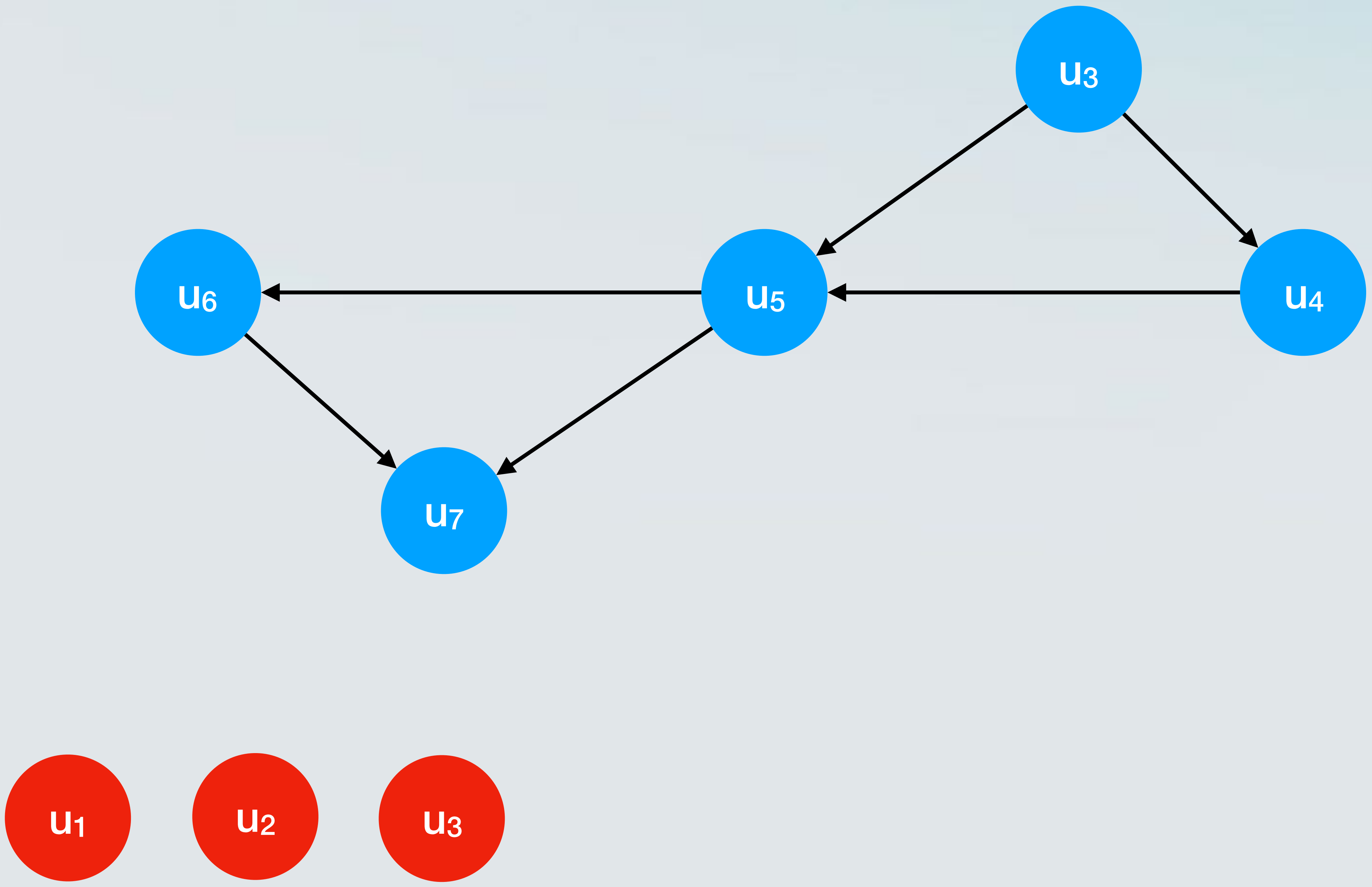

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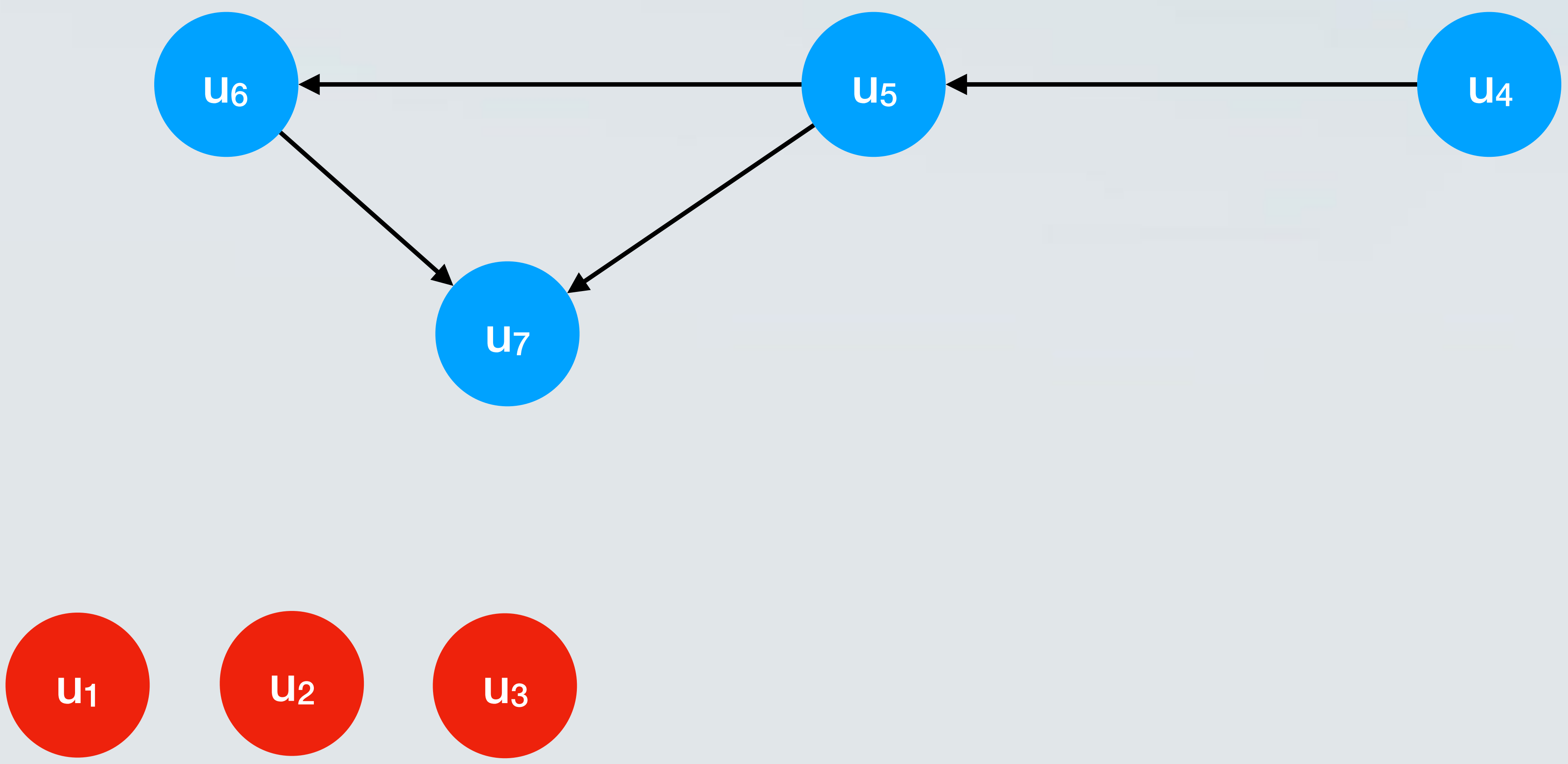

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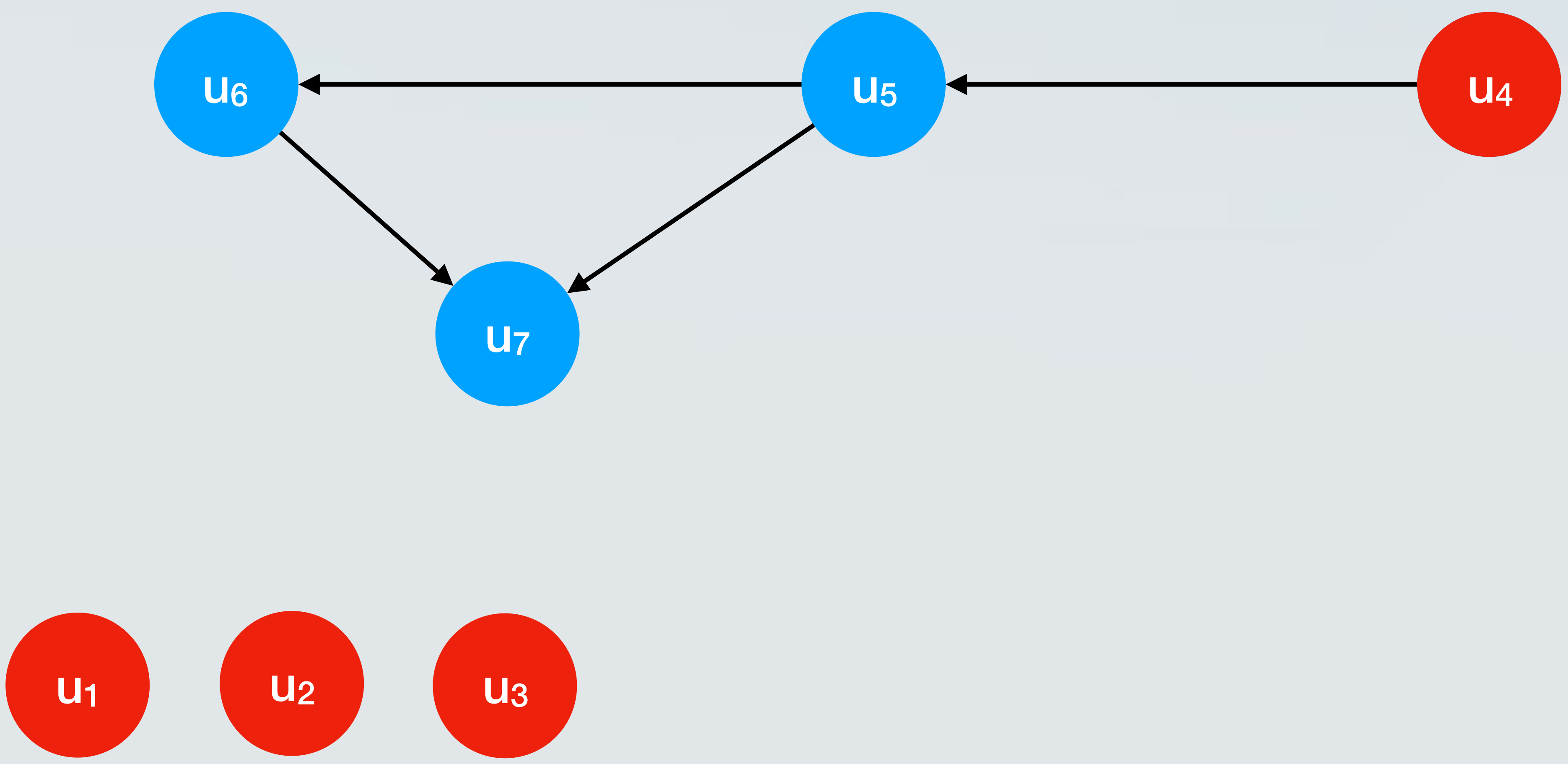

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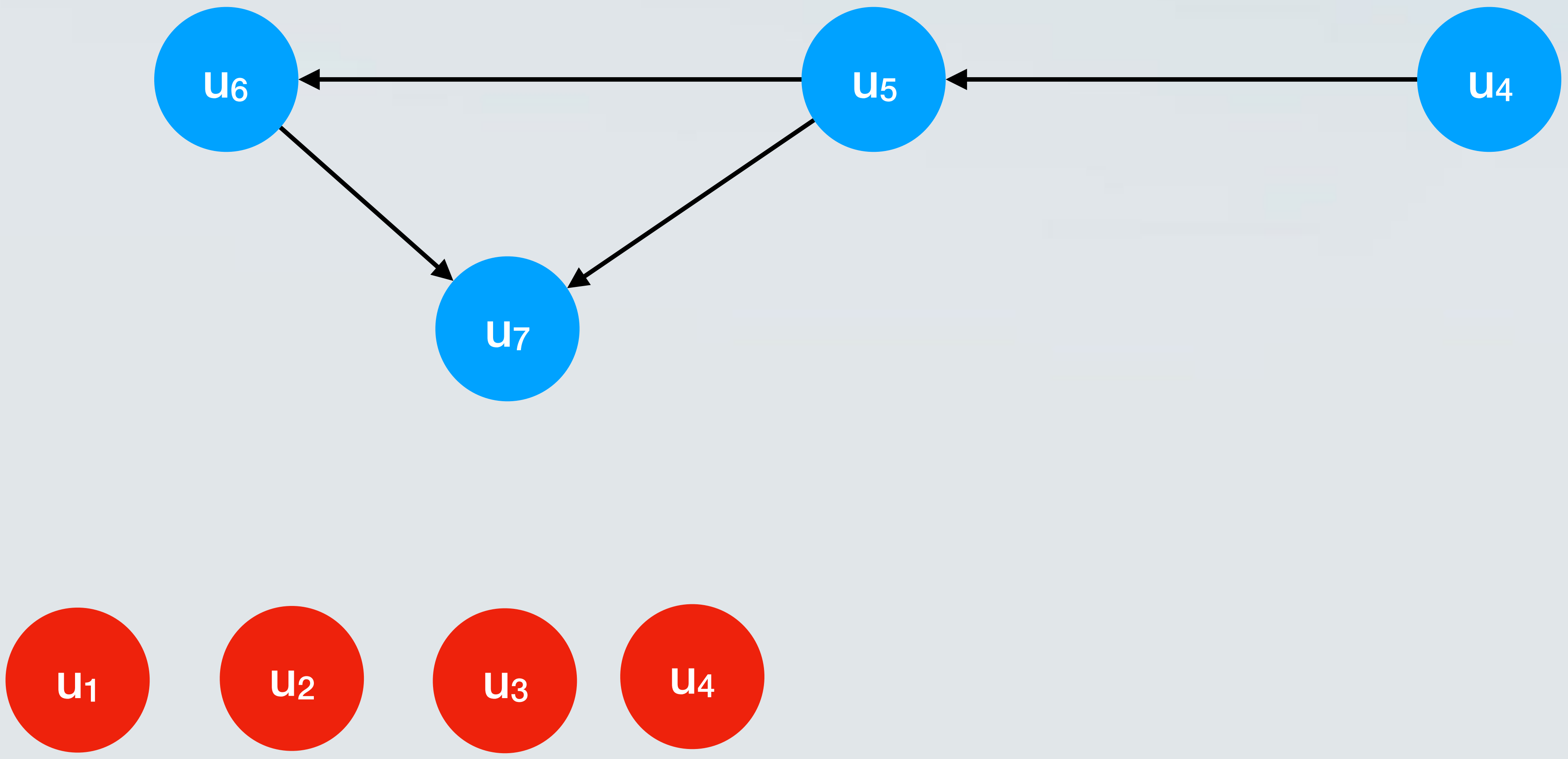

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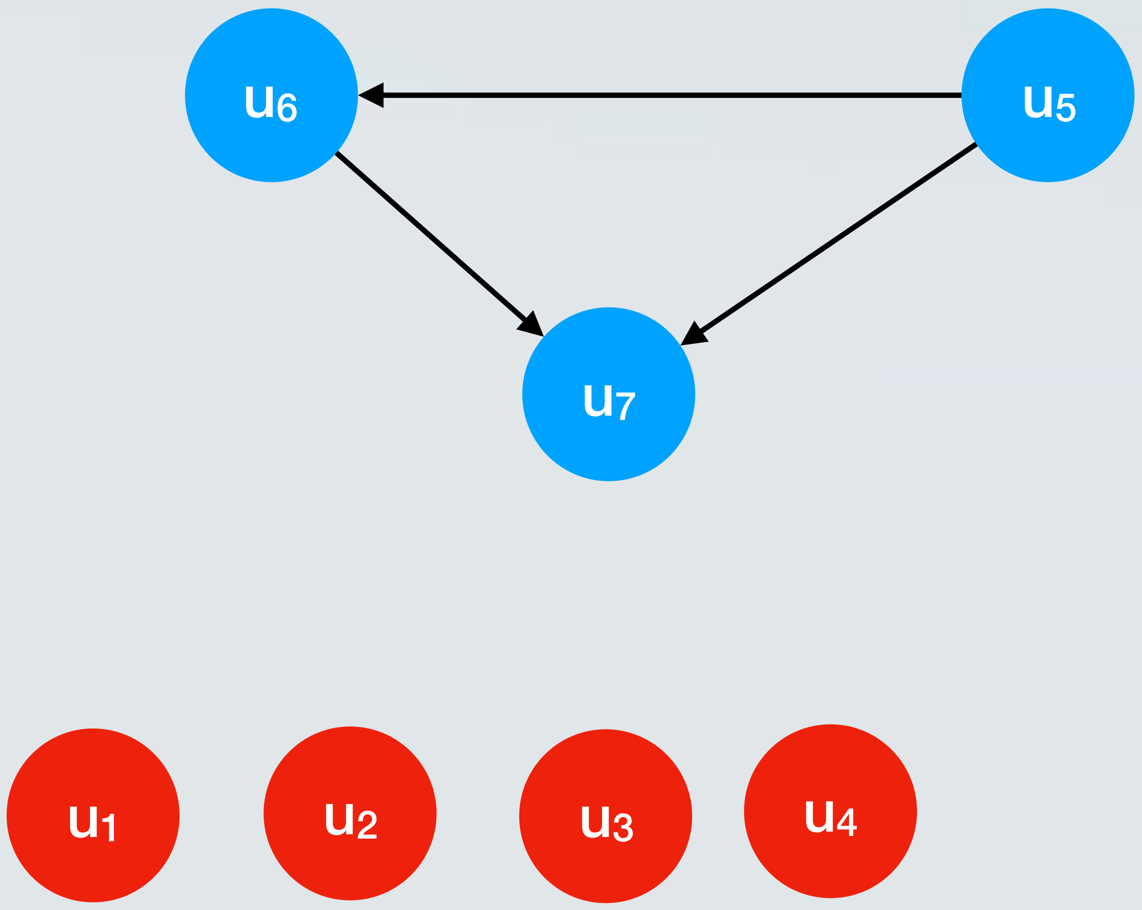

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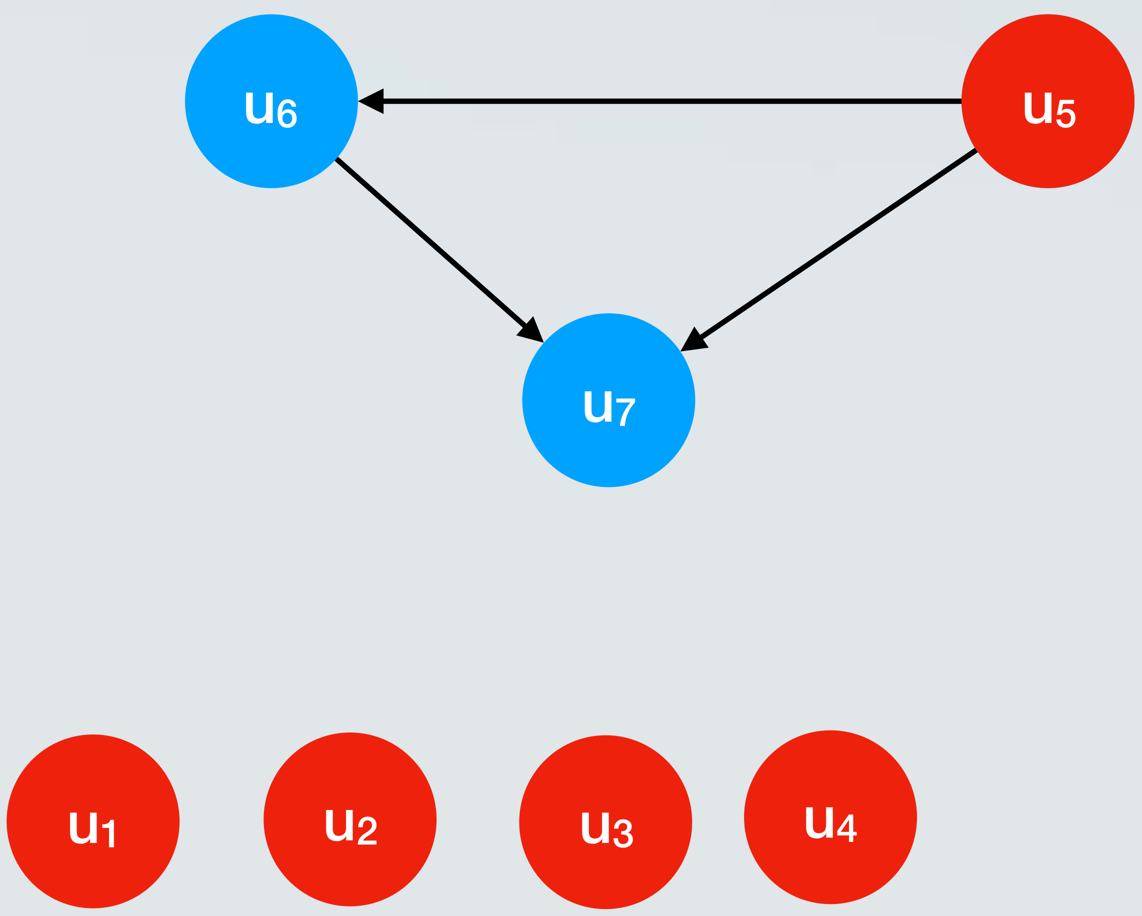

