COMP523 Tutorial 8 - Solutions

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Problem 1

Kleinberg and Tardos - Algorithm Design, Chapter 11, Solved Exercise 1.

Solution

See solution in the book, Kleinberg and Tardos - Algorithm Design, Chapter 11, Solved Exercise 1.

Problem 2

Prove that the optimal solutions to the LP-relaxation of the Vertex Cover ILP are half-integral, meaning that they take values in $\left\{0, \frac{1}{2}, 1\right\}$.

Solution

First, we remark that for a solution to be optimal, it can not be a convex combination of two other feasible solutions, where a convex combination is a sum weighted by probabilities, i.e., if $x = \alpha x_1 + (1-\alpha)x_2$, then x is a convex combination of x_1 and x_2. This follows from the fact that, if that was the case, then we could take the probability mass from the solution that has a larger value and move it to the solution that has a smaller value, obtaining a feasible solution with smaller overall value, contradicting the optimality of x.

We will now argue that if a solution to the Vertex Cover ILP is not half-integral, then it can be expressed as a convex combination of two feasible solutions; from the discussion above, this implies that it can not be optimal. Consider the set of vertices for which the solution x does not assign a half-integral value, and partition them into two sets V_+ and V_- such that:

$$V_+ = \left\{v : \frac{1}{2} < x_v < 1\right\} \quad \text{and} \quad V_- = \left\{v : 0 < x_v < \frac{1}{2}\right\}.$$

For $\varepsilon > 0$, defined the following two solutions:

$$y_v = \begin{cases} x_v + \varepsilon, & x_v \in V_+ \\ x_v - \varepsilon, & x_v \in V_- \\ x_v, & \text{otherwise.} \end{cases} \qquad z_v = \begin{cases} x_v - \varepsilon, & x_v \in V_+ \\ x_v + \varepsilon, & x_v \in V_- \\ x_v, & \text{otherwise.} \end{cases}$$

By assumption, we have that $V_+ \cup V_- \neq \emptyset$ and so x, y and z are all distinct. Additionally, it is not hard to see that $x = \frac{1}{2}(y+z)$ and therefore x is a convex combination of y and z. It remains to show that y and z are feasible solutions to the Vertex Cover LP-relaxation; we will achieve that via an appropriate choice of ε.

First, it is easy to ensure that $y_v, z_v \geq 0$; since $x_v > 0$, we can choose ϵ small enough to ensure that this holds. Next, consider the constraints $x_u + x_v \geq 1$ of the Vertex Cover LP-relaxation. Suppose that $x_u + x_v > 1$. Then again, by choosing ε to be small enough, we can ensure that $y_u + y_v \geq 1$ and $z_u + z_v \geq 1$. Finally, consider any edge for which $x_u + x_v = 1$. There, there are only three possibilities: either (a)

$x_u = x_v = \frac{1}{2}$, (b) $x_u = 0, x_v = 1$ (the case $x_u = 1, x_v = 0$ is completely symmetric) and (c) $u \in V_+, v \in V_-$ (the case $u \in V_-, v \in V_+$ is completely symmetric). In all three cases, for any choice of ε, we have that

$$x_u + x_v = y_u + y_v = z_u + z_v = 1.$$

This completes the proof.