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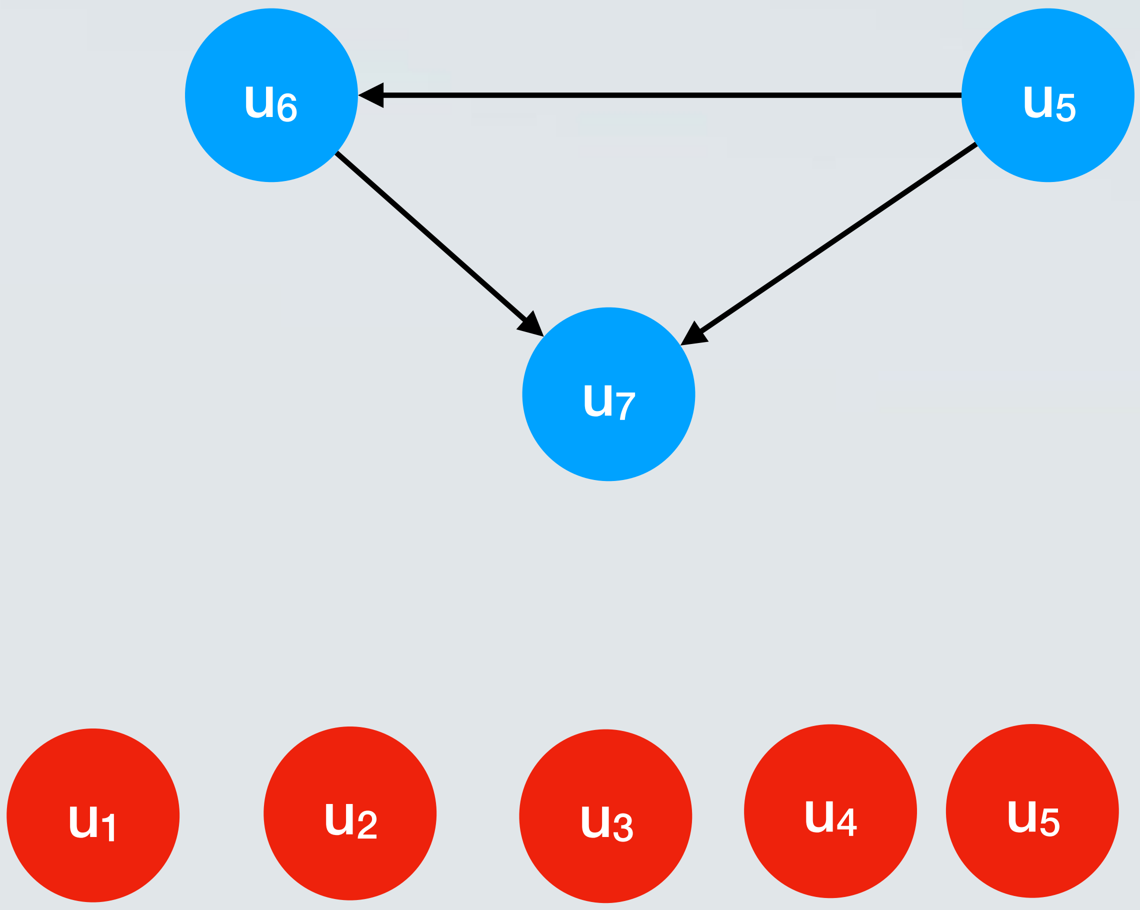

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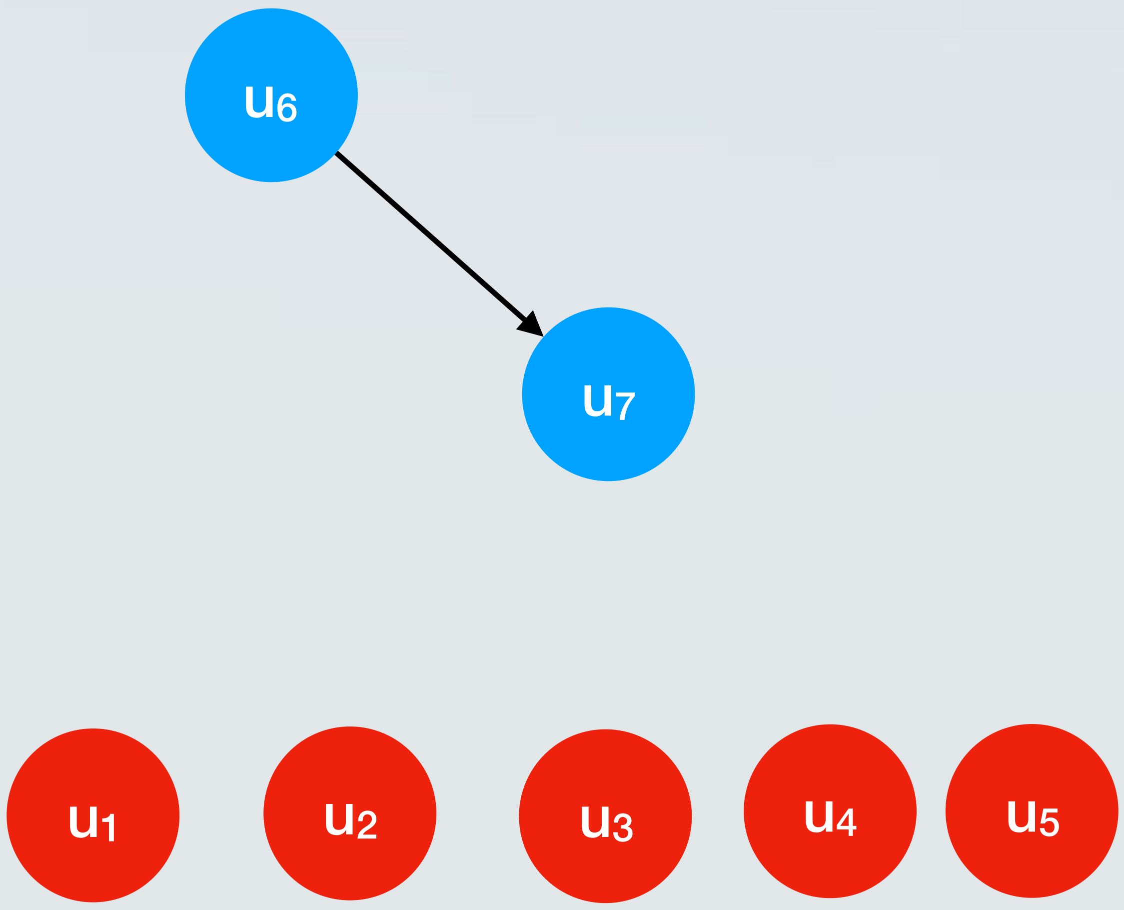

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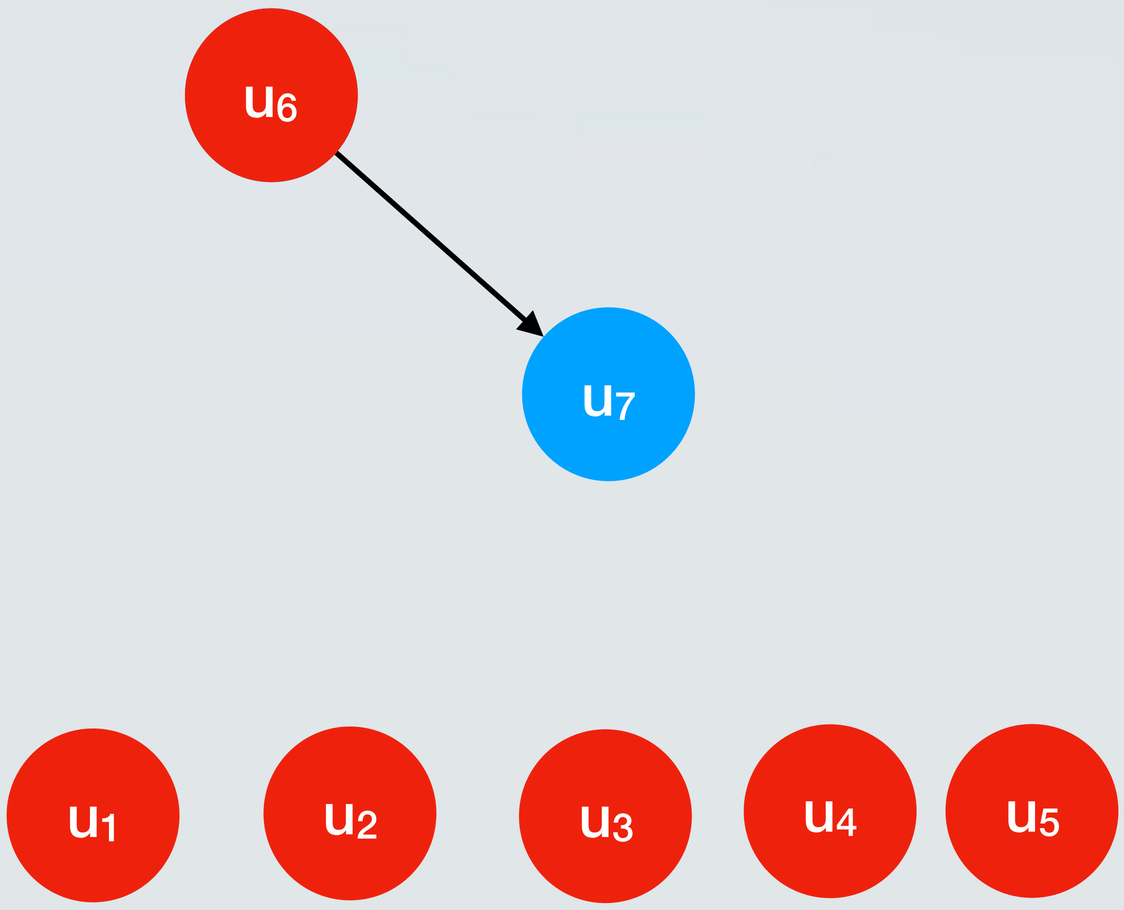

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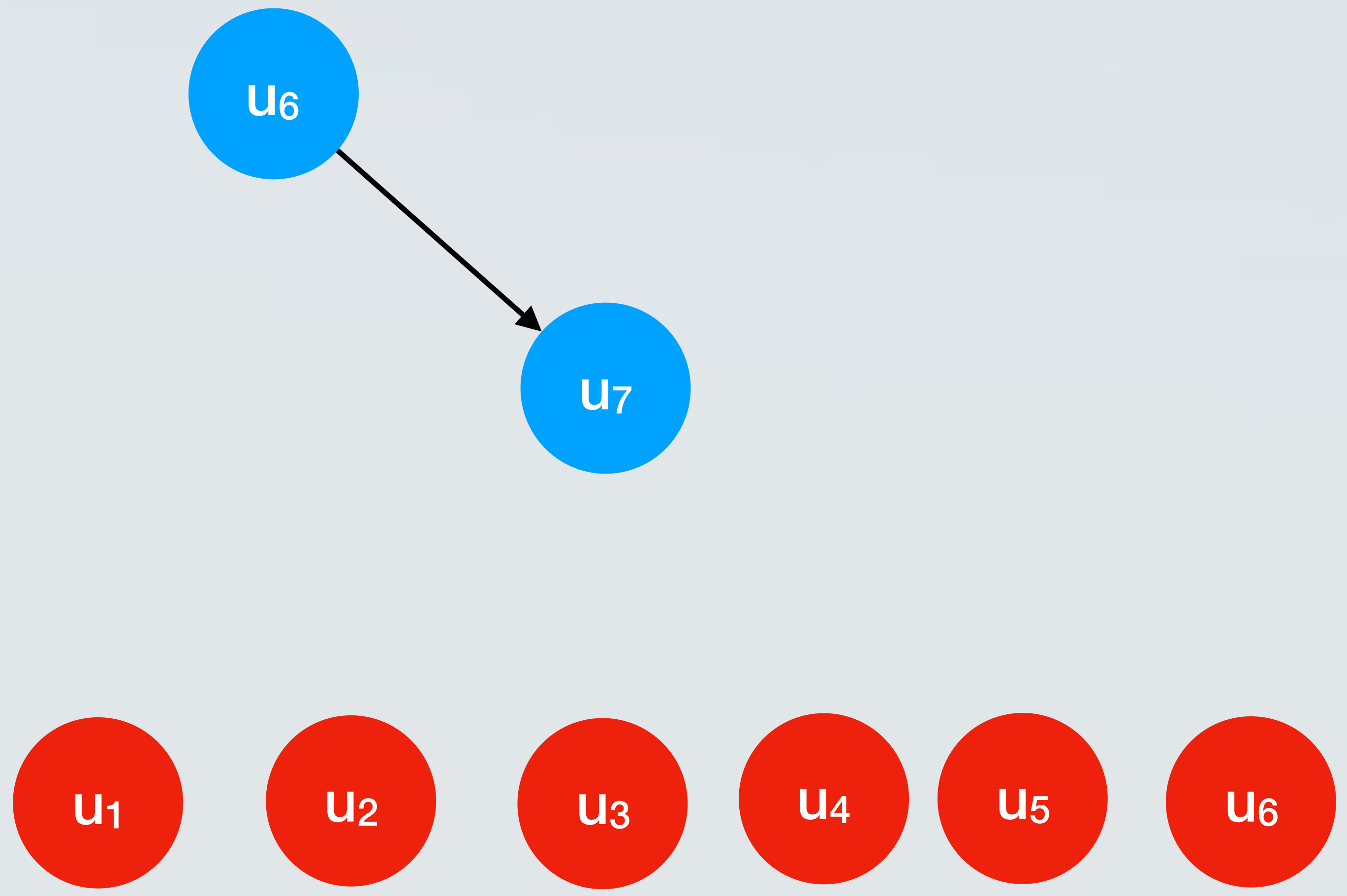

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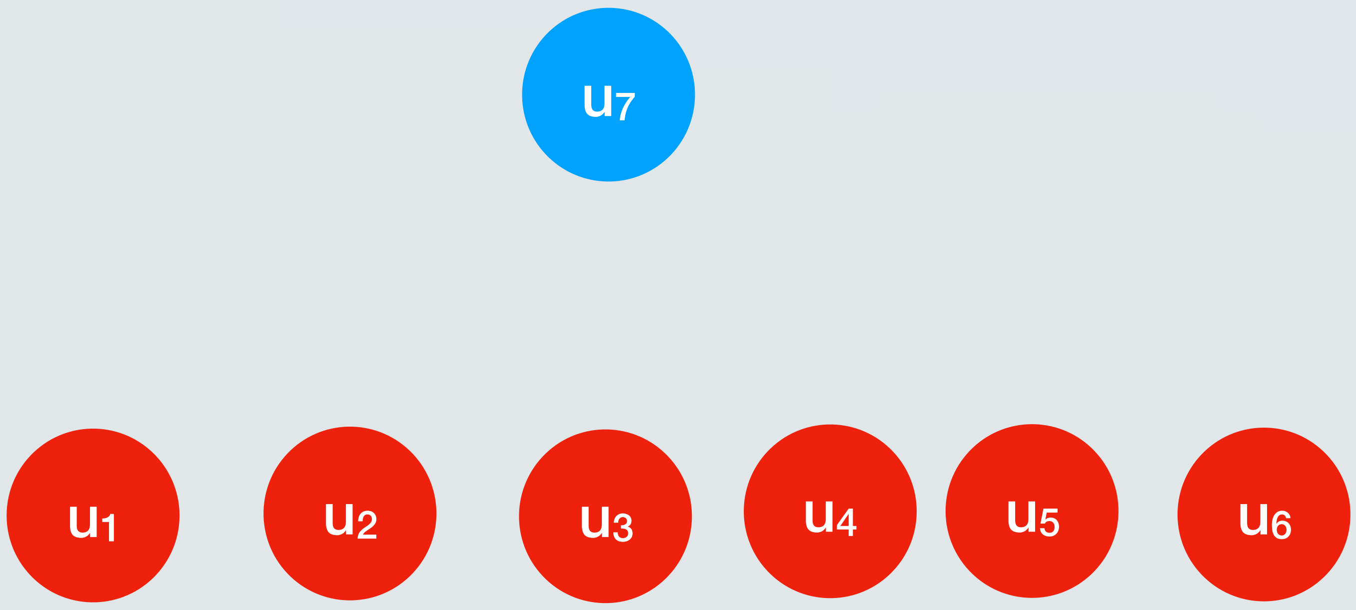

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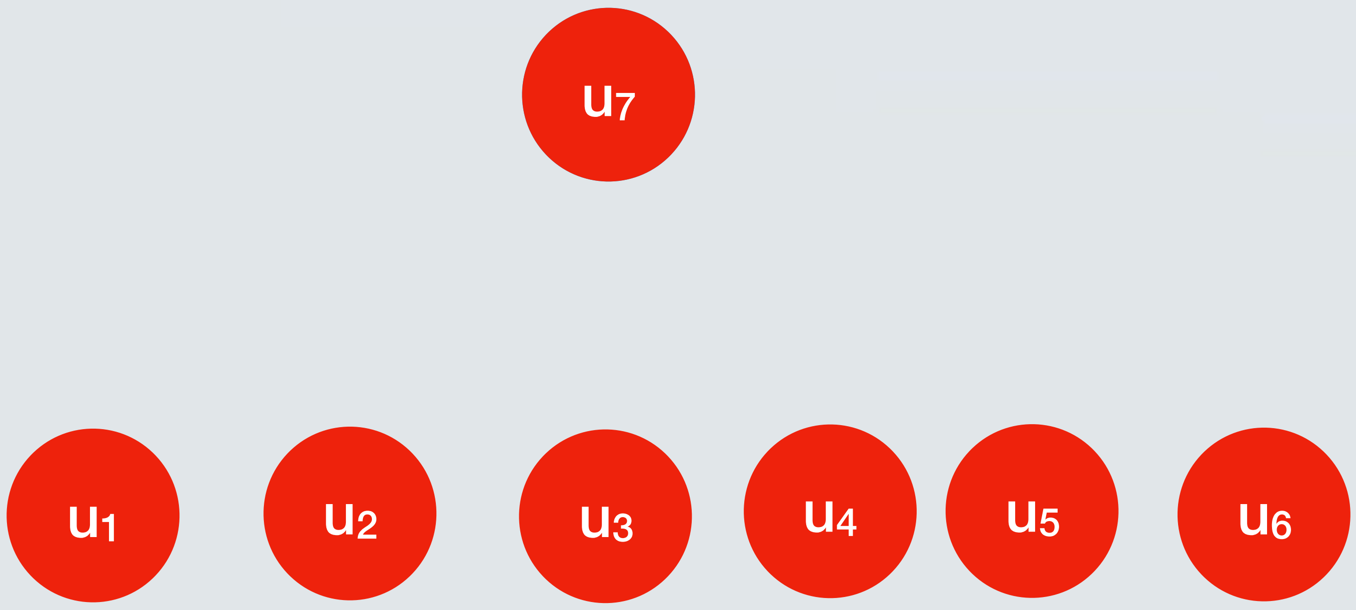

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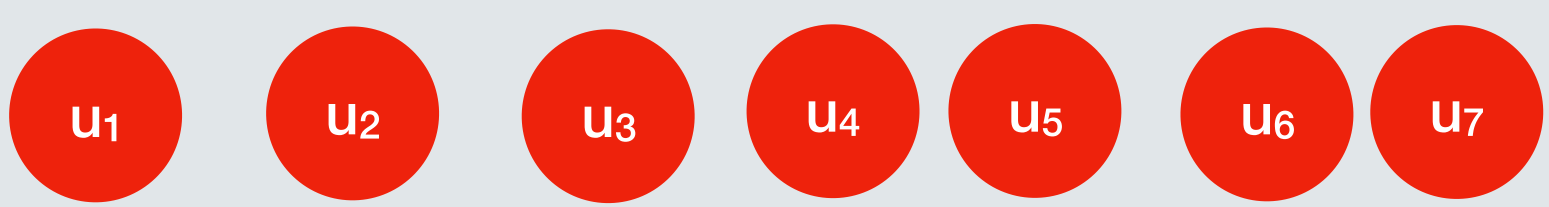

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- We check *n* nodes in the first iteration, *n-1* nodes in the second, and so on…
- What is the running time of this?
    - **$O(n^2)$**
- Can we do better?

A faster algorithm

- We will be more efficient in the choice of sources.
- We will say that a node is active, if it has not been selected (and therefore removed) as a source by the algorithm.
- We maintain two things:
  - (a) For each node w, the number of *incoming edges* from active nodes.
  - (b) The set S of all active nodes that have *no incoming edges* from other active nodes.

A faster algorithm

- In the beginning, all nodes are active and we can initialise (a) and (b) via a pass through the graph (time **O(m+n)**)
- In each iteration:
  - We select a node u from the set S.
  - We delete u.
  - We go through all the neighbours w of u and we reduce their value in (a) (i.e., number of incoming edges from active nodes) by 1.
  - When the value of (a) for some node w goes to 0, w is added to the set S.

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- Perform a DFS on G, starting from an arbitrary nodes s.
- Add the nodes that the DFS tree reaches to a stack.
  - A node is added to the stack when the DFS for that node is completed.
- Perform a DFS on G^{rev}, visiting the nodes in the order that they are popped from the stack.
- Output the DFS trees of the second DFS as the strongly connected components.

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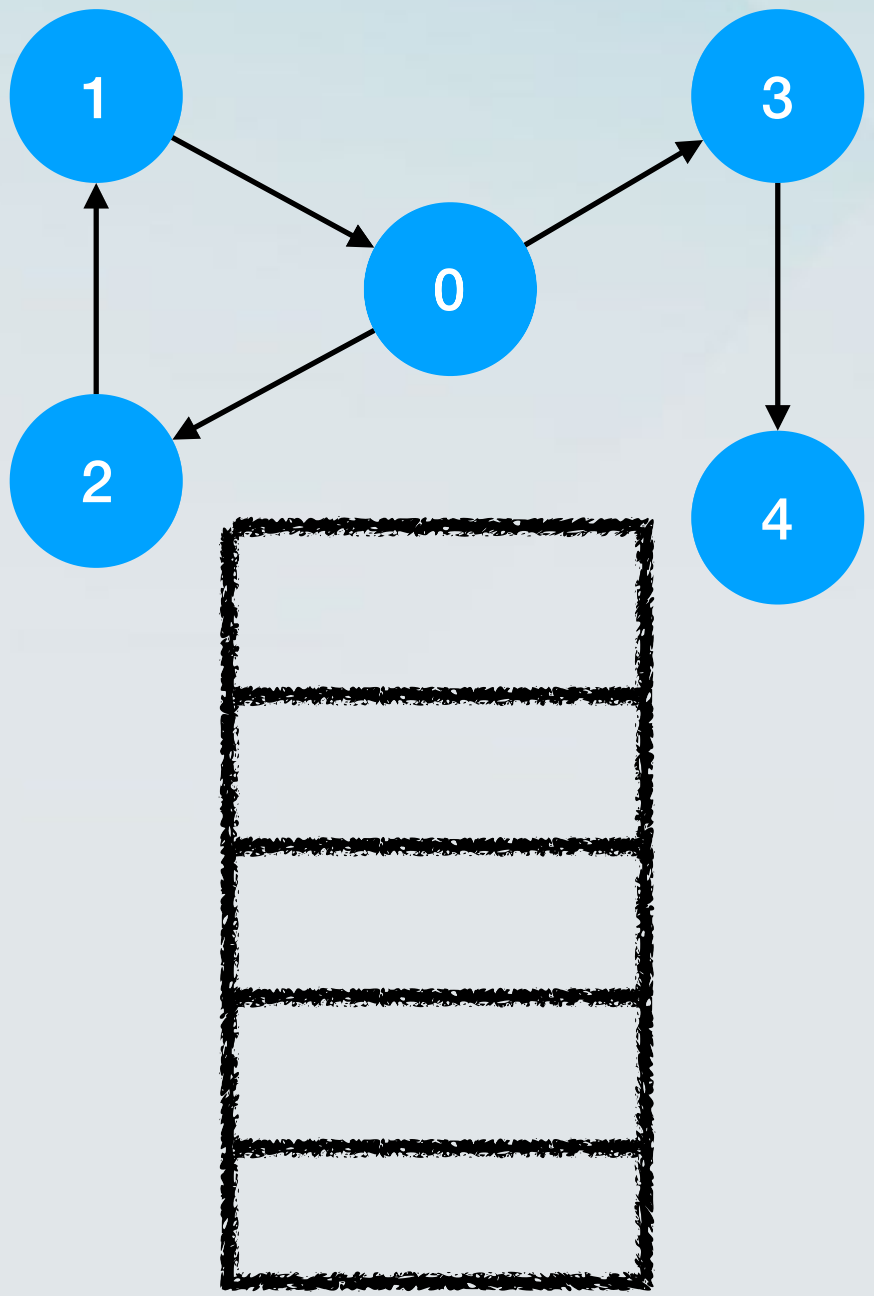

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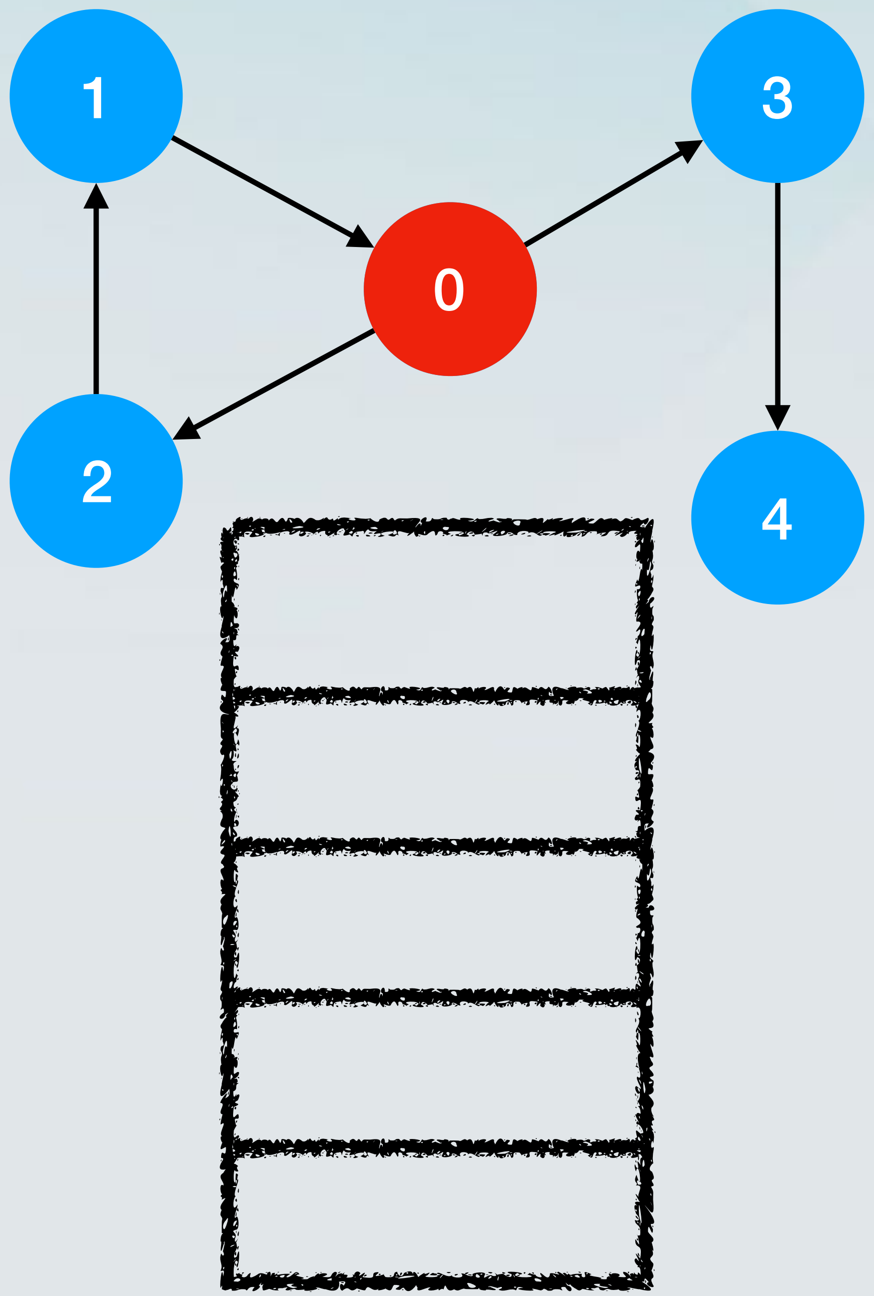

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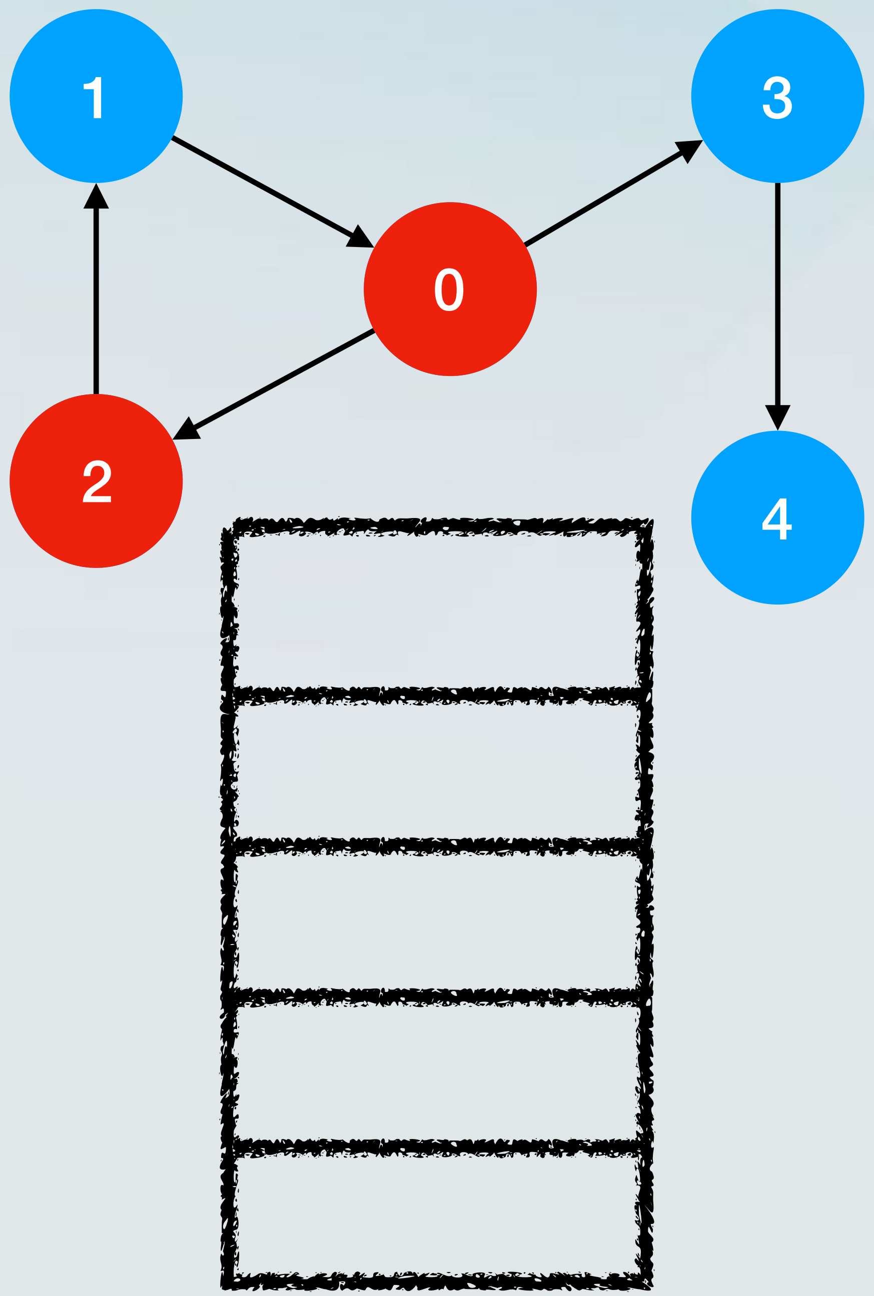

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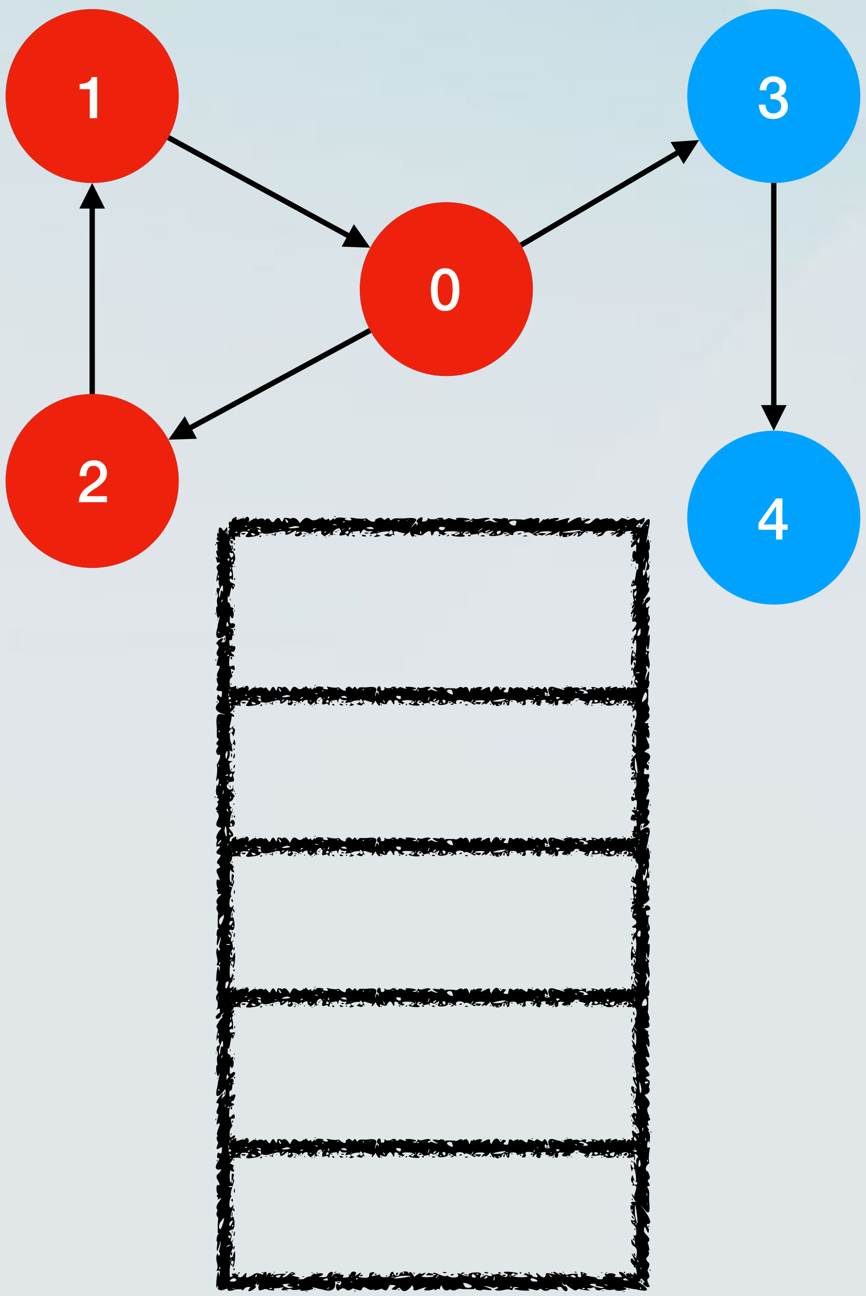

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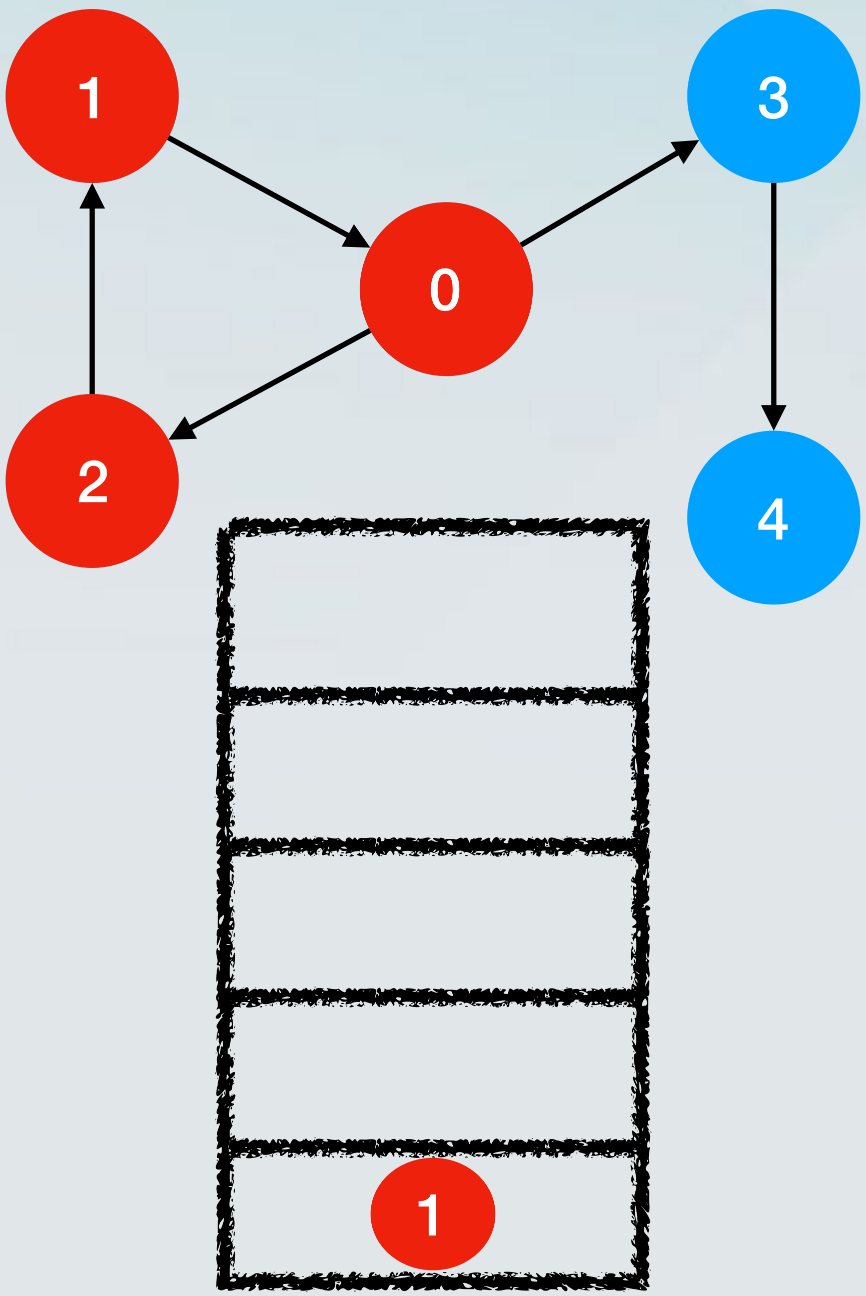

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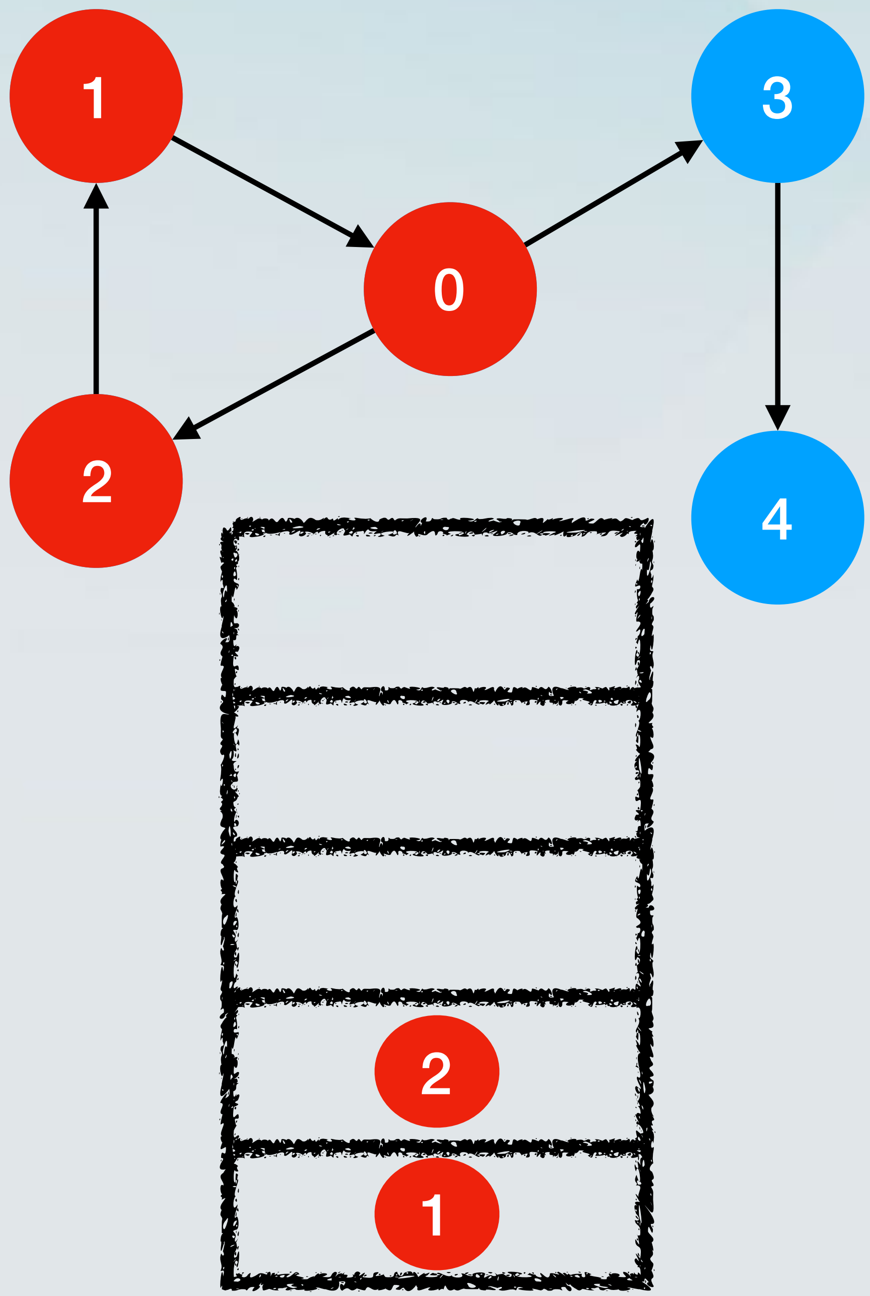

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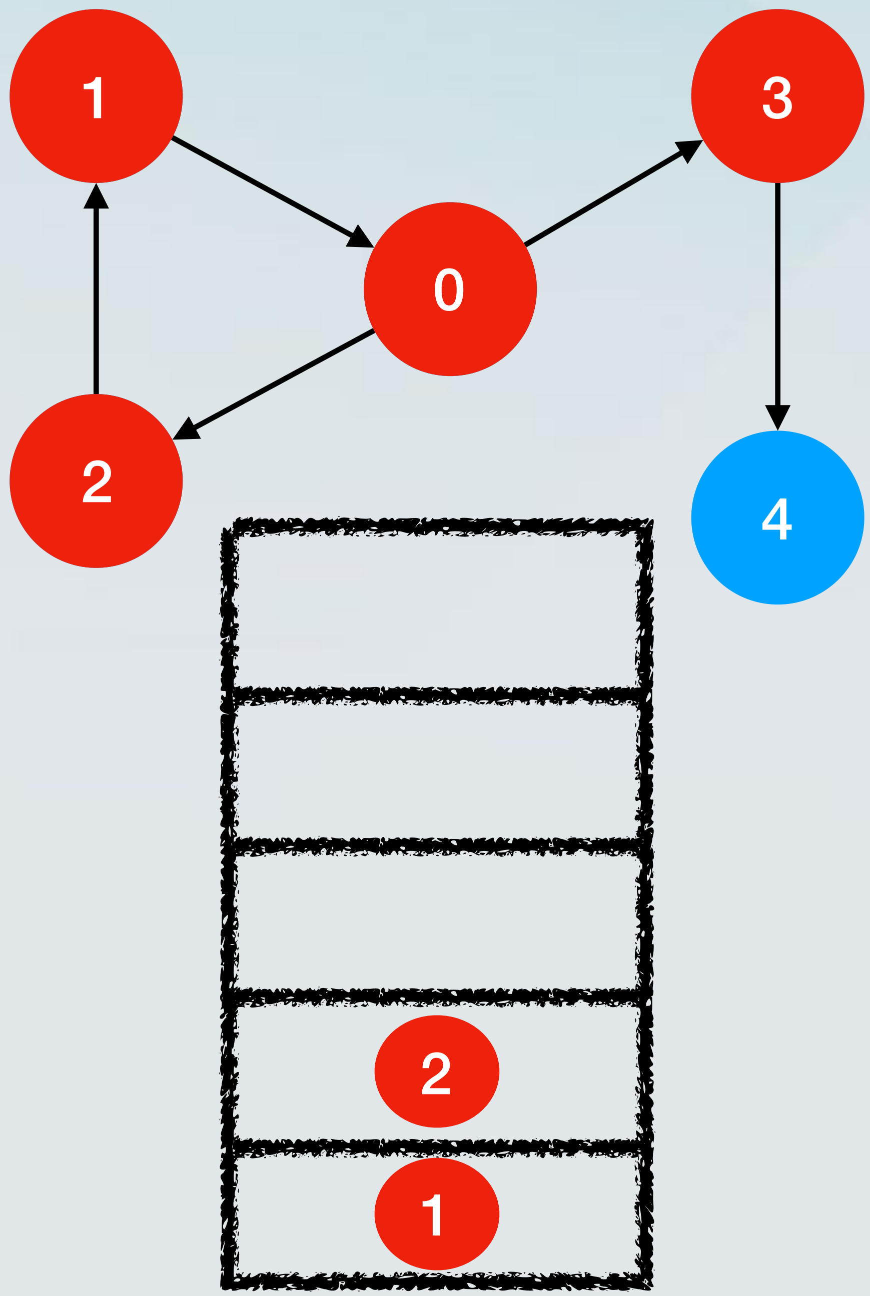

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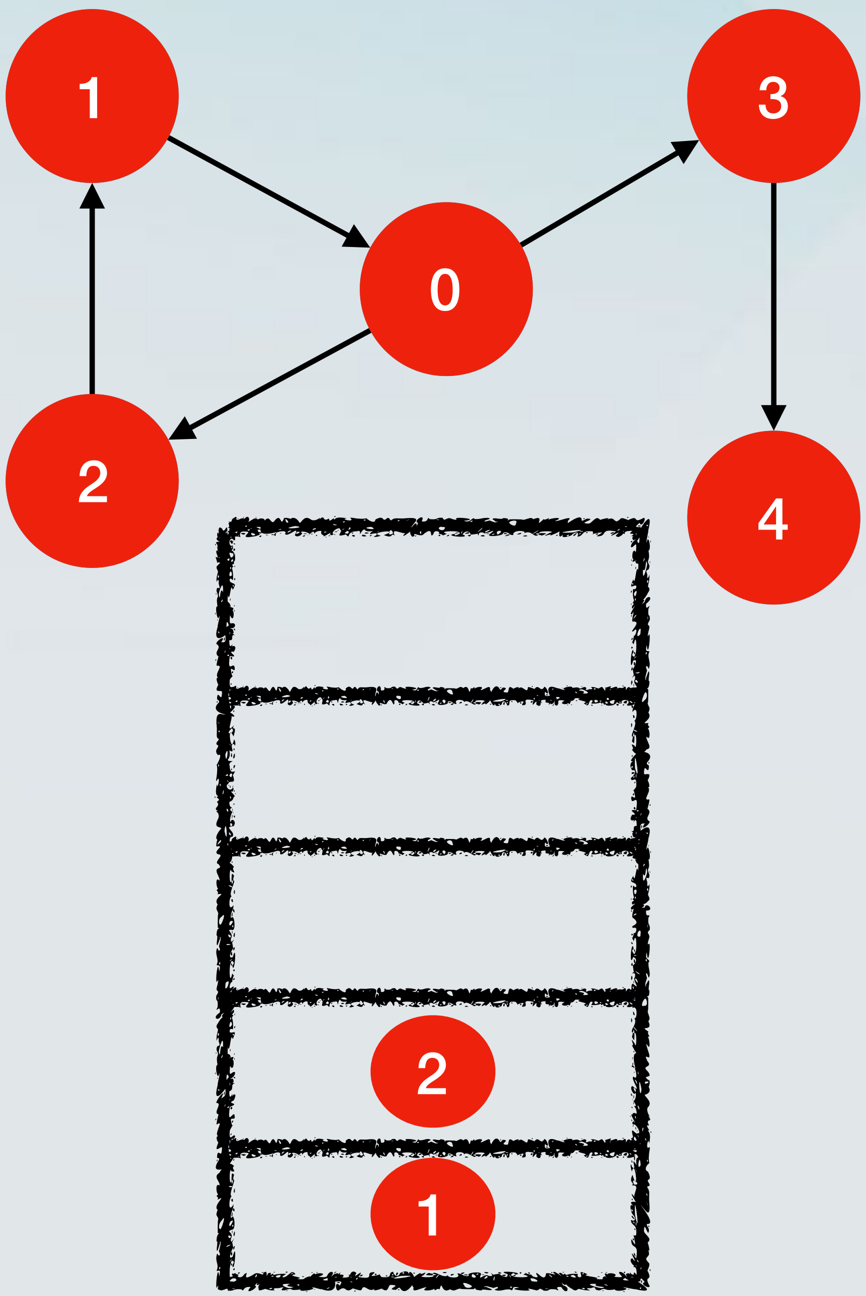

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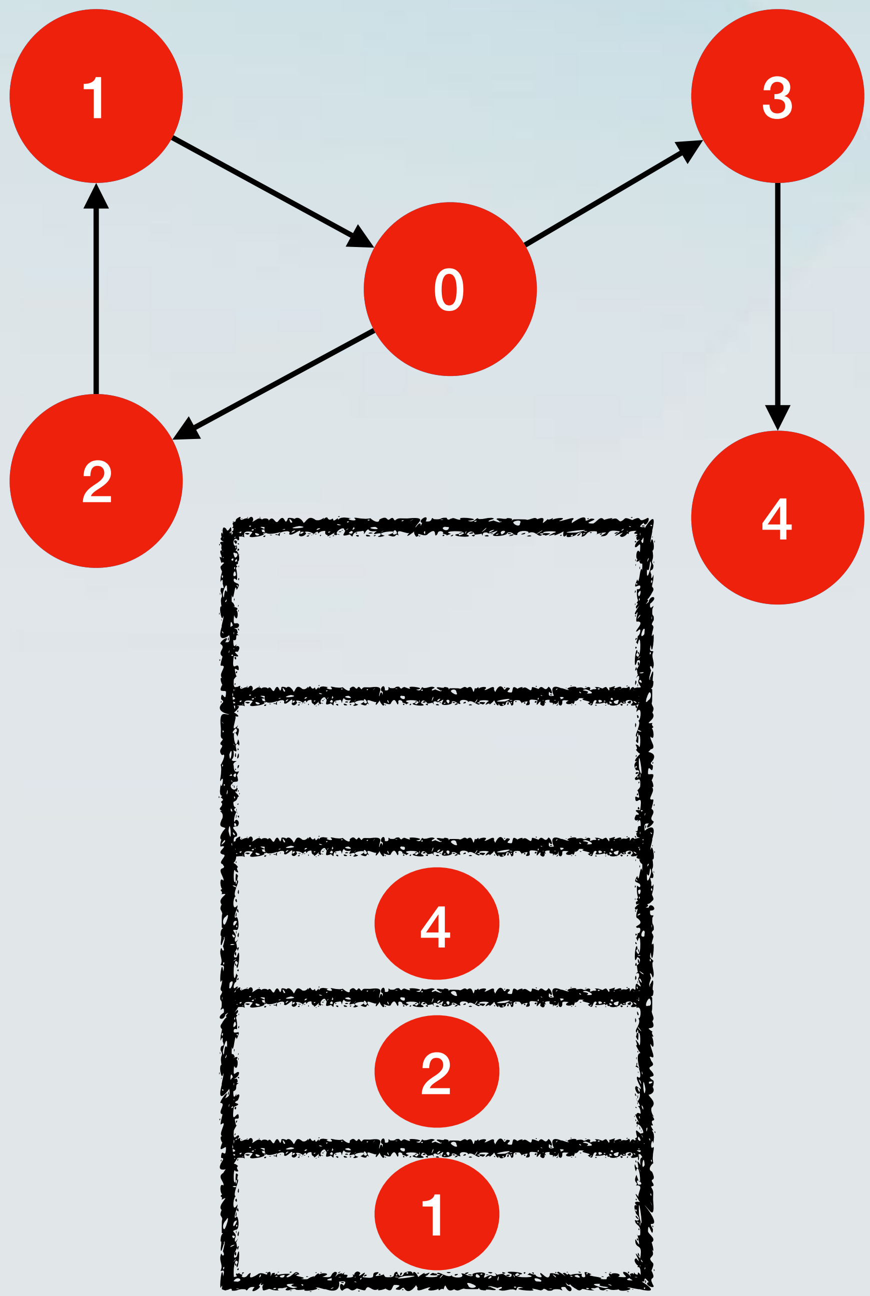

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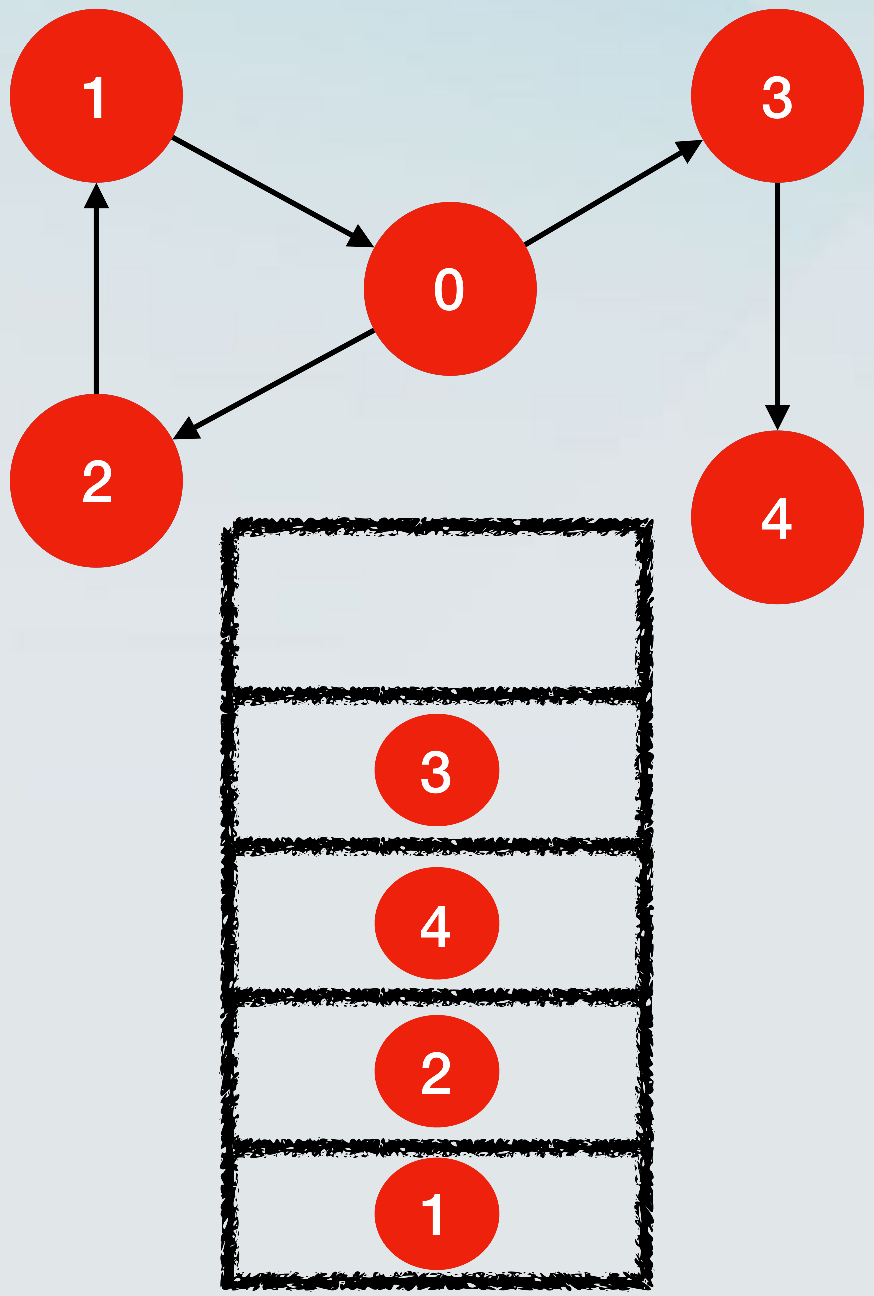

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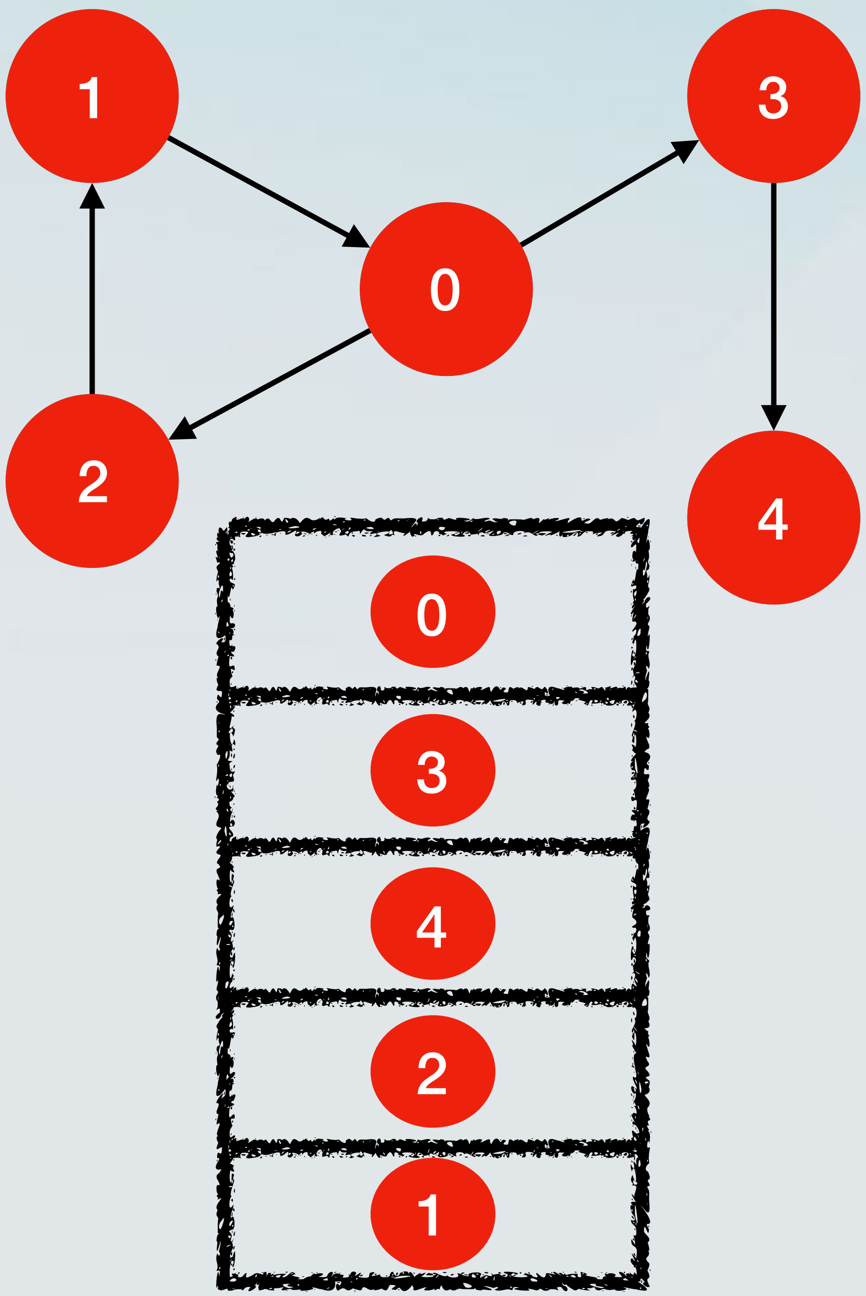

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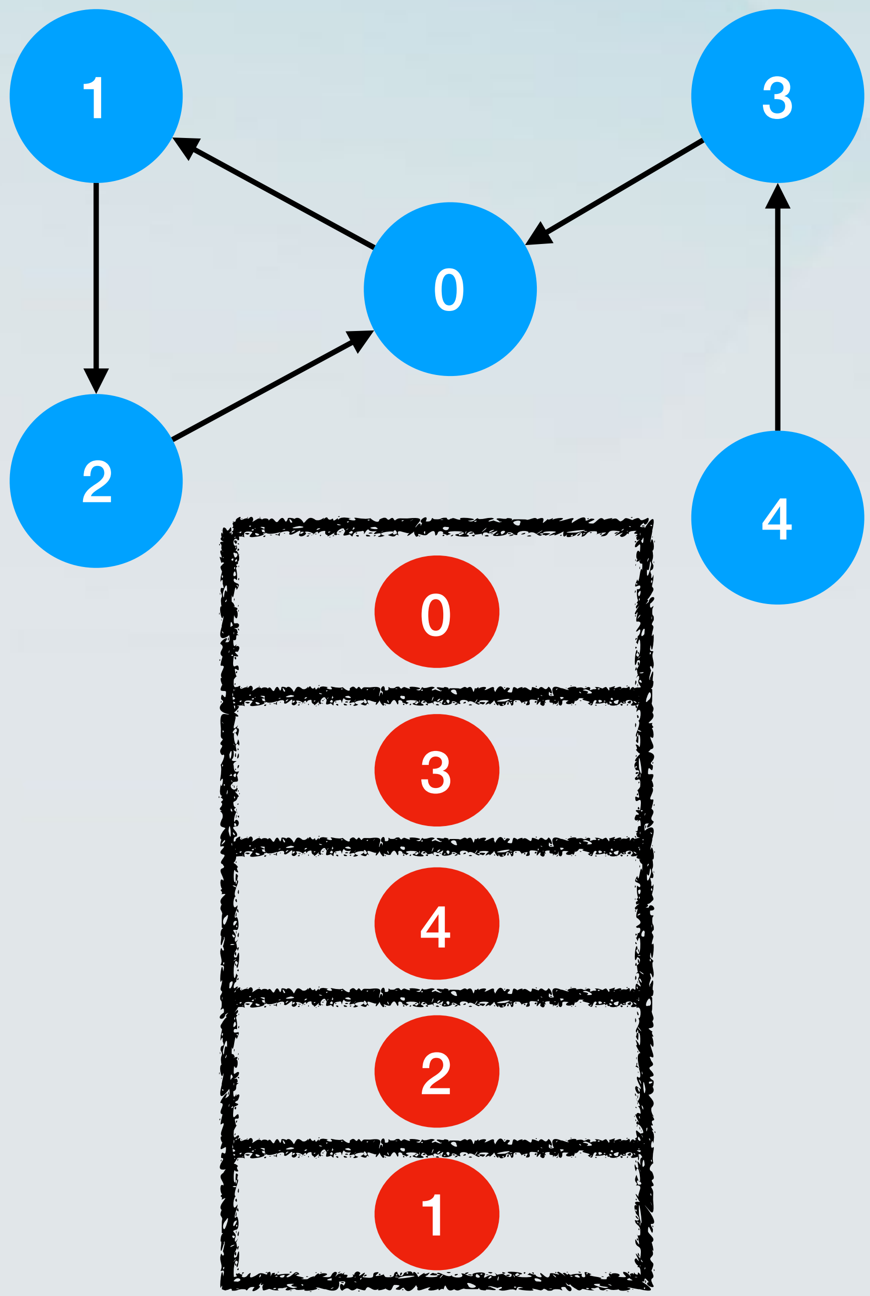

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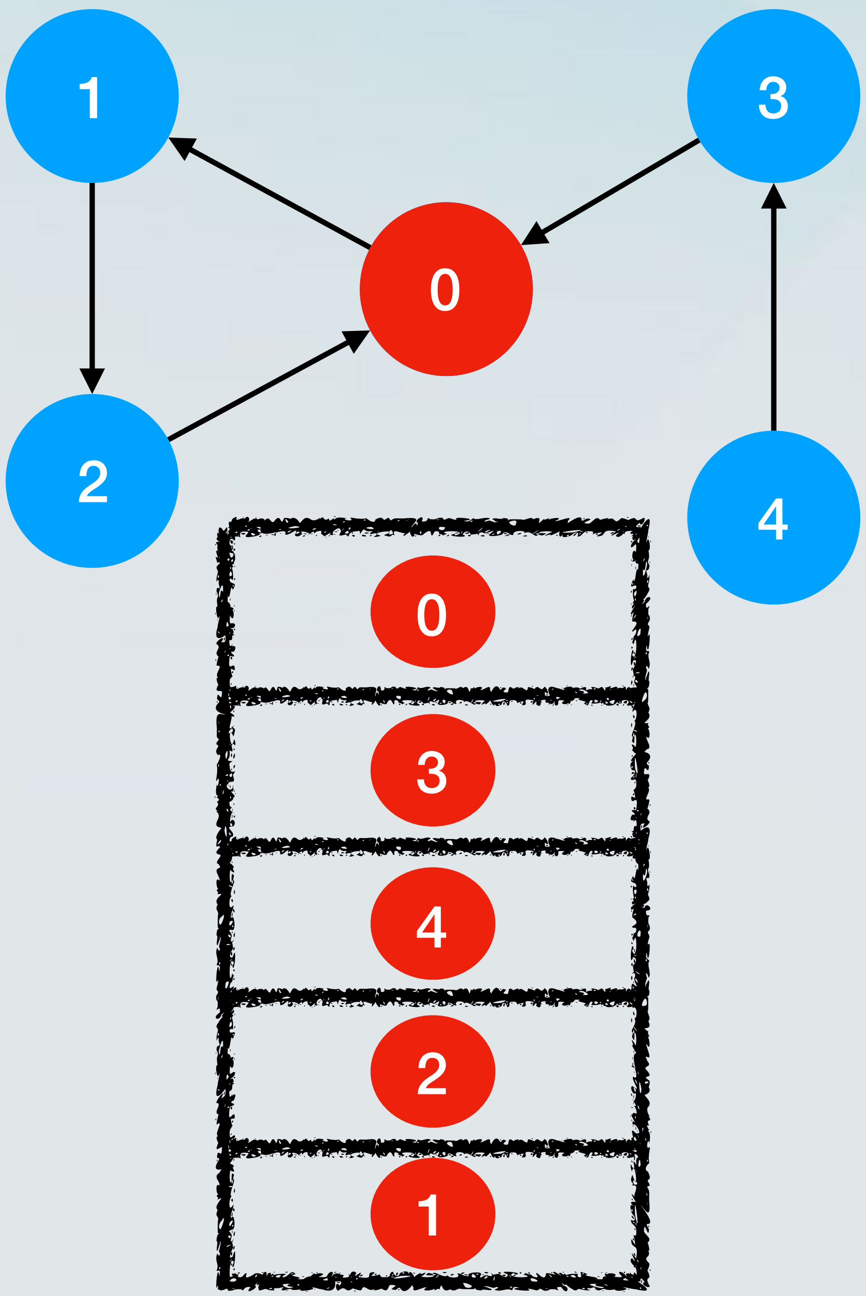

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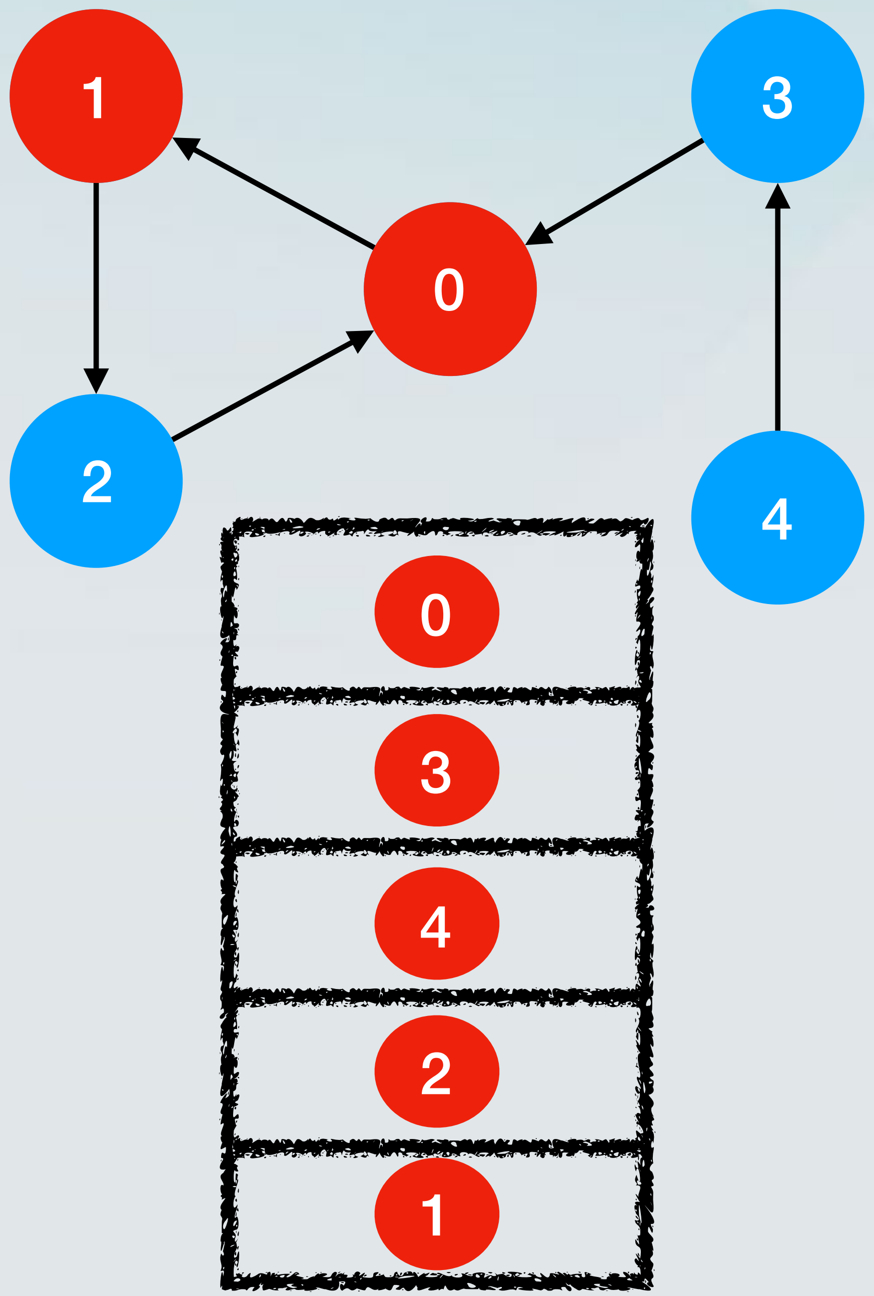

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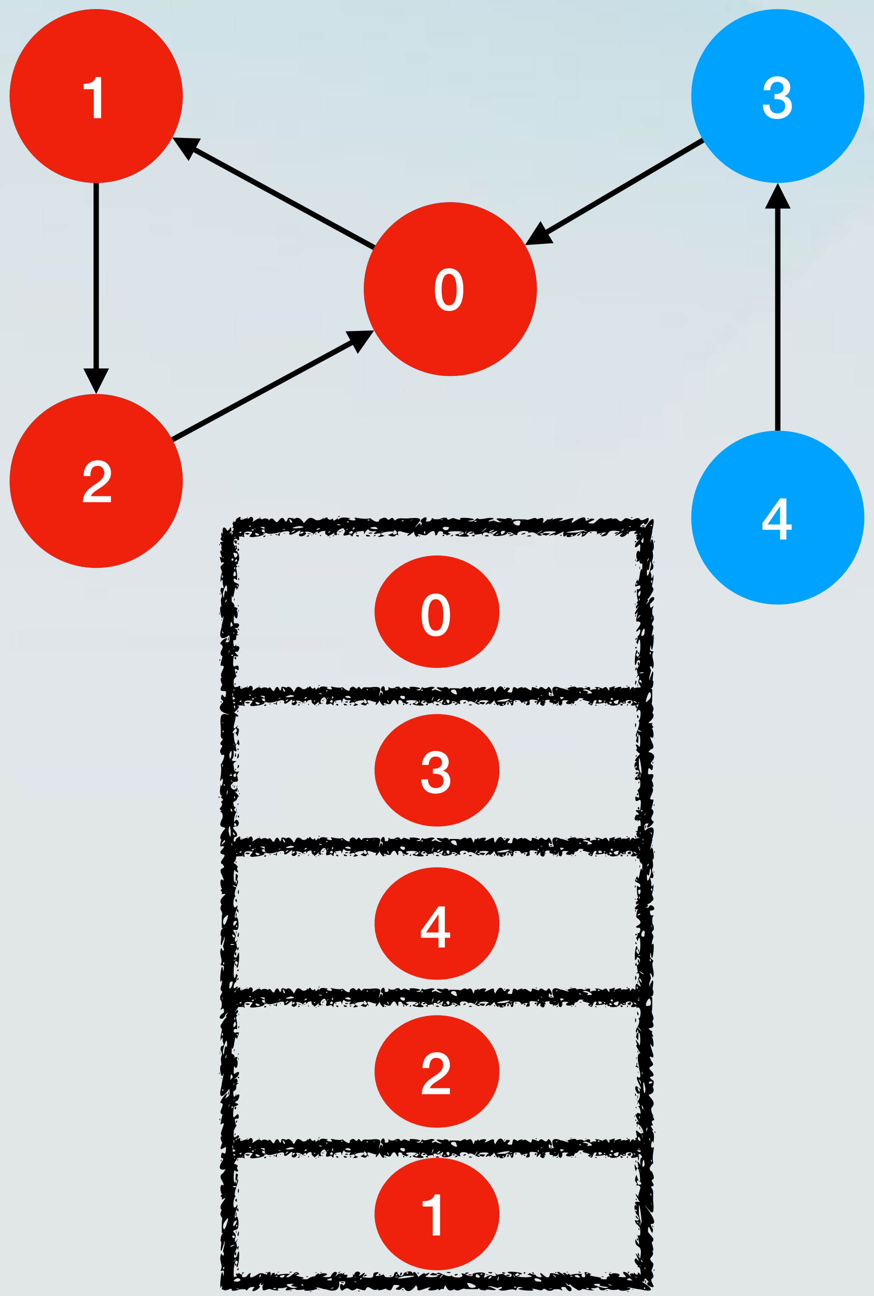

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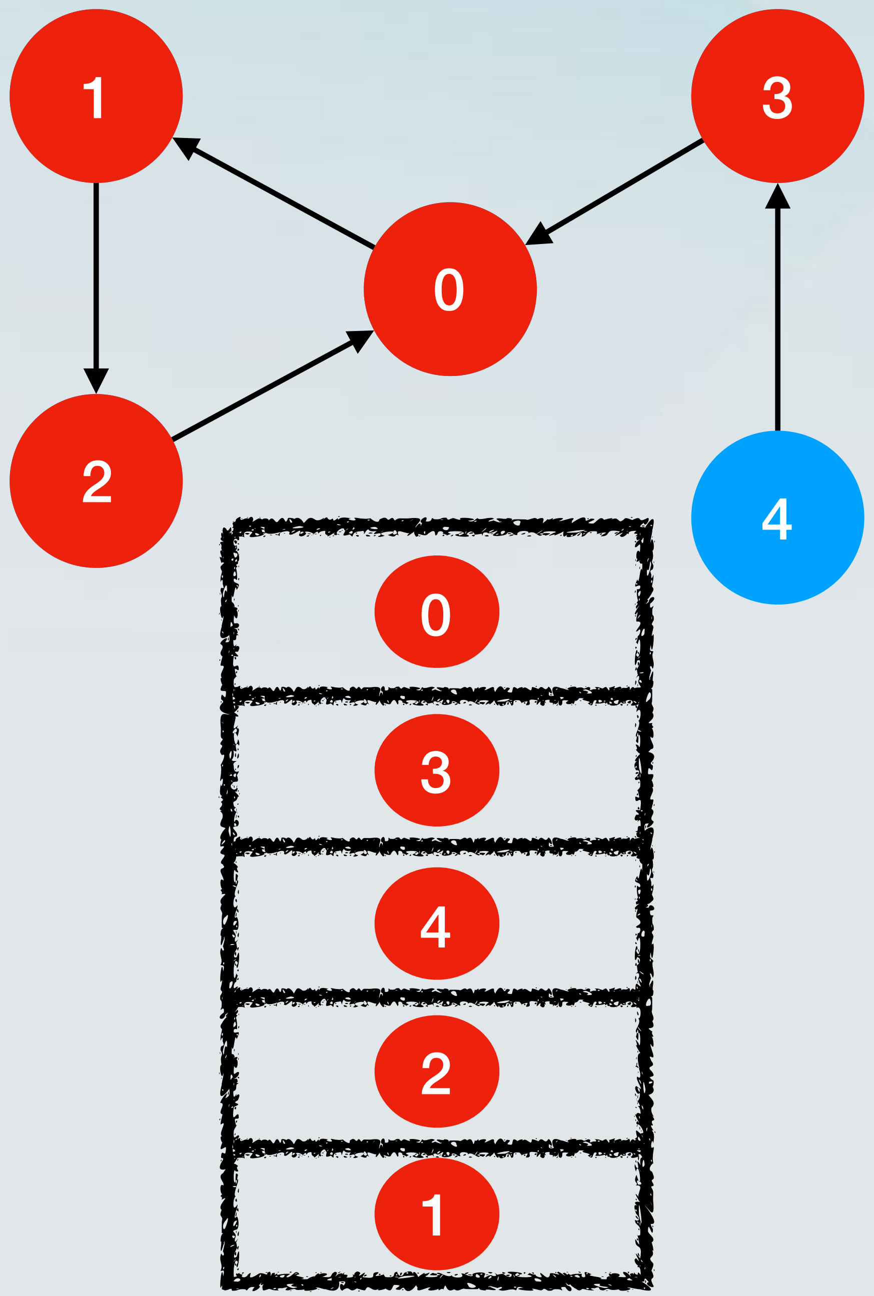

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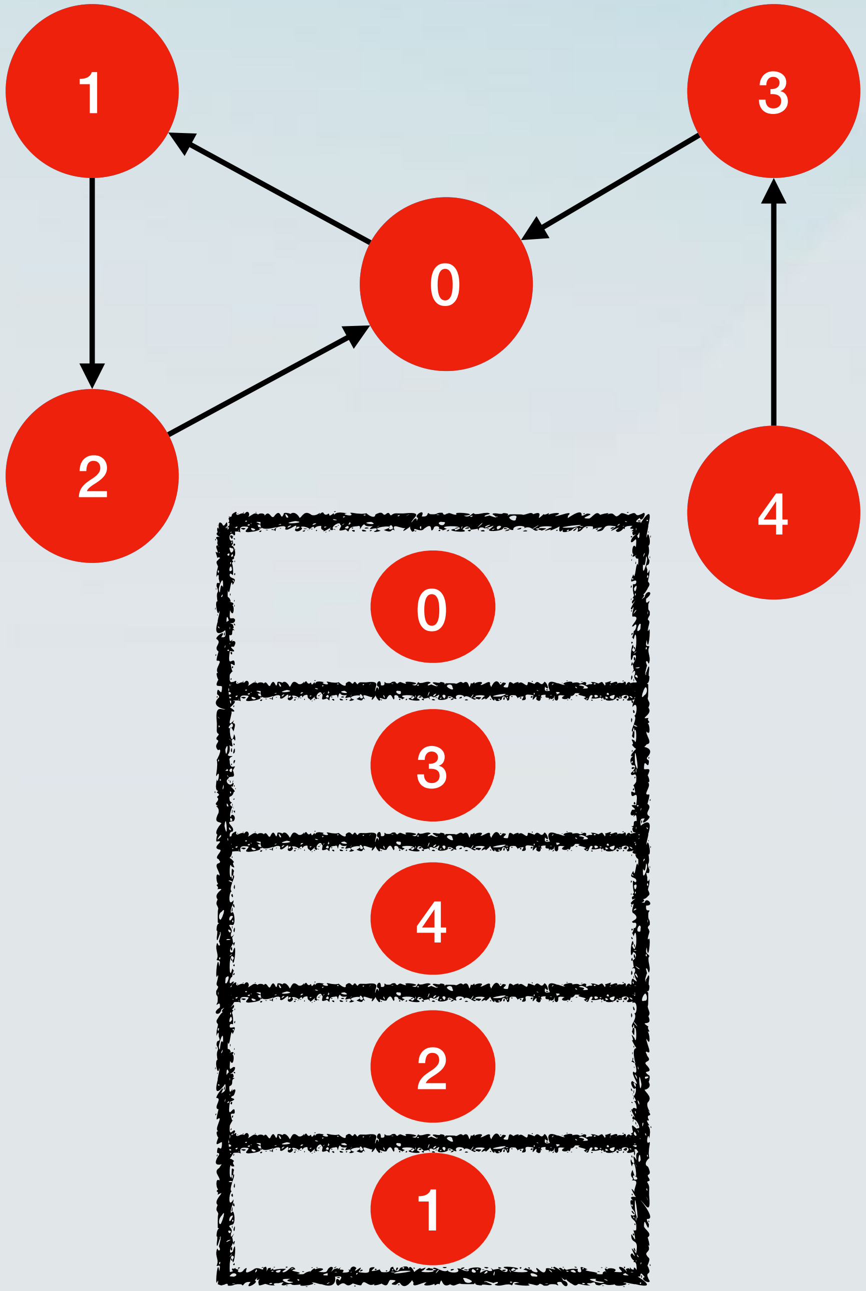

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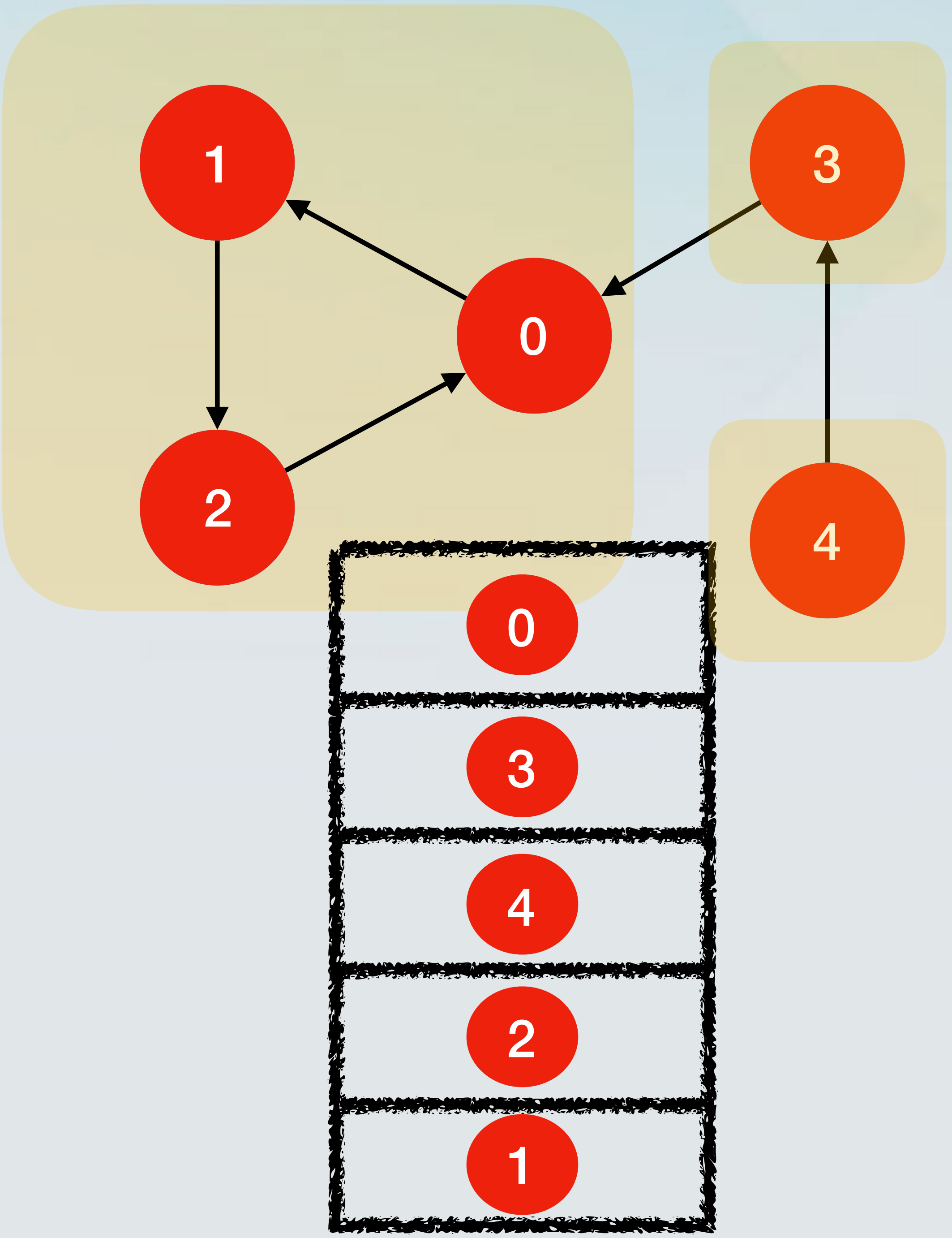


Running time

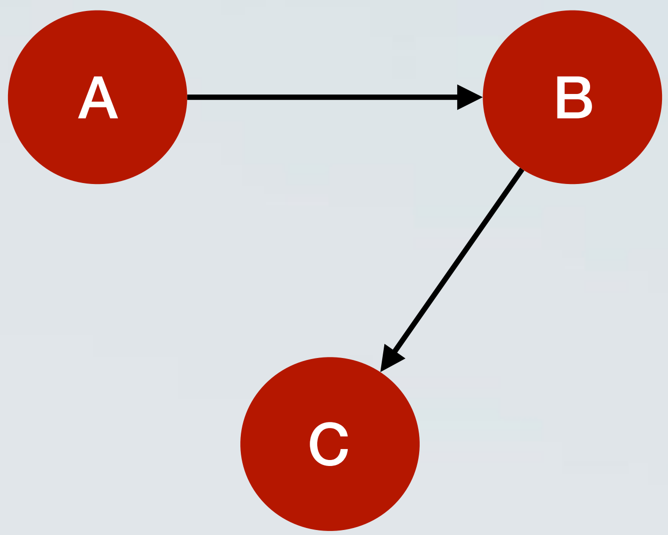
Running time

- We perform DFS twice.
- The running time is **O(m+n)**.

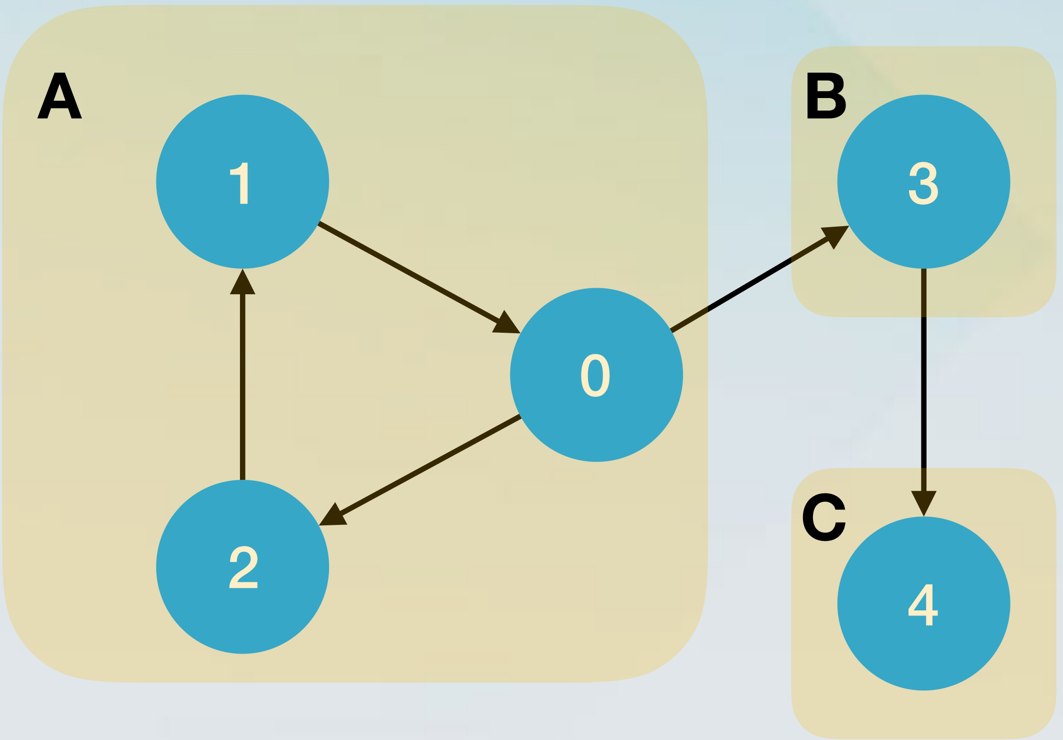
Correctness

- Define a *meta-graph* of the graph G, called the component graph $G^{SCC} = (V^{SCC}, E^{SCC})$
- Suppose that G has strongly connected components (SCCs) $C_1, C_2, \ldots, C_k$, for some k.
- $V^{SCC} = \{v_1, v_2, \ldots, v_k\}$ contains a vertex for each SCC of G.
- There is an edge (v_i, v_j) in E^{SCC} if G contains a directed edge (x,y) for some x in C_i and some y in C_j (i.e., an edge *crossing* two different components.

Example



Component Graph G^{SCC}



Graph G

Simple but key lemma

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- Let C and C' be distinct SCCs in a directed graph G.
  Let u, v in C
  Let u',v' in C'
  Suppose that G contains a path from u to u'. (1)
  Then G ***cannot*** contain a path from v' to v.

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  - There is a path from v to u (same SCC).
  - There is path from v’ to u (because of (2)).
  - This means that u and v’ are mutually reachable, hence in the same SCC.

Component graph

Component graph

- What does the simple-but-key lemma imply for the component graph?

Component graph

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- For two distinct connected components, there can be a path from the first to the second, or vice-versa, but not both!

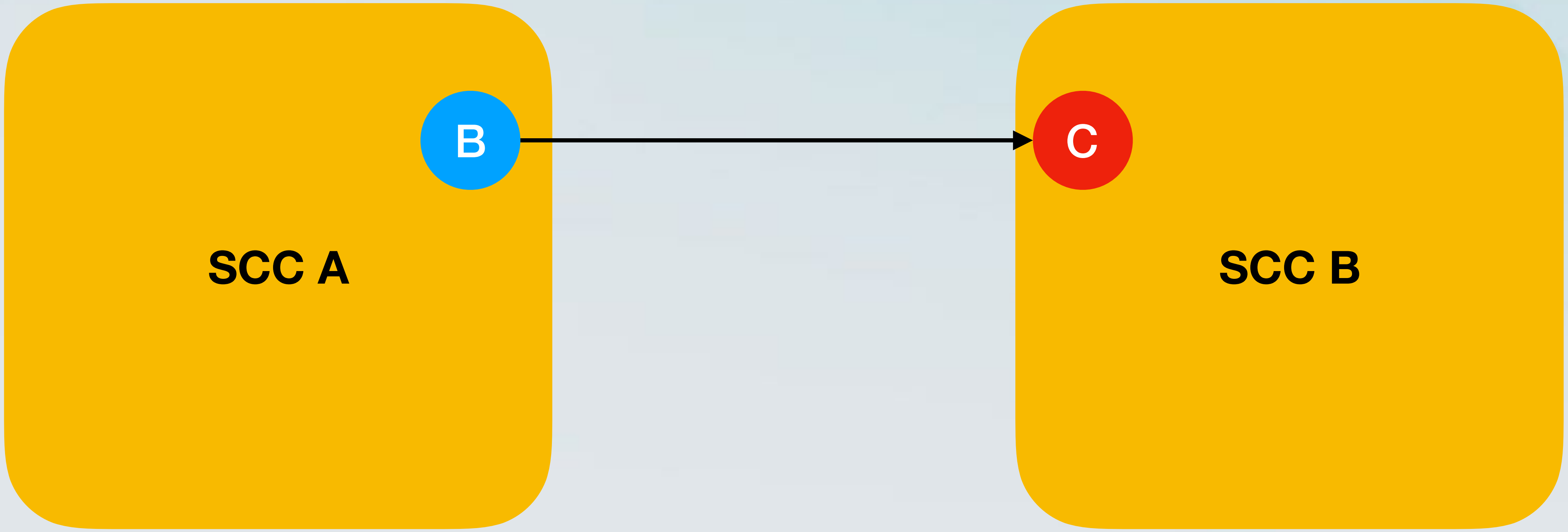
Component graph

- What does the simple-but-key lemma imply for the component graph?
- For two distinct connected components, there can be a path from the first to the second, or vice-versa, but not both!
- **The component graph is a DAG.**

Lemma

- Let C and C’ be distinct SCCs in G. Suppose there is a directed edge ***crossing*** C and C’. Then the DFS on the nodes of C finishes later than the DFS on the nodes of C’.

Proof by picture



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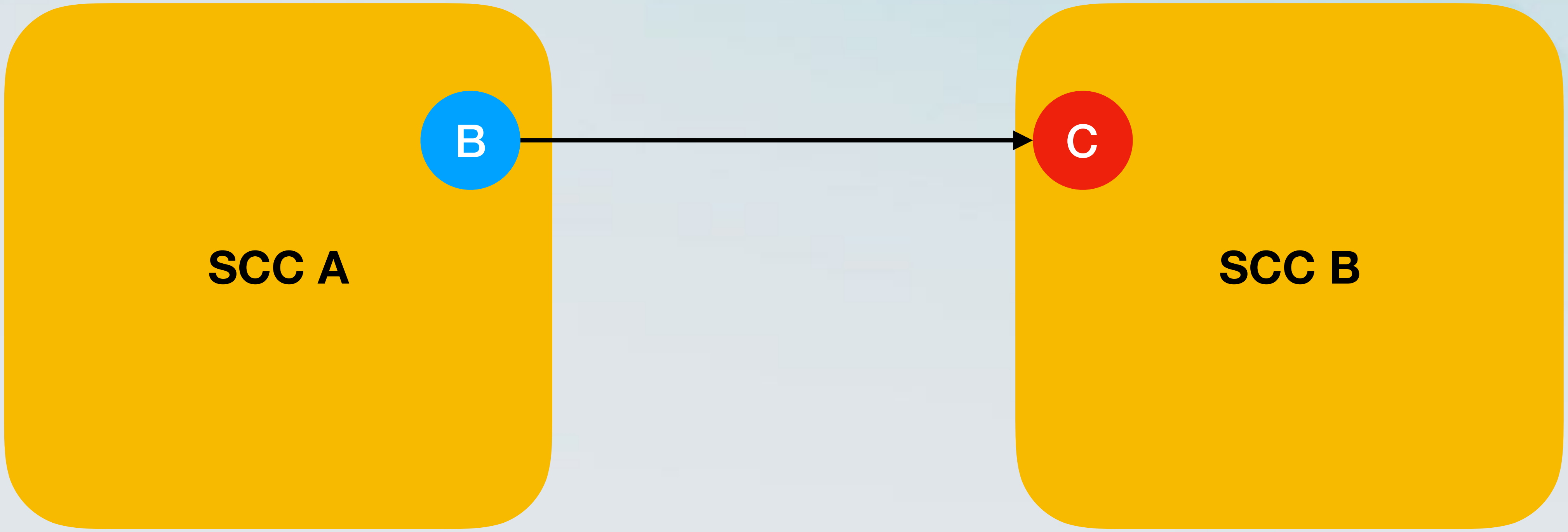
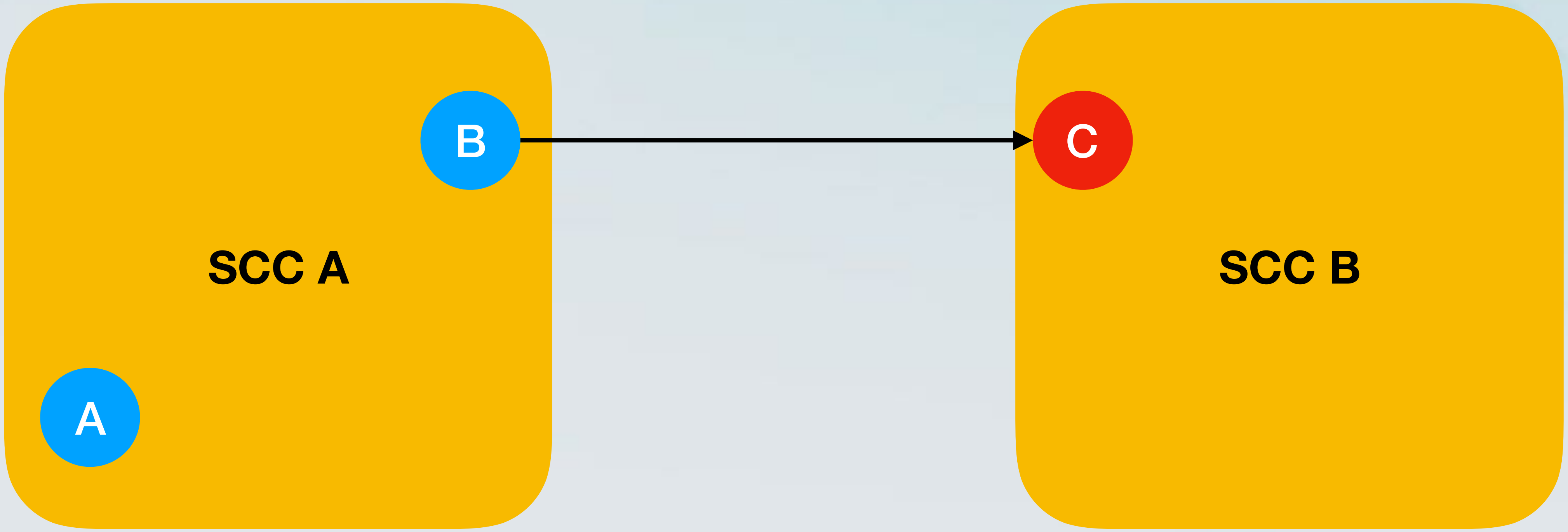


Case 1: We start here

Proof by picture



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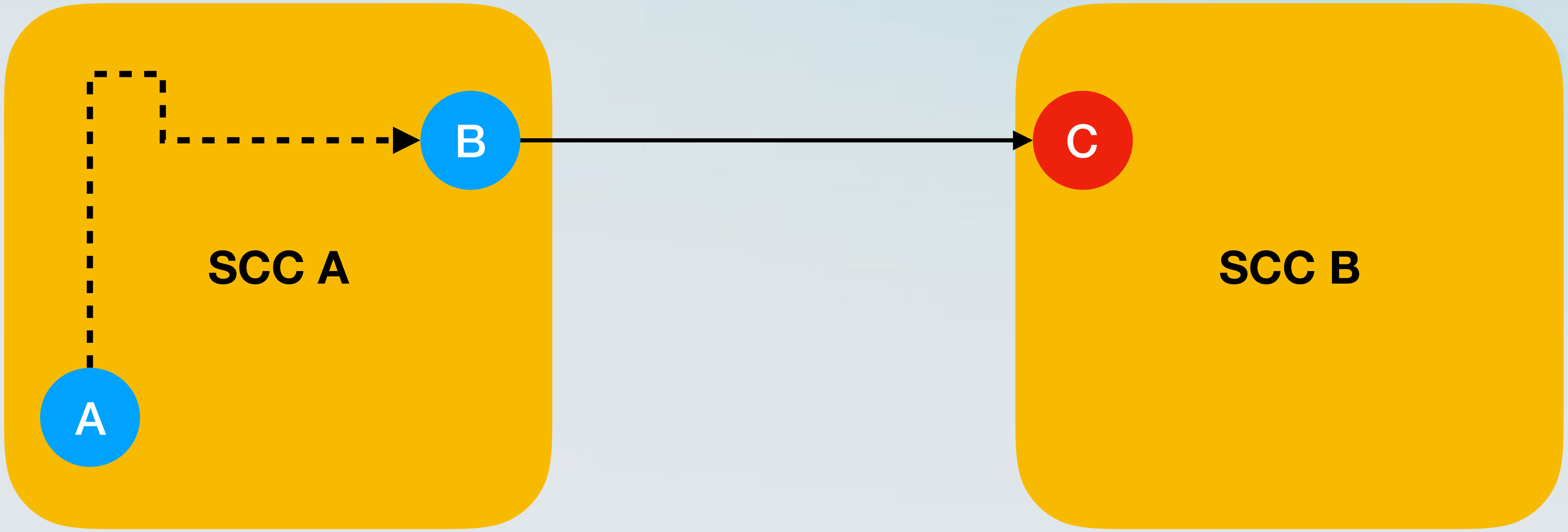

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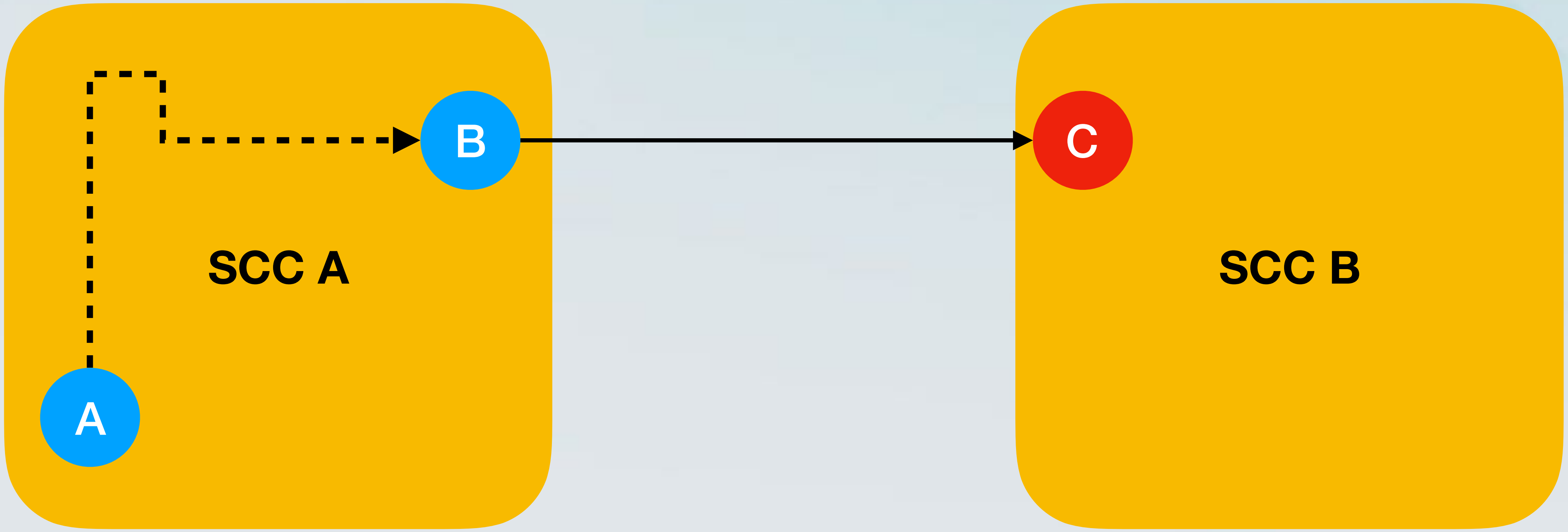

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There is a path A-B
because of same SCC

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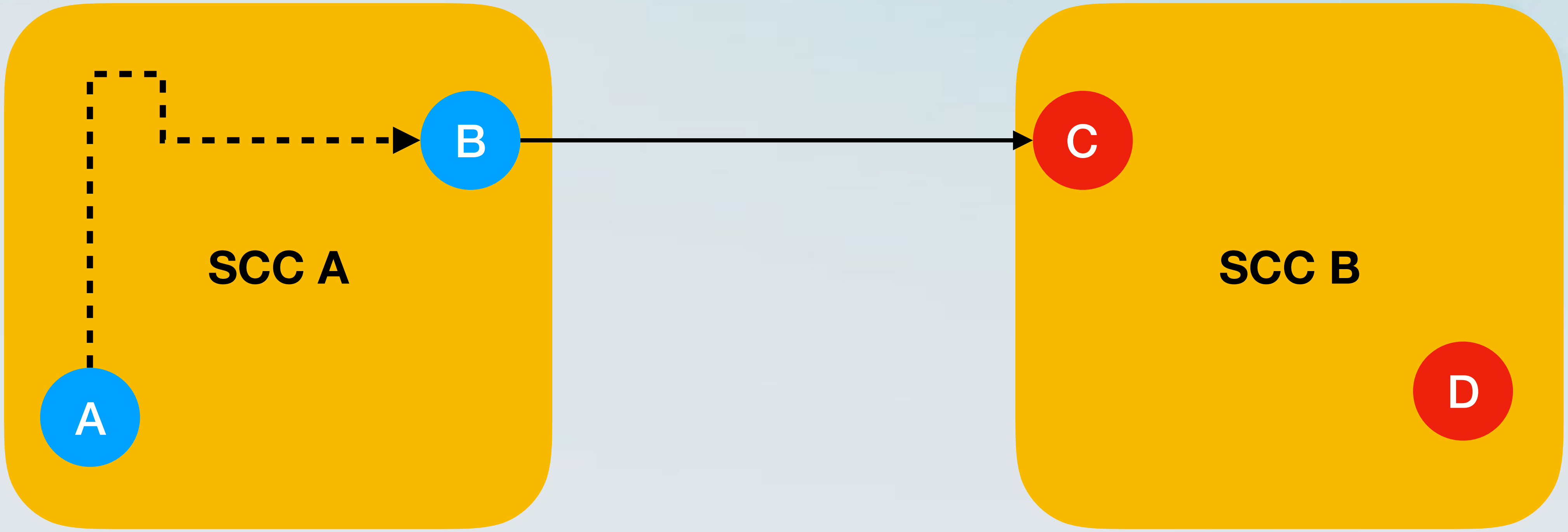

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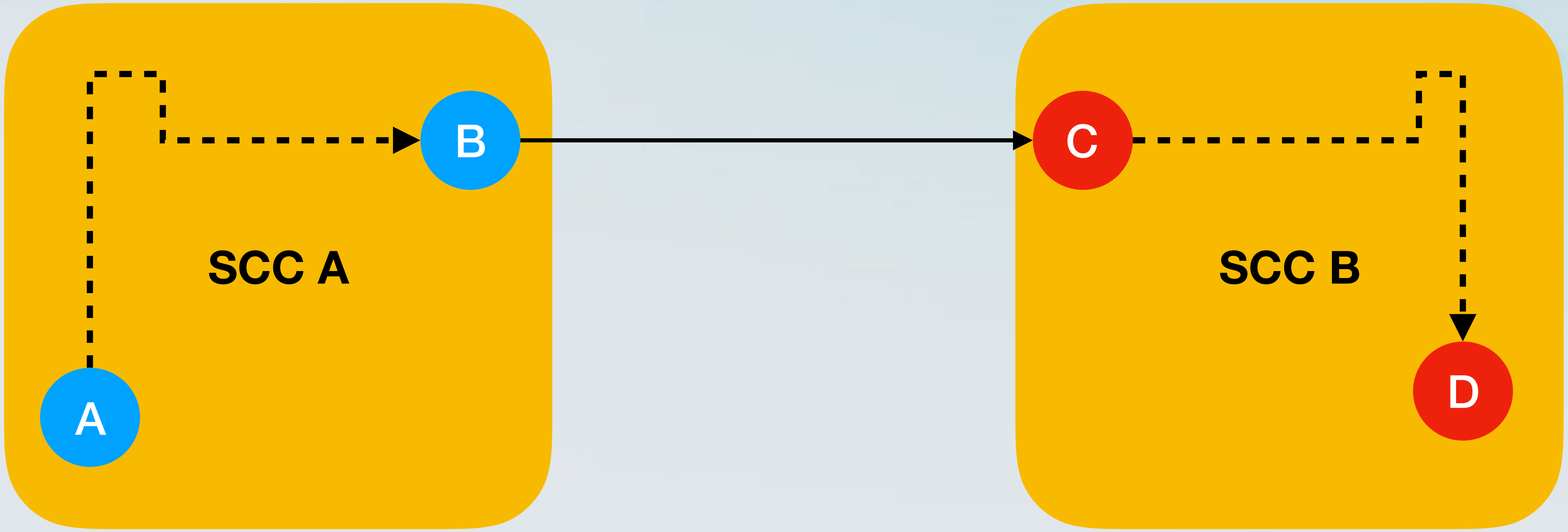

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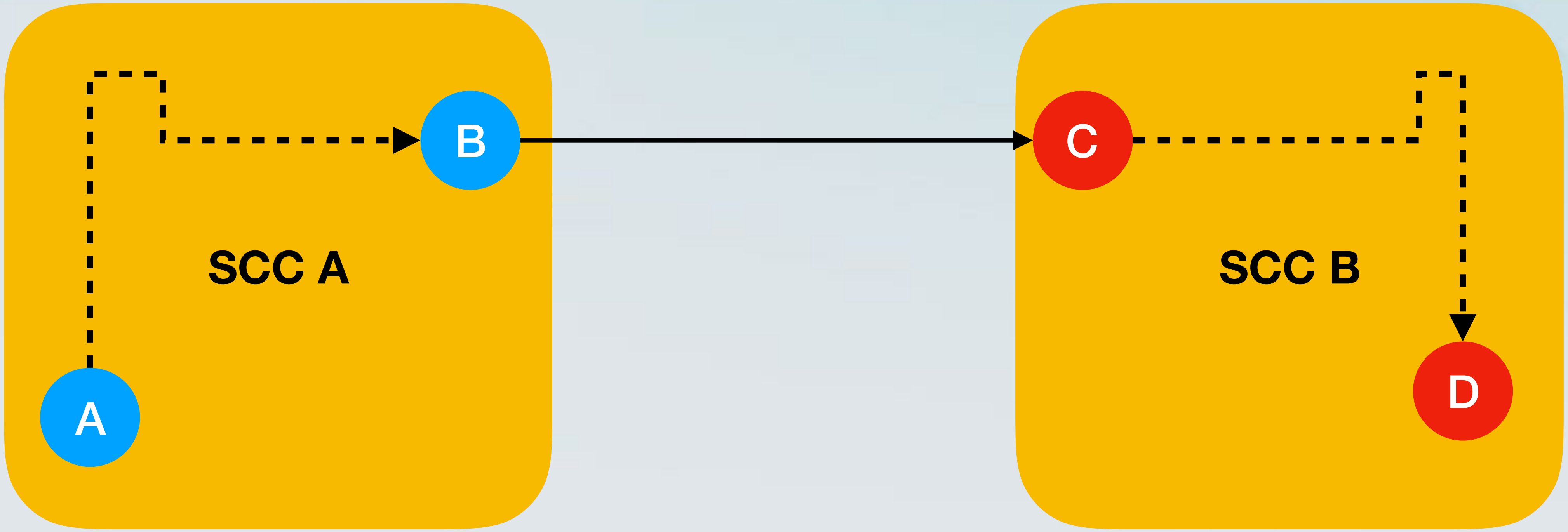

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There is a path A-B
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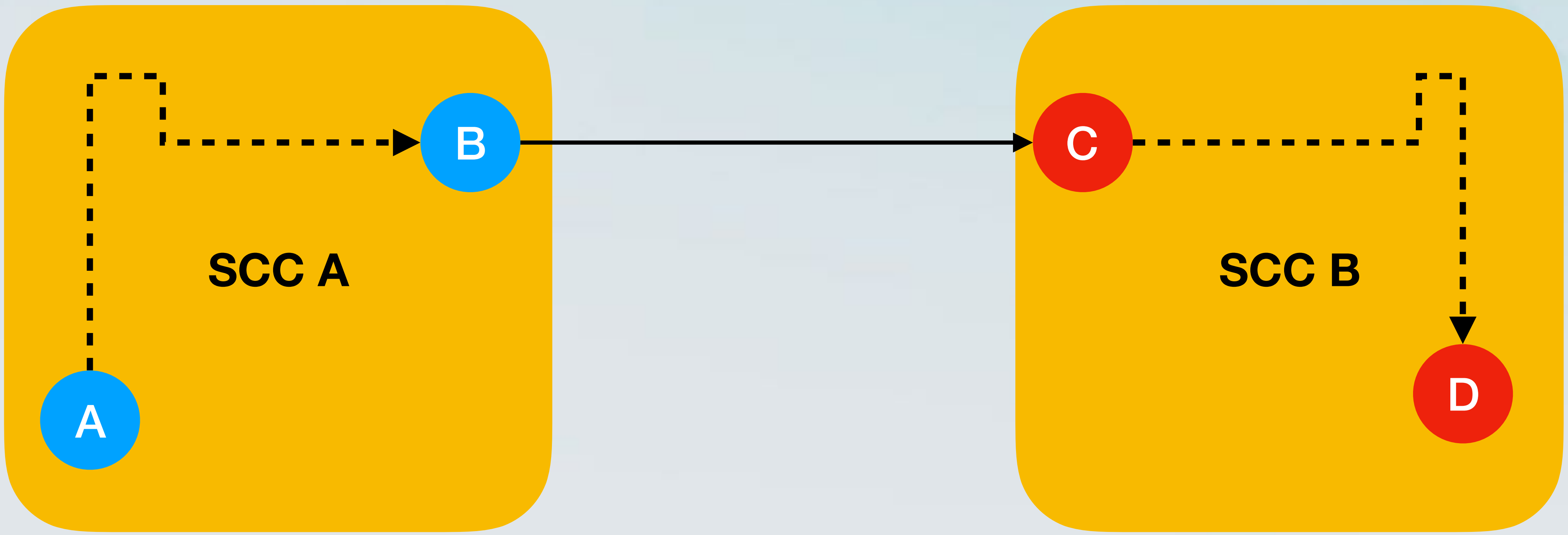
There is a path C-D
because of same SCC

Proof by picture



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There is a path C-D
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There is path A-D

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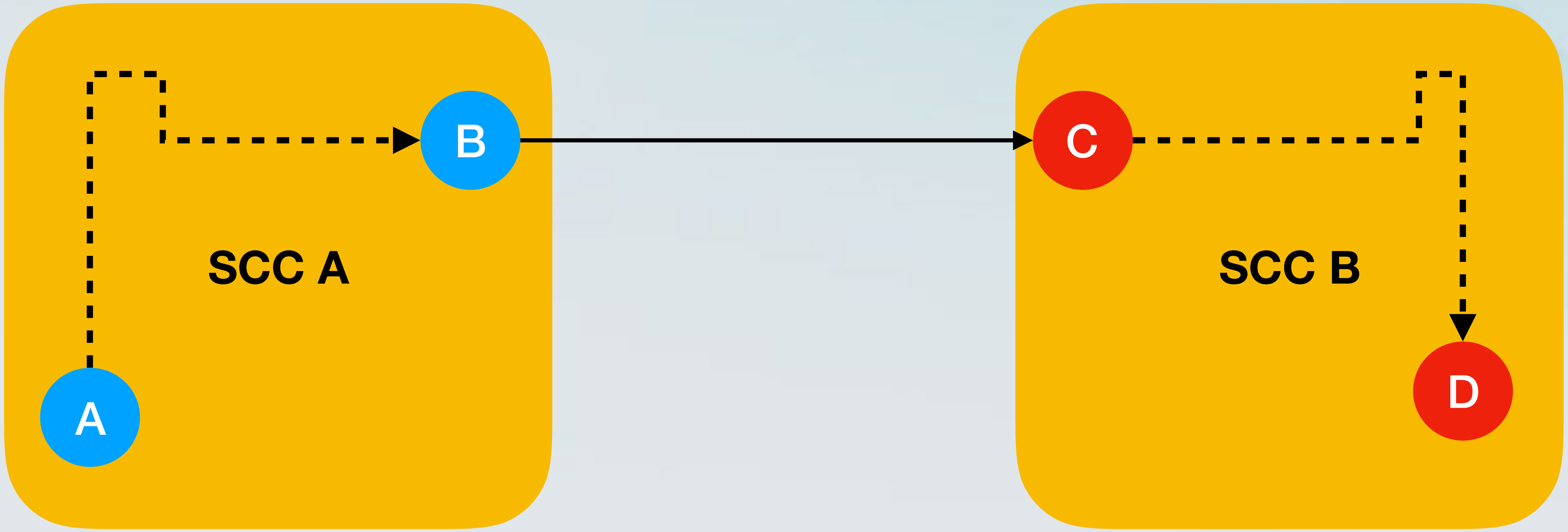

Case 1: We start here

There is a path A-B
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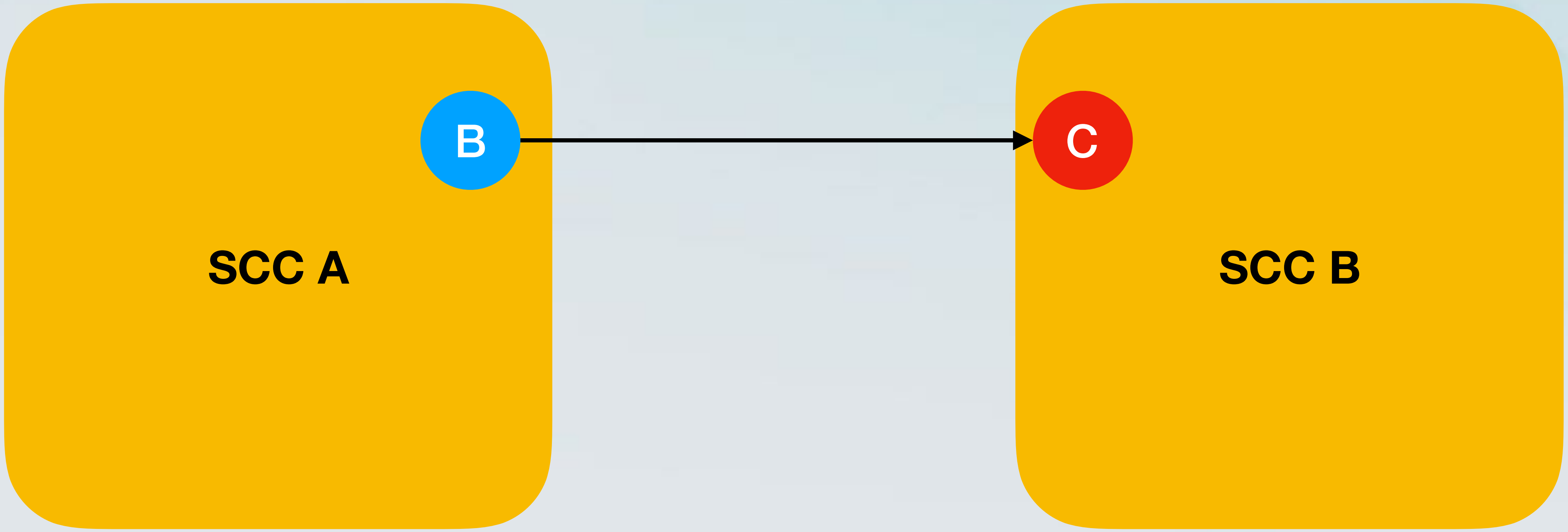
There is a path C-D
because of same SCC

There is path A-D

DFS will explore this
path and A will finish last.

Proof by picture



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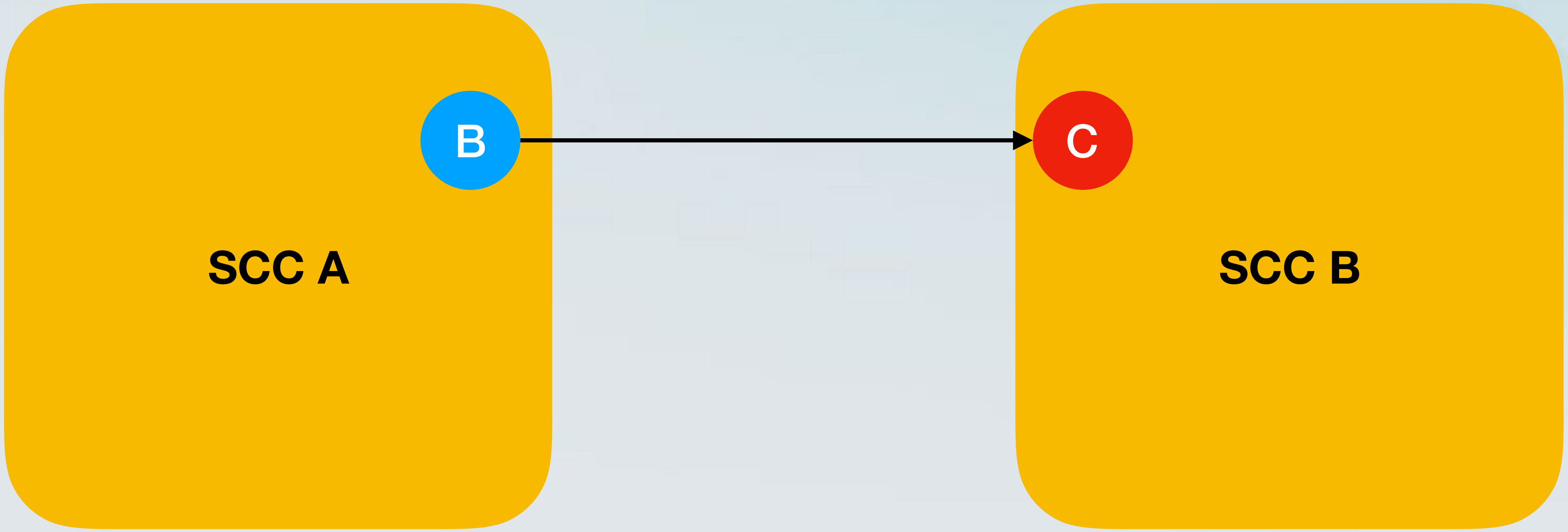

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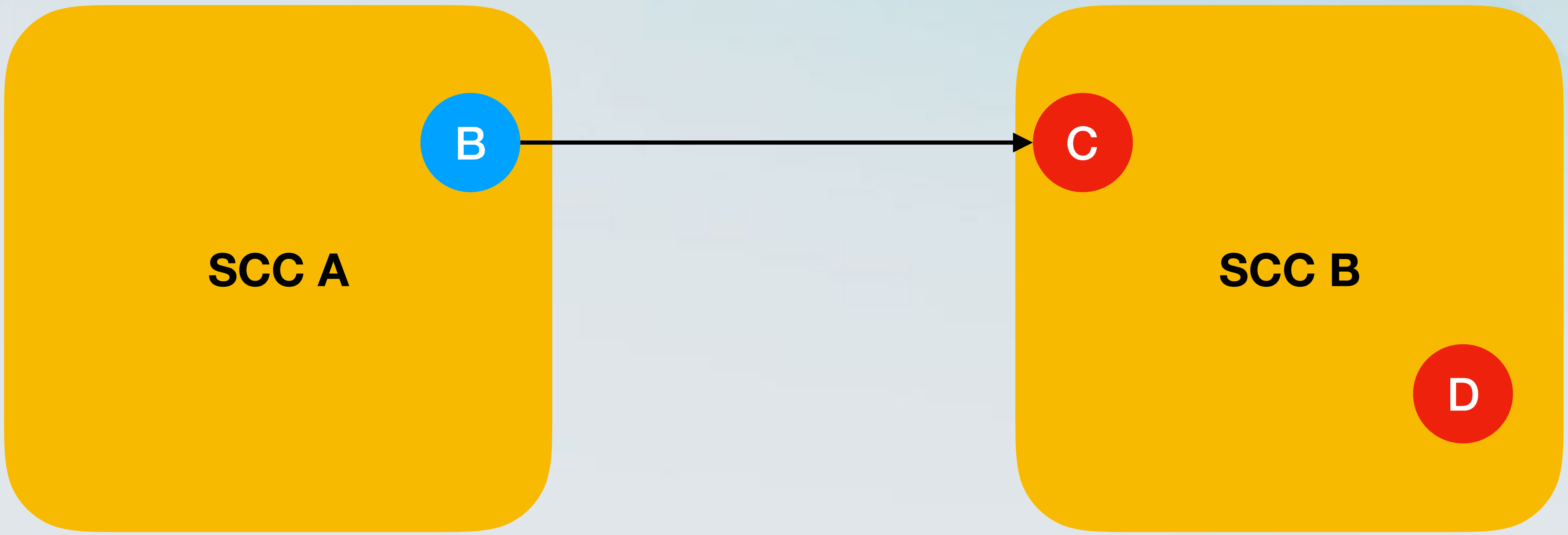

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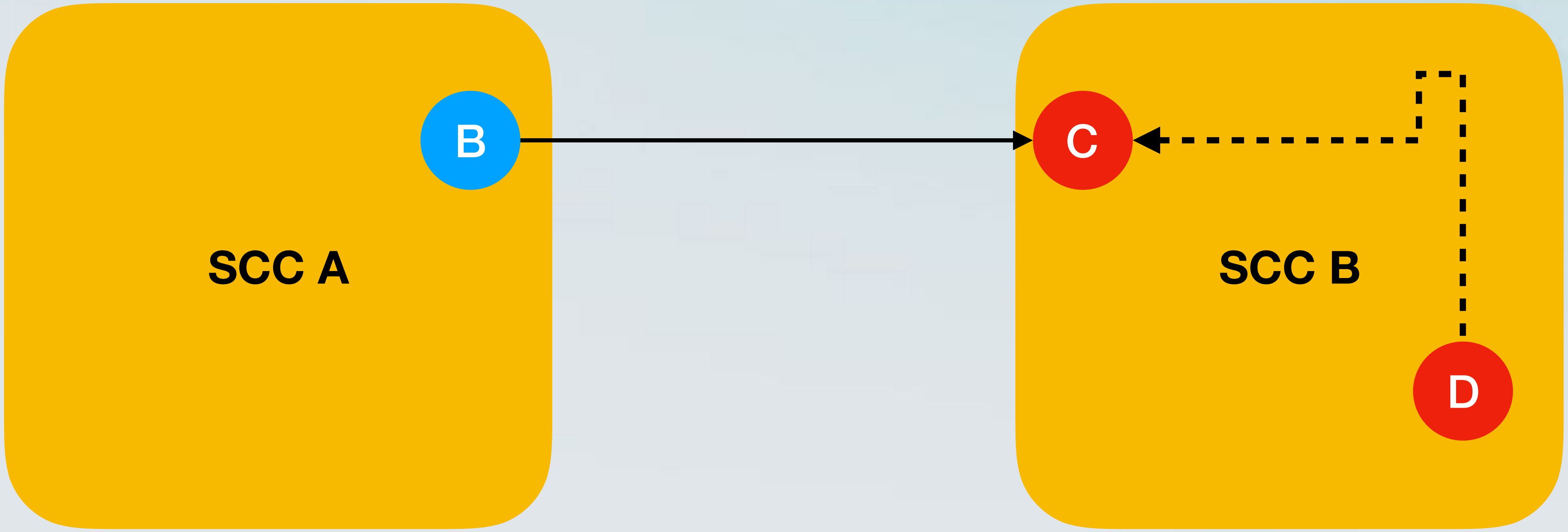

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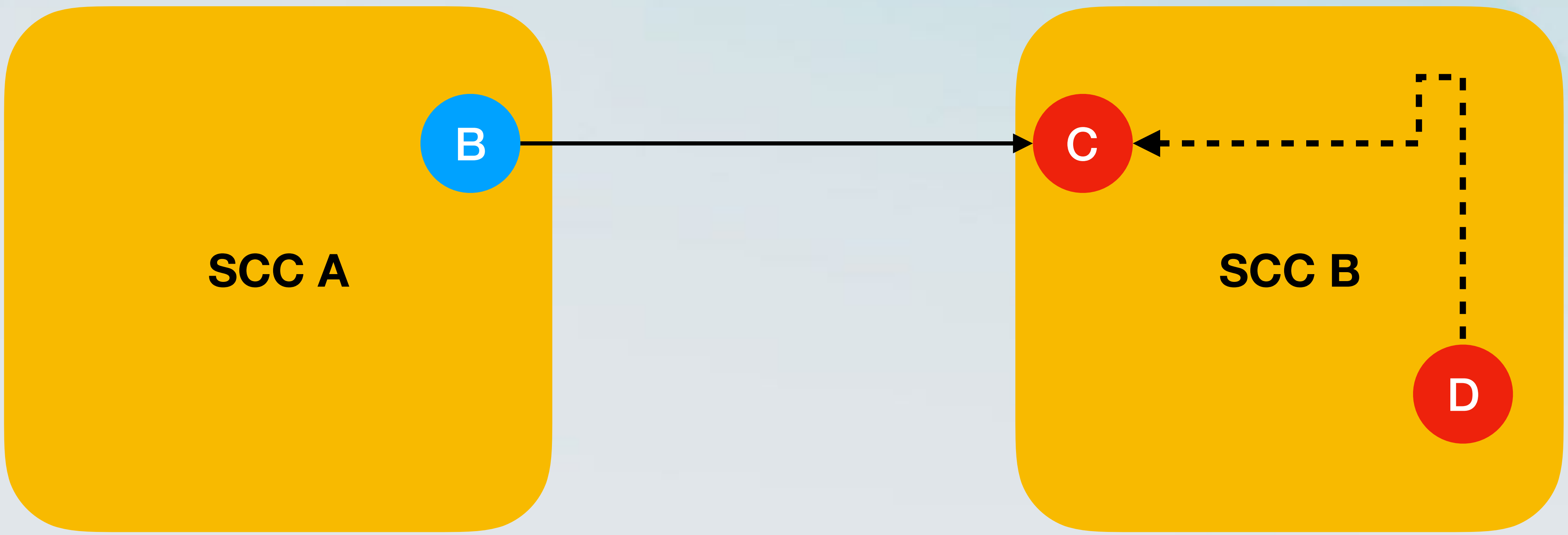

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There is a path C-D because of same SCC

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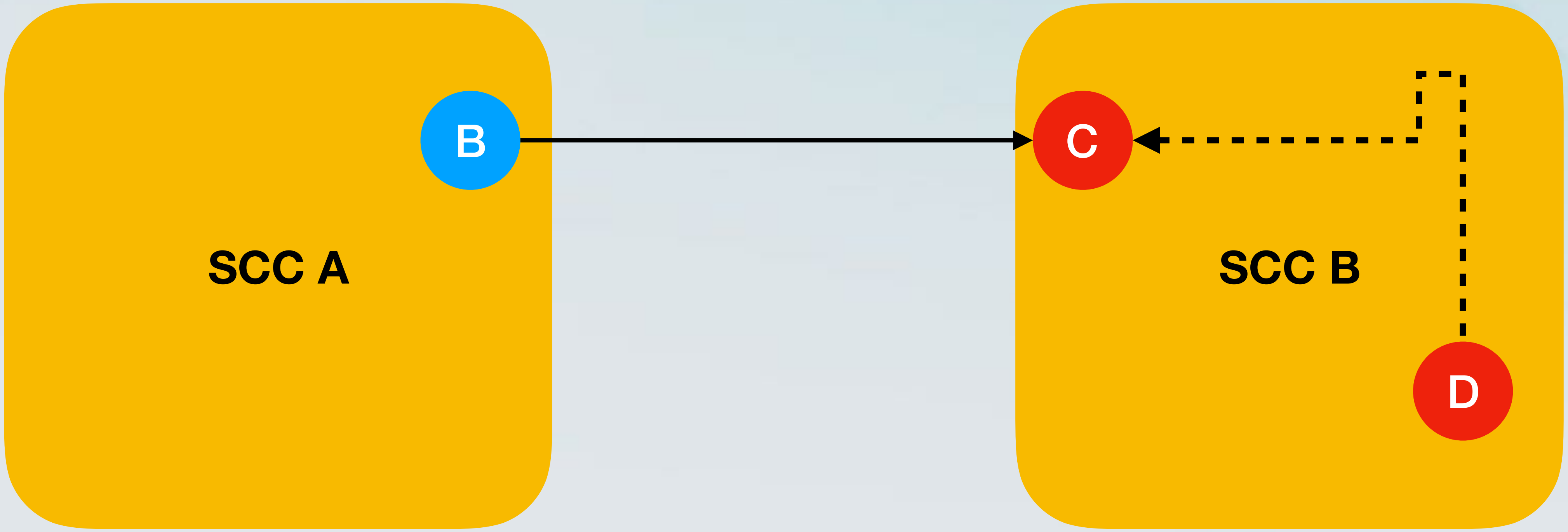

There is no path from SCC B to SCC A, by the simple-but-key lemma.

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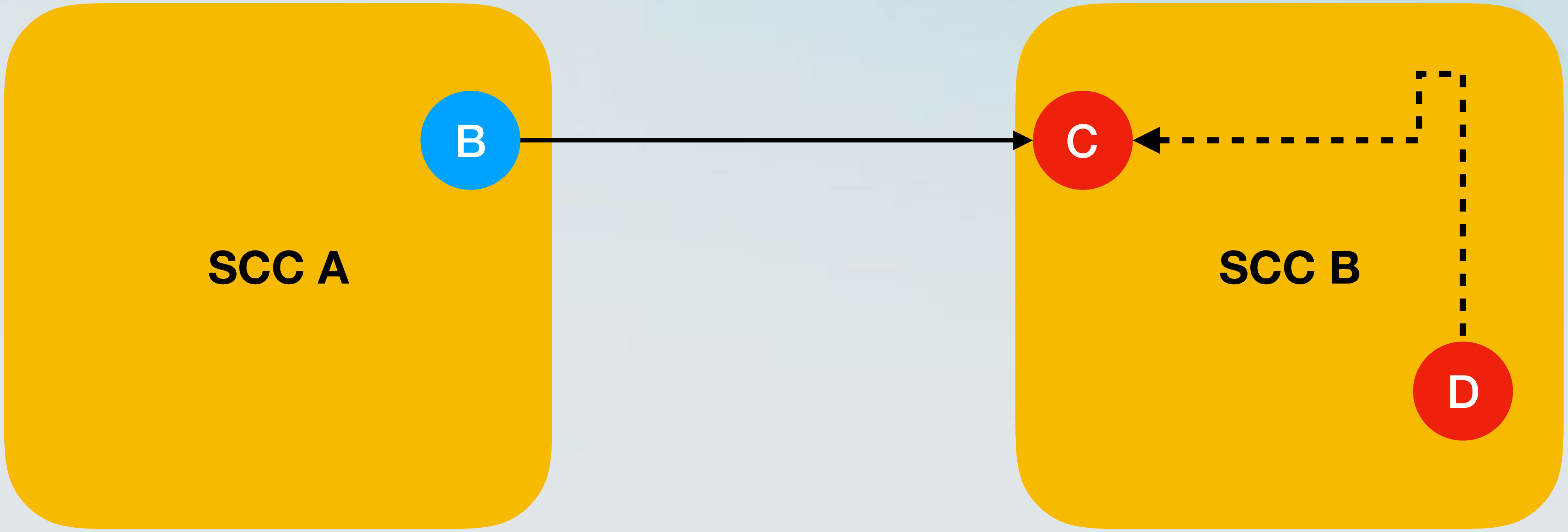

There is no path from SCC B to SCC A, by the simple-but-key lemma.

Case 2: We start here

There is a path C-D because of same SCC

DFS will finish with SCC B before it moves to SCC A.

Lemma and Corollary

- Lemma: Let C and C' be distinct SCCs in G. Suppose there is a directed edge ***crossing*** C and C'. Then the DFS on the nodes of C finishes later than the DFS on the nodes of C'.

- Corollary: If the forward DFS finishes on component C later than component C', then

  - there is no edge crossing from C' to C in G.

  - there is no edge crossing from C to C' in G^{rev}.

- This means that in the backward DFS on G^{rev}, if we start with the SCC that finishes last in the forward DFS of G, we will not find edges to other SCCs.

Intuition

Intuition

DFS finished here first in the forward pass.

Intuition

SCC A

DFS finished here first in the forward pass.

SCC B

This means there are no edges from SCC A to SCC B in the reverse graph.

Intuition

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DFS finished here first in the forward pass.

DFS will finish here before it moves to any other SCC.

SCC B

This means there are no edges from SCC A to SCC B in the reverse graph.

Intuition

SCC A

SCC B

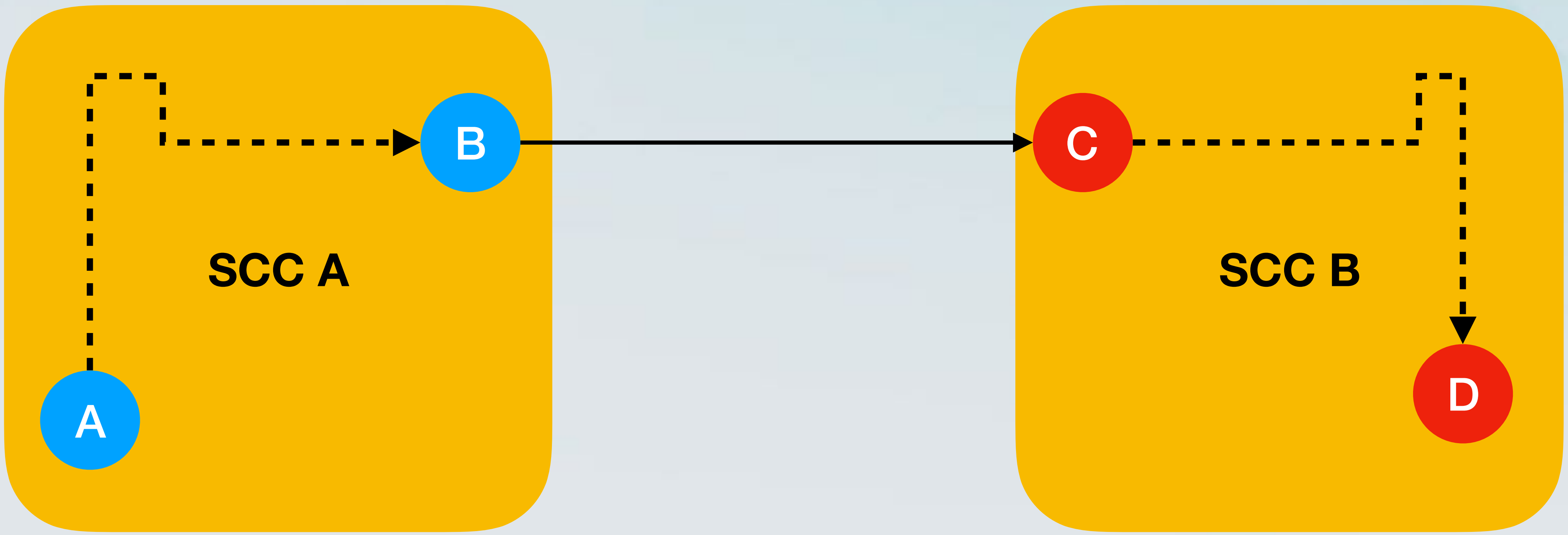
DFS finished here first in the forward pass.

This means there are no edges from SCC A to SCC B in the reverse graph.

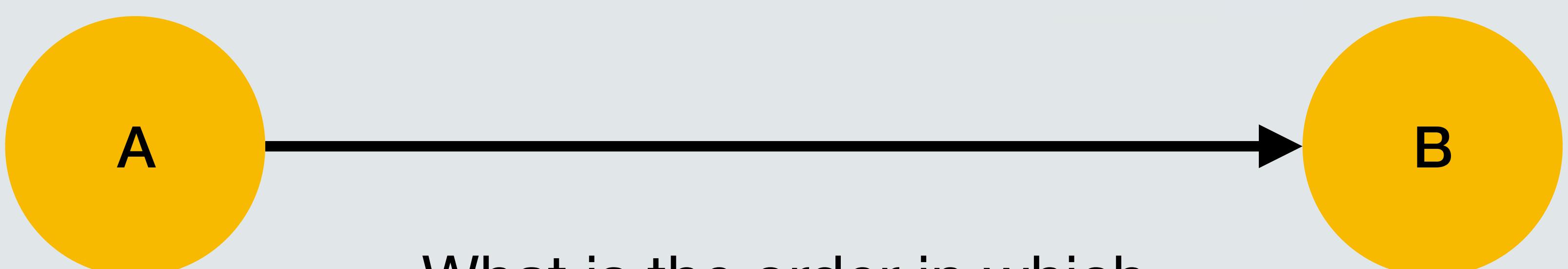
DFS will finish here before it moves to any other SCC.

SCC A will be correctly indentified.

Back to the component graph



This finishes first



What is the order in which the backward DFS visits the nodes?

Back to the component graph

Back to the component graph

- The backward DFS visits the nodes of G^{SCC} in topological order.

Back to the component graph

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Back to the component graph

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- Alternative viewpoint:

  - Produce a topological order of G^{SCC}.

  - Run a DFS on G^{rev} considering SCCs according to that topological order.